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Student Edition

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1 2 3 4 5 6 7 8 073 15 14 13 12 11 10 09 08

Table of Contents

Section 1: Lessons 1-10, Investigation 1

	LESSON		PAGE
	1	Using Properties of Real Numbers	2
	2	Evaluating Expressions and Combining Like Terms	8
	3	Using Rules of Exponents	13
	LAB 1	**Graphing Calculator:** Graphing a Function and Building a Table	19
	4	Identifying Functions and Using Function Notation	21
	LAB 2	**Graphing Calculator:** Storing and Recalling Data in a Matrix	27
	5	Using Matrices to Organize Data and to Solve Problems	29
	6	Finding Percent of Change	36
	7	Solving Linear Equations **Exploration:** Solving Equations Using Algebra Tiles	42
	8	Finding Direct Variation	48
	9	Multiplying Matrices	54
	10	Solving and Graphing Inequalities	61
	INV 1	**Investigation:** Logic and Truth Tables	69

DISTRIBUTED STRANDS

- Number Sense and Foundations of Algebra
- Linear Functions
- Matrices
- Polynomials and Polynomial Functions
- Rational and Radical Functions
- Linear Systems
- Probability and Statistics
- Quadratic Functions
- Trigonometry
- Exponential and Logarithmic Functions
- Conic Sections
- Sequences, Series, and Logic

Section 2: Lessons 11-20, Investigation 2

	LESSON		PAGE
	11	Understanding Polynomials	72
	12	Solving Inverse Variation Problems	77
	LAB 3	**Graphing Calculator:** Calculating Points on a Graph	84
	13	Graphing Linear Equations I	86
	14	Finding Determinants	93
	15	Solving Systems of Equations by Graphing	100
	16	Using Cramer's Rule	107
	LAB 4	**Graphing Calculator:** Changing the Line and Window of a Graph	114
	17	Solving Equations and Inequalities with Absolute Value **Exploration:** Transforming $f(x) = \lvert x \rvert$	116
	18	Calculating with Units of Measure	124
	19	Multiplying Polynomials	129
	20	Performing Operations with Functions	136
	INV 2	**Investigation:** Solving Parametric Equations	143

Section 3: Lessons 21-30, Investigation 3

LESSON		PAGE
21	Solving Systems of Equations Using the Substitution Method	146
LAB 5	**Graphing Calculator:** Storing and Plotting a List of Data	153
22	Analyzing Continuous, Discontinuous, and Discrete Functions	155
23	Factoring Polynomials	163
24	Solving Systems of Equations Using the Elimination Method	170
LAB 6	**Graphing Calculator:** Calculating 1- and 2-Variable Statistical Data	178
25	Finding Measures of Central Tendency and Dispersion	180
26	Writing the Equation of a Line	187
27	Connecting the Parabola with the Quadratic Function	194
28	Simplifying Rational Expressions	201
29	Solving Systems of Equations in Three Variables	207
30	Applying Transformations to the Parabola and Determining the Minimum or Maximum	215
INV 3	**Investigation:** Graphing Three Linear Equations in Three Variables	222

DISTRIBUTED STRANDS

- Number Sense and Foundations of Algebra
- Linear Functions
- Matrices
- Polynomials and Polynomial Functions
- Rational and Radical Functions
- Linear Systems
- Probability and Statistics
- Quadratic Functions
- Trigonometry
- Exponential and Logarithmic Functions
- Conic Sections
- Sequences, Series, and Logic

Section 4: Lessons 31-40, Investigation 4

LESSON		PAGE
31	Multiplying and Dividing Rational Expressions	226
32	Solving Linear Systems with Matrix Inverses **Exploration:** Exploring Matrix Inverses	233
33	Applying Counting Principles	240
34	Graphing Linear Equations II	246
35	Solving Quadratic Equations I	253
36	Using Parallel and Perpendicular Lines	259
37	Adding and Subtracting Rational Expressions	266
38	Dividing Polynomials Using Long Division	273
39	Graphing Linear Inequalities in Two Variables	279
40	Simplifying Radical Expressions	286
INV 4	**Investigation:** Understanding Cryptography	293

Section 5: Lessons 41-50, Investigation 5

LESSON		PAGE
41	Using the Pythagorean Theorem and the Distance Formula **Exploration:** Visualizing the Pythagorean Theorem	296
LAB 7	**Graphing Calculator:** Calculating Permutations and Combinations	303
42	Finding Permutations and Combinations **Exploration:** Pascal's Triangle and Combinations	304
43	Solving Systems of Linear Inequalities	312
44	Rationalizing Denominators	318
LAB 8	**Graphing Calculator:** Applying Linear and Median Regression	323
45	Finding the Line of Best Fit **Exploration:** Collecting and Analyzing Data	325
46	Finding Trigonometric Functions and their Reciprocals	331
47	Graphing Exponential Functions	337
48	Understanding Complex Fractions	343
49	Using the Binomial Theorem	348
50	Finding Inverses of Relations and Functions **Exploration:** Graphing a Function and its Inverse	355
INV 5	**Investigation:** Finding the Binonomial Distribution	361

DISTRIBUTED STRANDS

- Number Sense and Foundations of Algebra
- Linear Functions
- Matrices
- Polynomials and Polynomial Functions
- Rational and Radical Functions
- Linear Systems
- Probability and Statistics
- Quadratic Functions
- Trigonometry
- Exponential and Logarithmic Functions
- Conic Sections
- Sequences, Series, and Logic

Section 6: Lessons 51-60, Investigation 6

LESSON		PAGE
51	Using Synthetic Division	364
LAB 9	**Graphing Calculator:** Using the Trigonometry Keys and Adjusting to a Trigonometric Window	370
52	Using Two Special Right Triangles	372
53	Performing Compositions of Functions	378
54	Using Linear Programming	384
55	Finding Probability	391
56	Finding Angles of Rotation	399
57	Finding Exponential Growth and Decay	406
58	Completing the Square **Exploration:** Modeling Completing the Square	413
59	Using Fractional Exponents	420
60	Distinguishing Between Mutually Exclusive and Independent Events	427
INV 6	**Investigation:** Deriving the Quadratic Formula	433

Section 7: Lessons 61-70, Investigation 7

LESSON		PAGE
61	Understanding Advanced Factoring	436
62	Using Complex Numbers	442
63	Understanding the Unit Circle and Radian Measures **Exploration:** Exploring the Unit Circle	447
LAB 10	**Graphing Calculator:** Using the Log Keys	455
64	Using Logarithms	457
65	Using the Quadratic Formula	462
66	Solving Polynomial Equations	469
67	Finding Inverse Trigonometric Functions	476
68	Finding Conditional Probability	483
69	Simplifying Complex Expressions	489
70	Solving Radical Equations	495
INV 7	**Investigation:** Collecting Data	502

DISTRIBUTED STRANDS

- Number Sense and Foundations of Algebra
- Linear Functions
- Matrices
- Polynomials and Polynomial Functions
- Rational and Radical Functions
- Linear Systems
- Probability and Statistics
- Quadratic Functions
- Trigonometry
- Exponential and Logarithmic Functions
- Conic Sections
- Sequences, Series, and Logic

Section 8: Lessons 71-80, Investigation 8

LESSON		PAGE
71	Using the Law of Sines	506
72	Using the Properties of Logarithms **Exploration:** Discovering the Properties of Logarithms	512
LAB 11	**Graphing Calculator:** Calculating Confidence Intervals	519
73	Using Sampling	521
74	Finding the Discriminant	528
75	Graphing Radical Functions	534
76	Finding Polynomial Roots I	540
77	Using the Law of Cosines	546
78	Solving Quadratic Equations II	552
79	Understanding Piecewise Functions	558
LAB 12	**Graphing Calculator:** Calculating Normal Distribution Areas and Z-Scores	563
80	Finding the Normal Distribution	565
INV 8	**Investigation:** Finding the Area Under a Curve	572

Section 9: Lessons 81-90, Investigation 9

LESSON		PAGE
81	Using Natural Logarithms	574
82	Graphing Sine and Cosine Functions	580
83	Writing Quadratic Equations from Roots	586
84	Solving Rational Equations **Exploration:** Exploring Rational Functions	592
85	Finding Polynomial Roots II	598
86	Translating Sine and Cosine Functions	606
87	Evaluating Logarithmic Expressions	612
88	Solving Abstract Equations	617
89	Solving Quadratic Inequalities	623
90	Graphing the Tangent Function	630
INV 9	**Investigation:** Understanding Step Functions	635

DISTRIBUTED STRANDS

- Number Sense and Foundations of Algebra
- Linear Functions
- Matrices
- Polynomials and Polynomial Functions
- Rational and Radical Functions
- Linear Systems
- Probability and Statistics
- Quadratic Functions
- Trigonometry
- Exponential and Logarithmic Functions
- Conic Sections
- Sequences, Series, and Logic

Section 10: Lessons 91-100, Investigation 10

LESSON		PAGE
LAB 13	**Graphing Calculator:** Graphing in Circles and Polar Equations	638
91	Making Graphs and Solving Equations of Circles	640
92	Finding Arithmetic Sequences **Exploration:** Exploring Sequences	646
93	Solving Exponential Equations and Inequalities	652
94	Solving Rational Inequalities	658
95	Factoring Higher-Order Polynomials	665
96	Using Polar Coordinates	671
97	Finding Geometric Sequences	678
98	Making Graphs and Using Equations of Ellipses	684
99	Using Vectors	690
100	Graphing Rational Functions I	696
INV 10	**Investigation:** Graphing Polar Models	703

Section 11: Lessons 101-110, Investigation 11

LESSON		PAGE
101	Making Graphs of Polynomial Functions	706
102	Solving Logarithmic Equations and Inequalities	714
103	Graphing Reciprocal Trigonometric Functions	719
104	Finding Transformations	725
105	Finding Arithmetic Series	732
106	Using the Fundamental Theorem of Algebra **Exploration:** Making Conjectures About Roots of Polynomial Functions	738
107	Graphing Rational Functions II	745
108	Using Fundamental Trigonometric Identities	752
109	Making Graphs and Using Equations of Hyperbolas	757
110	Graphing Logarithmic Functions	764
INV 11	**Investigation:** Using De Moivre's Theorem	770

DISTRIBUTED STRANDS

- Number Sense and Foundations of Algebra
- Linear Functions
- Matrices
- Polynomials and Polynomial Functions
- Rational and Radical Functions
- Linear Systems
- Probability and Statistics
- Quadratic Functions
- Trigonometry
- Exponential and Logarithmic Functions
- Conic Sections
- Sequences, Series, and Logic

Section 12: Lessons 111-120, Investigation 12

LESSON		PAGE
111	Transforming Polynomial Functions	774
112	Using Sum and Difference Identities	780
113	Using Geometric Series **Exploration:** Exploring an Infinite Geometric Series	786
114	Identifying Conic Sections	793
115	Finding Double-Angle and Half-Angle Identities	798
LAB 14	**Graphing Calculator:** Determining Regression Models	804
116	Finding Best Fit Models	806
117	Solving Systems of Nonlinear Equations	813
118	Recognizing Misleading Data	819
119	Solving Trigonometric Equations	825
INV 12	**Investigation:** Using Mathematical Induction	831

Appendix 834

LESSON		PAGE
1	Changes in Measure	834
2	Computer Spreadsheets	838
3	Precision and Accuracy	842
4	Predictions	846
5	Scale Factor	850
6	Regions and Solids	854
7	Apply Scientific Notation	858

Skills Bank 862

LESSON		PAGE
1	Estimation	862
2	Mental Math	863
3	Exponents	864
4	Operations with Decimals	865
5	Compare and Order Rational Numbers	866
6	Operations with Fractions	867
7	Negative Numbers/Operations with Integers	868
8	Ratios, Proportions, Percents	869
9	Time, Rate, Distance	870
10	Coordinate Plane/Ordered Pairs	871
11	Plane Figures and Coordinate Geometry	872
12	Parallel Lines and Transversals	873
13	Measure Angles	874
14	Angle Relationships	875
15	Properties of Polygons	876
16	Geometric Formulas	877
17	Area of Polygons, Circles, and Composite Figures	878
18	Angle Relationships in Circles and Polygons	879
19	Views of Solid Figures	880
20	Geometric Patterns and Tessellations	881
21	Stem and Leaf Plots	882
22	Statistical Graphs	883
23	Proofs	884
24	Venn Diagrams	885

Properties and Formulas 886

		PAGE
	Properties	886
	Formulas	890
	Symbols	891
	Metric Measures	892
	Customary Measures	892

Glossary 894

Index 957

LESSON 1

Using Properties of Real Numbers

Warm Up

Start off each lesson by practicing prerequisite skills and math vocabulary that will make you more successful with today's new concept.

1. (SB) **Vocabulary** A number whose value is less than 0 is called a _____ number. negative

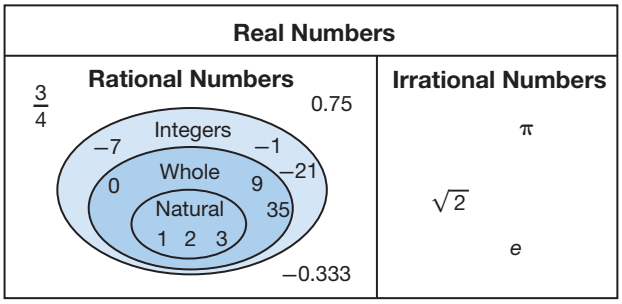
2. (SB) Add $12 + (-12)$. 0

3. (SB) Multiply $\frac{2}{3} \cdot \frac{3}{2}$. 1

4. (SB) Subtract $10 - 9.85$. 0.15

New Concepts

A set is a collection of objects. If all of the objects in one set are also members of a second set, then the first set is a subset of the second. The real numbers are a set of numbers consisting of several subsets of numbers. The **real numbers** consist of the **rational numbers** and the **irrational numbers.** The rational numbers include the natural numbers $\{1, 2, 3, \ldots\}$, the **whole numbers** $\{0, 1, 2, 3, \ldots\}$, and the **integers** $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.



Rational numbers can be written as a fraction of integers. Rational numbers can also be written as decimals that terminate or repeat. Irrational numbers cannot be written in any of these ways. Pi and many roots of real numbers are common examples of irrational numbers.

Example 1 Classifying Real Numbers

Caution

The negative integers are not considered whole numbers even though they do not contain fractions.

Identify the subsets of real numbers of which each number is a member.

a. -5

SOLUTION -5 is a real number, a rational number, and an integer.

b. 25

SOLUTION 25 is a real number, a rational number, an integer, a whole number, and a natural number.

c. $-\sqrt{5}$

SOLUTION $-\sqrt{5}$ is a real number and an irrational number.

Online Connection
www.SaxonMathResources.com

There are a number of properties that can be used when adding and multiplying real numbers.

Properties of Addition and Multiplication

Let a, b, and c be real numbers.

Property	Addition	Multiplication
Closure	$a + b$ is a real number	ab is a real number
Commutative	$a + b = b + a$	$ab = ba$
Associative	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$

The following property involves both addition and multiplication.

Distributive	$a(b + c) = ab + ac$

Each day brings you a **New Concept** where a new topic is introduced and explained through thorough **Examples** — using a variety of methods and real-world applications.

You will be reviewing and building on this concept throughout the year to gain a solid understanding and ensure mastery on the test.

Example 2 Identifying Properties of Real Numbers

Identify which property of real numbers is being demonstrated.

a. $3 \cdot 8 = 8 \cdot 3 = 24$

SOLUTION Commutative Property of Multiplication

b. $5(7 + 11) = 5 \cdot 7 + 5 \cdot 11 = 35 + 55 = 90$

SOLUTION Distributive Property

c. $(12 + 20) + 30 = 12 + (20 + 30) = 12 + 50 = 62$

SOLUTION Associative Property of Addition

Example 3 Using the Properties of Real Numbers

Simplify each expression. Identify which property you used for each step.

a. $12 + 4 + 18 + 56$

SOLUTION

$12 + 4 + 18 + 56$

$= 12 + 18 + 4 + 56$ Commutative Property of Addition

$= (12 + 18) + (4 + 56)$ Associative Property of Addition

$= 30 + 60$ Add.

$= 90$ Add.

b. $5(23)$

SOLUTION

$5(23)$

$= 5(20 + 3)$

$= (5 \cdot 20) + (5 \cdot 3)$ Distributive Property

$= 100 + 15$ Multiply.

$= 115$ Add.

Hint

The Distributive Property makes multiplying easier by breaking down a number into numbers where mental math can be used.

More Properties of Addition and Multiplication		
Property	**Addition**	**Multiplication**
Identity	$a + 0 = a, 0 + a = a$	$a \cdot 1 = a, 1 \cdot a = a$
Inverse	$a + (-a) = 0$	$a \cdot \frac{1}{a} = 1, a \neq 0$

Math Language

The **additive inverse** of a number is sometimes called the opposite of the number.

In some lessons, **Explorations** allow you to go into more depth with the mathematics by investigating math concepts with manipulatives, through patterns, and in a variety of other ways.

Example 4 Finding Inverses of Real Numbers

a. Find the additive inverse of $-7a$.

SOLUTION The additive inverse of $-7a$ is $7a$ since $-7a + 7a = 0$.

b. Find the multiplicative inverse of $\frac{5n}{12p}$.

SOLUTION The multiplicative inverse is the reciprocal of $\frac{5n}{12p}$ which is $\frac{12p}{5n}$.

$$\frac{5n}{12p} \cdot \frac{12p}{5n} = 1$$

Example 5 Application: Finance

DVDs are on sale at Tech World for \$11.95 each including tax. Use the Distributive Property to mentally calculate the total cost of buying 5 DVDs.

SOLUTION Think of \$11.95 as \$12.00 − \$0.05.

Write an expression for the cost of 5 DVDs.

$5(12 - 0.05)$

$= 5(12) - 5(0.05)$ Use the Distributive Property.

$= 60 - 0.25$ Multiply.

$= 59.75$ Subtract.

The total cost of 5 DVDs is \$59.75.

The **Lesson Practice** lets you check to see if you understand today's new concept.

The italic numbers refer to the Example in this lesson in which the major concept of that particular problem is introduced. You can refer to lesson examples if you need additional help.

Lesson Practice

a. *(Ex 1)* Identify the subsets of real numbers of which $\sqrt{3}$ is a member.

real numbers and irrational numbers

b. *(Ex 1)* Identify the subsets of real numbers of which $-\frac{2}{3}$ is a member.

real numbers and rational numbers

c. *(Ex 1)* Identify the subsets of real numbers of which 1 is a member.

real numbers, rational numbers, integers, whole numbers, and natural numbers

d. *(Ex 2)* Identify which property of real numbers is being demonstrated.

Distributive Property

$$-2(5 + 9) = -2 \cdot 5 + (-2) \cdot 9 = -10 - 18 = -28$$

e. *(Ex 2)* Identify which property of real numbers is being demonstrated.

Associative Property of Multiplication

$$(12 \cdot 6) \cdot 2 = 12 \cdot (6 \cdot 2) = 12 \cdot 12 = 144$$

f. *(Ex 2)* Identify which property of real numbers is being demonstrated.

$3 + 21 = 21 + 3 = 24$ Commutative Property of Addition

g. *(Ex 3)* Simplify the expression $(43 + 21) + 9$. Identify which property you used for each step.

$43 + (21 + 9)$ Associative Property of Addition

$= 43 + 30$ Add.

$= 73$ Add.

h. $(16 \cdot 4) \cdot 4$ Commutative Property of Multiplication
$16 \cdot (4 \cdot 4)$ Associative Property of Multiplication
$16 \cdot 16$ Multiply.
256 Multiply.

h. *(Ex 3)* Simplify the expression $(4 \cdot 16) \cdot 4$. Identify which property you used for each step.

i. *(Ex 4)* Find the additive inverse of $\frac{3q}{2}$. $-\frac{3q}{2}$

j. *(Ex 4)* Find the multiplicative inverse of -4. $-\frac{1}{4}$

k. *(Ex 5)* Lucinda drove four hours from her house to a family reunion. Her average speed was 55 mph. Use the Distributive Property to mentally calculate the distance between Lucinda's house and the reunion. 220 miles

Practice Distributed and Integrated

The ***italic numbers*** refer to the lesson(s) in which the major concept of that particular problem is introduced. You can refer to the examples or practice in that lesson, if you need additional help.

Simplify each expression.

1. *(SB)* $-3 - 6 + 1$ -8

2. *(SB)* $-4 + 6 - 8 =$ -6

Simplify each expression using the properties of real numbers.

***3.** *(1)* $-2[(5 - 7 - 2) - (-2 - 7) - 2]$ -6

***4.** *(1)* $4[3 - (-2)] + 5(-2 + 1)$ 15

Simplify each expression.

5. *(SB)* $14.6 - 9.03$ 5.57

6. *(SB)* $39.75 + 49.2$ 88.95

Order the given numbers from least to greatest.

7. *(SB)* 2.1, 2.3, 2.09 2.09, 2.1, 2.3

8. *(SB)* $\frac{3}{5}, \frac{4}{7}, \frac{1}{2}$ $\frac{1}{2}, \frac{4}{7}, \frac{3}{5}$

Find the distance traveled given the rate and time below.

9. *(SB)* rate = 40 miles per hour, time = $3\frac{1}{2}$ hours 140 miles

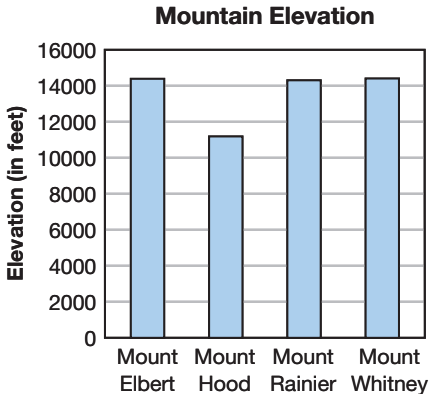
10. *(SB)* rate = 55 miles per hour, time = 0.5 hours 27.5 miles

***11.** *(1)* **Write** Explain the steps you would use to simplify $-2(28 - 19) + 6$. Then find the value. First subtract $28 - 19$, then multiply by -2, then add 6. -12

***12.** *(1)* **Justify** Simplify $2 \cdot 3 \cdot 6$. Justify your answer by identifying which property you used for each step. Possible answer: $2(3 \cdot 6)$, Associative Property of Multiplication; 2(18), Multiply; 36, Multiply.

13. *(SB)* **Geography** The following graph represents the elevations of mountains located in the United States.

Which mountain has the lowest elevation? Mount Hood



14. (SB) **Sports** A football field is 120 yards long and 160 feet wide. What is the area in square yards? 6400 square yards

***15.** (1) **Verify** Simplify the expression $(6 \cdot 5) \cdot 2$ using only one property of real numbers. Identify the property that you used. Verify your result using two properties of real numbers.

***16.** (1) **Multiple Choice** Identify the subset of real numbers of which 7 is not a member. **A**

A irrational number **B** integer

C whole number **D** real number

17. (SB) **Multi-Step** Suzy can read 12 pages in four minutes. Jonas can read 9 pages in three minutes.

a. Find the number of pages Suzy and Jonas can read per minute.

b. Compare the rates. Suzy and Jonas read at the same rate.

***18.** (1) **Multiple Choice** Identify which property or properties of real numbers are being demonstrated. **D**

$$27 \cdot 3 = (20 \cdot 3) + (7 \cdot 3) = 60 + 21 = 81$$

A Commutative Property of Addition

B Associative Property of Addition

C Both A and B

D Distributive Property

19. (SB) **Sports** The specified circumference of an official professional baseball is from 9 to 9.25 inches.

a. Find the approximate radius of the smallest official baseball. 1.43 inches

b. Find the approximate surface area of the smallest official baseball. 25.78 square inches

σ ***20.** (1) **Statistics** The table below shows the number of homeruns hit by a softball player over a three-year period. Use properties of real numbers to mentally calculate the average annual homeruns that she hit during this period. 11 homeruns

	Year 1	Year 2	Year 3
Homeruns	6	13	14

***21.** (1) **Interest Rates** The average credit card annual interest rate is around 19%. Use the Distributive Property to mentally calculate the amount of interest paid annually on a $500 balance.

22. (SB) **Biology** Some large species of bamboo plants can grow at a rate of 1 meter per day.

a. If a 2-meter tall bamboo plant grows at this rate, how tall will it be after 2 weeks? 16 m

b. How long will it take for the bamboo plant to reach a height of 50 meters? 48 days

15. $(6 \cdot 5) \cdot 2 = 6 \cdot (5 \cdot 2) = 6 \cdot 10 = 60$, Associative Property of Multiplication; $(6 \cdot 5) \cdot 2 = (5 \cdot 6) \cdot 2 = 5 \cdot (6 \cdot 2) = 5 \cdot 12 = 60$, Commutative Property of Multiplication and Associative Property of Multiplication

17a. Suzy: $\frac{3 \text{ pages}}{1 \text{ minute}}$; Jonas: $\frac{3 \text{ pages}}{1 \text{ minute}}$.

In the **Practice,** you will review today's new concept as well as math you learned in earlier lessons. By practicing problems from many lessons every day, you will begin to see how math concepts relate and connect to each other and to the real world.

Also, because you practice the same topic in a variety of ways over several lessons, you will have "time to learn" the concept and will have opportunities to show that you understand.

The mixed set of Practice is just like the mixed format of your state test. You'll be practicing for the "big" test every day!

21. Think of 19% as 0.20 − 0.01. Write an expression for the annual interest: 500(0.20 − 0.01). Use the Distributive Property: 500(0.20) − 500(0.01). Multiply: 100 − 5. Subtract: 95. The annual interest is $95.

*23. (1) **Geometry** The base of a triangle equals $\frac{1}{m}$ and the height equals m. The area of the triangle can be calculated as shown below. Which properties of real numbers are demonstrated? Inverse Property of Multiplication, Identity Property of Multiplication

$$A = \frac{1}{2}(bh) = \frac{1}{2}\left(\frac{1}{m} \cdot m\right) = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

The starred problems usually cover challenging or recently presented content. Because of that, it is suggested that these exercises be worked first, in case you might want help.

24. (SB) Carlos measures an angle x. He then finds that the complement of the angle is 35°.

a. What is the measure of angle x? 55°

b. What is the supplement of angle x? 125°

*25. (1) **Error Analysis** Explain the error in the statement that follows and correct the statement. "−7 is a real number, a rational number, and a whole number."

25. Negative integers are not considered whole numbers. The statement should be "–7 is a real number, a rational number, and an integer."

26. (SB) **Estimate** The chart below shows the total amount of hybrid vehicles sold by a manufacturer over a 4-year time span. Round to the nearest ten-thousand to estimate:

a. the total number of hybrid vehicles sold from 2003 to 2006. 590,000

b. the sales increase from 2003 to 2006. 200,000

Year	2003	2004	2005	2006
Cars Sold	47,525	83,153	209,711	246,642

*27. (1) **Graphing Calculator** Use the Distributive Property to mentally calculate 12(18). Check using a graphing calculator. 216

28. (SB) **Error Analysis** Two students calculated the total cost for 3 CDs. Each CD cost $14.95. Which student has the correct answer? Explain the other student's error.
Student B is correct. Student A did not distribute the 3 to the second term.

Student A
$3(15 - 0.05)$
$3(15) - (0.05)$
$45 - 0.05$
44.95

Student B
$3(15 - 0.05)$
$3(15) - 3(0.05)$
$45 - 0.15$
44.85

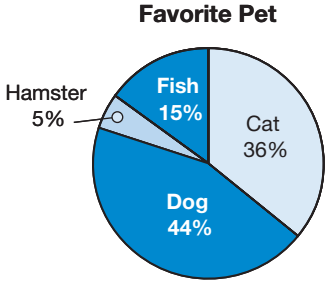
29. (SB) A cube has a side length of 5 centimeters.

a. Find the surface area of the cube. 150 cm^2

b. Find the volume of the cube. 125 cm^3

30. (SB) **Analyze** John asked 100 people to choose their favorite pet.

Use the chart to determine how many people chose dogs as their favorite pet. 44 people chose dogs as their favorite pet.



LESSON 2

Evaluating Expressions and Combining Like Terms

Warm Up

1. (SB) **Vocabulary** In the term $3xy$, 3 is the _____.
2. (1) True or False: $23 + 2(4) = 5(6) + 7$.
3. (SB) Simplify: $8 + 12 \div 4 - 5$.

New Concepts

An **algebraic expression** can contain letters that represent unspecified numbers. These letters are called **variables.** The value of the algebraic expression $x + 4$ depends on the number you use as a replacement for x. If you replace x with -32, the expression will have a value of -28.

$$x + 4$$
$$(-32) + 4 = -28$$

When you replace the variables in an expression with selected numbers and simplify using the order of operations, you have **evaluated** the expression.

Order of Operations
1. Parentheses and grouping symbols
2. Exponents
3. Multiply and divide from left to right
4. Add and subtract from left to right

Example 1 Evaluating Expressions with Exponents

Evaluate each expression if $x = -2$ and $y = -4$.

a. $x^2y - y$

SOLUTION

$x^2y - y$

$= (-2)^2(-4) - (-4)$ Replace x with -2 and y with -4.

$= (+4)(-4) - (-4)$ Perform operations with exponents.

$= -16 - (-4)$ Multiply.

$= -16 + 4 = -12$ Subtract.

b. $2xy + 3y^2$

SOLUTION

$2xy + 3y^2$

$= 2(-2)(-4) + 3(-4)^2$ Replace x with -2 and y with -4.

$= 2(-2)(-4) + 3(16)$ Perform operations with exponents.

$= 16 + 48 = 64$ Multiply and add.

Hint

Subtracting a number is the same as adding its opposite.

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Example 2 Evaluating Expressions with Parentheses

Evaluate each expression if $a = -2$ and $b = 4$.

a. $a(-b - a) - ab$

SOLUTION

$a(-b - a) - ab$

$= -2[-(4) - (-2)] - (-2)(4)$ Replace a with -2 and b with 4.

$= -2(-2) - (-2)(4)$ Perform operations inside parentheses.

$= 4 - (-8)$ Multiply.

$= 4 + 8$ Subtract.

$= 12$

b. $ab - a + \dfrac{a + b}{2}$

SOLUTION

$ab - a + \dfrac{a + b}{2}$

$= (-2)(4) - (-2) + \dfrac{(-2) + 4}{2}$ Replace a with -2 and b with 4.

$= (-2)(4) - (-2) + \dfrac{2}{2}$ Perform operations inside parentheses.

$= (-8) - (-2) + 1$ Divide.

$= -8 + 2 + 1$ Add and subtract.

$= -5$

Example 3 Using a Calculator to Evaluate Expressions

Caution

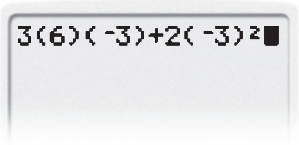
When using your calculator to square negative numbers you need to use parentheses. $(-3)^2 = 9$, but $-3^2 = -9$.

Use a calculator to evaluate $3mp + 2p^2$ if $m = 6$ and $p = -3$.

SOLUTION Replace m with 6 and p with -3.

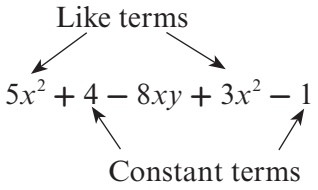
$$3(6)(-3) + 2(-3)^2$$

Enter this expression into the calculator as shown.



Press ENTER to find the answer is -36.

The **terms of an algebraic expression** are separated by addition and subtraction symbols. **Like terms** have the same variable raised to the same power. **Constant terms** are like terms that always have the same value.



Add like terms by adding the coefficients of the terms, as shown in the following examples.

Example 4 Simplifying Expressions

Simplify the expression by adding like terms.

$$3xy - 2x + 4 - 6yx + 3x$$

SOLUTION Rearrange the expression so that the like terms are next to each other. Then add like terms.

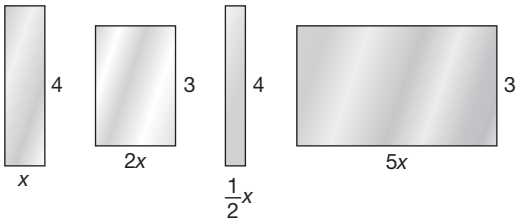
$$3xy - 2x + 4 - 6yx + 3x$$
$$= 3xy - 6yx - 2x + 3x + 4$$
$$= -3xy + x + 4$$

Hint

The order of the variables that a term is composed of doesn't change its value (*xyz* equals *zyx*).

Example 5 Application: Metalworking

A metal worker is using the four sheets of metal shown below with their dimensions. What is the total area of the four sheets?



SOLUTION

First find the individual areas.

$$4 \cdot x = 4x \text{ square units}$$
$$3 \cdot 2x = 6x \text{ square units}$$
$$4 \cdot \frac{1}{2}x = 2x \text{ square units}$$
$$3 \cdot 5x = 15x \text{ square units}$$

Now add to find the total area.

$$4x + 6x + 2x + 15x = 27x \text{ square units}$$

Lesson Practice

a. (Ex 1) Evaluate $2x - 5x^2y$ if $x = 3$ and $y = -2$.

b. (Ex 1) Evaluate $4xy^2 + x^2$ if $x = -3$ and $y = -1$.

c. (Ex 2) Evaluate $\frac{2(a+b)+4}{4} - a$ if $a = -4$ and $b = 6$.

d. (Ex 2) Evaluate $-3ab - b(4 - a)$ if $a = -2$ and $b = 7$.

e. (Ex 3) Use a calculator to evaluate $p(-p + 1) - m$ if $m = 5$ and $p = -3$.

f. (Ex 4) Simplify the expression by adding like terms.

$$-xy + 9 - 7x - 9xy + 7$$

g. (Ex 5) At the local drugstore, notebooks cost $7x$ dollars, pens cost $2x$ dollars, folders cost $3x$ dollars, and highlighters cost $4x$ dollars. Trent bought 1 notebook, 2 pens, 3 folders, and 1 highlighter. What was the total cost?

Practice Distributed and Integrated

Find the volume of the rectangular prisms with the given dimensions.

1. (SB) $l = 2$ inches, $w = 4$ inches, $h = 9$ inches

2. (SB) $l = 0.5$ cm, $w = 1.2$ cm, $h = 6$ cm

3. (1) Golf Tiger Woods obtained an average score of 68 during the four rounds of golf at the PGA Championship in 2007. Use the Distributive Property to mentally calculate his total score for the four rounds of golf.

***4.** (2) Amusement Parks An amusement park in Atlanta, Georgia offers a child's ticket for $29.99 and an adult ticket for $39.99.

a. Round individual ticket prices to the nearest dollar and write an expression for the total ticket cost for a family with A adults and C children.

b. Which would pay more, a family with one adult and four children, or a family with three adults and one child? Explain.

Evaluate each expression for the given values of x and y.

***5.** (2) $x - xy^2 - xy$ if $x = -2$ and $y = -3$

***6.** (2) $(x - y) - x(-y)$ if $x = -5$ and $y = 3$

***7.** (2) $x^2(x - xy)$ if $x = -2$ and $y = 3$

***8.** (2) $x^2 - y(x - y)$ if $x = -\frac{1}{2}$ and $y = \frac{1}{4}$

***9.** (2) $xy(1 - y)$ if $x = \frac{1}{5}$ and $y = -10$

***10.** (2) $xy - (x^2 - y)$ if $x = -\frac{1}{3}$ and $y = \frac{1}{2}$

Identify the subsets of real numbers of which each number is a member.

11. (1) $\sqrt{8}$

12. (1) -3

13. (1) $\frac{2}{3}$

*14. (2) **Budgets** Van budgets $12 a day for groceries for weekdays and $15 a day for weekends. Write an expression in simplest form to show his grocery budget for w weeks.

*15. (2) **Multi-Step** For the expression $2qr(3 + r)$

a. evaluate if $q = 2$ and $r = -1$.

b. explain how the result would be affected if q was doubled.

c. show this is true by evaluating the expression if $q = 4$ and $r = -1$.

*16. (2) **Sports** A football is kicked from ground level. The height h in feet of the ball after t seconds can be modeled by the expression $-16t^2 + 48t$. Find the height of the ball after 2 seconds.

Measurement Find the measure of the complementary angle given the measures of the angles below.

17. (SB) 35°

18. (SB) 68°

19. (SB) 5°

20. (SB) 89°

*21. (2) **Multiple Choice** Evaluate $4x^2 + 7y$ if $x = 3$ and $y = -2$.

A 10 **B** 41 **C** 50 **D** 22

22. (1) **Error Analysis** Two students multiplied $-13 \cdot 18$ using the Distributive Property. Which student is correct? Find and explain the other student's error.

Student A	Student B
$-13 \cdot 18 = -13(20 - 2)$ $= -13(20) + 13(2)$ $= -260 + 26$ $= -234$	$-13 \cdot 18 = -13(20 - 2)$ $= -13(20) - 13(2)$ $= -260 - 26$ $= -286$

*23. (2) **Geometry** A rectangle has length c and width $2d$. Another rectangle has length 4 and width $(5 - c)$. Write a simplified expression of their combined area.

*24. (2) **Verify** Is $a^2 + b^2 = (a + b)^2$? Support your answer with an example.

*25. (2) **Consumer** A local sandwich shop has $6 lunches, and you receive a $5 discount on the tenth lunch. Calculate your total cost for ten lunches at this shop.

26. (1) **Write** Does zero have a multiplicative inverse? Explain your answer.

*27. (2) **Graphing Calculator** Use a calculator to evaluate the expression $fg^2 - (2f - g^2)$ if $f = 1$ and $g = -2$. Then, verify the answer by simplifying and solving.

Simplify by combining like terms.

*28. (2) $5ab + 7a - 3ab + 4b$ *29. (2) $y - 9x^2y + 4 + 3x^2y + 12$ *30. (2) $2(x + 3) - x$

LESSON 3

Using Rules of Exponents

Warm Up

1. (SB) **Vocabulary** In the expression 3^2, 2 is the _____.
2. (SB) True or False: $5^4 = 5 + 5 + 5 + 5$.
3. (SB) Evaluate $n \cdot n \cdot n \cdot n$ for $n = -3$.

New Concepts

In an expression such as x^n the **exponent** tells the number of times that the base is a factor. So 2^3 is defined as 2 times 2 times 2. Two to the negative third power is defined as 1 over 2 to the third power.

$$2^{-3} = \frac{1}{2^3}$$

Reading Math

In the expression 5^4, 5 is the base and 4 is the exponent. The expression is read "5 raised to the 4th power."

The formal definition of **negative exponents** is as follows:

Definition of x^{-n}

If n is any real number and x is any real number that is not zero,

$$x^{-n} = \frac{1}{x^n}.$$

This definition says that when an exponential expression is written in reciprocal form, the sign of the exponent must be changed. If the exponent is negative, it is positive in reciprocal form; and if it is positive, it is negative in reciprocal form.

Example 1 Simplifying Negative Exponents

Simplify.

a. $\frac{1}{3^{-2}}$ **b.** 3^{-3}

SOLUTION

a. $\frac{1}{3^{-2}} = 3^2 = 9$ **b.** $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

c. -3^{-2} **d.** $(-3)^{-2}$

SOLUTION

c. $-3^{-2} = -\frac{1}{3^2} = -\frac{1}{9}$ **d.** $(-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}$

e. $-(-3)^{-3}$

SOLUTION

e. $-(-3)^{-3} = -\frac{1}{(-3)^3} = -\frac{1}{-27} = -\left(-\frac{1}{27}\right) = \frac{1}{27}$

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Two rules for exponents are the product rule and the power rule. These rules simplify operations with bases.

Math Reasoning

Connect Apply the definition of negative exponents to explain why the product theorem states that $x \neq 0$.

The product of powers whose bases are the same can be found by writing each power as repeated multiplication. $x^2 \cdot x^3$ means $x \cdot x$ times $x \cdot x \cdot x$ which equals x^5. The exponents may be added to obtain the same result: $x^2 \cdot x^3 = x^{2+3} = x^5$. This demonstrates the product rule for exponents.

Product Rule for Exponents

If m, n, and x are real numbers and $x \neq 0$,

$$x^m \cdot x^n = x^{m+n}.$$

Example 2 Simplifying Expressions Using the Product Rule

Simplify.

a. $x^2yx^{-5}y^{-4}x^5x$

SOLUTION

Simplify by adding the exponents of like bases.

$$x^2yx^{-5}y^{-4}x^5x$$
$$= x^2x^{-5}x^5xyy^{-4}$$
$$= x^{2+(-5)+5+1}y^{1+(-4)}$$
$$= x^3y^{-3}$$

x^3y^{-3} may also be written with positive exponents as $\frac{x^3}{y^3}$.

b. $\dfrac{yy^{-3}x^4y^5x^{-10}}{y^{-6}x^{-3}y^{10}x^2}$

SOLUTION

Step 1: Simplify the numerator and denominator by adding the exponents of like bases.

Step 2: Apply the definition of x^{-n}.

Step 3: Repeat step 1.

$$\frac{yy^{-3}x^4y^5x^{-10}}{y^{-6}x^{-3}y^{10}x^2} = \frac{x^{4+(-10)}y^{1+(-3)+5}}{x^{(-3)+2}y^{(-6)+10}} = \frac{x^{-6}y^3}{x^{-1}y^4} = x^{-6} \cdot x^1 \cdot y^3 \cdot y^{-4} = x^{-5}y^{-1} \text{ or } \frac{1}{x^5y}$$

Math Reasoning

Analyze In Example 2 the definition of x^{-n} is used to simplify each expression. What mathematical operation would have the same result? Explain.

The product rule can be used to expand $(x^2)^3$ as

$$(x^2)^3 = x^2x^2x^2 = x^{2+2+2} = x^6$$

This demonstrates the power rule for exponents.

Power Rule for Exponents

If m, n, and x are real numbers,

$$(x^m)^n = x^{m \cdot n}.$$

This rule extends to any number of exponential factors, so that

$$(x^ay^bz^c\ldots)^n = x^{an}y^{bn}z^{cn}\ldots.$$

Example 3 Simplifying Using the Power Rule of Exponents

Simplify.

a. $\dfrac{x(x^{-3})^2y(xy^{-2})^{-3}}{(x^2)^3y^{-3}(x^2)^3}$

SOLUTION

Use the power rule in both the numerator and denominator, then simplify the expressions using the product rule.

$$\frac{x(x^{-3})^2y(xy^{-2})^{-3}}{(x^2)^3y^{-3}(x^2)^3} = \frac{xx^{-6}yx^{-3}y^6}{x^6y^{-3}x^6} = \frac{x^{-8}y^7}{x^{12}y^{-3}} = x^{-20}y^{10} = \frac{y^{10}}{x^{20}}.$$

b. $\dfrac{(x^a)^b(y^a)^{b+2}}{x^{-a}}$

SOLUTION

$$\frac{(x^a)^b(y^a)^{b+2}}{x^{-a}} = \frac{x^{ab}y^{ab+2a}}{x^{-a}} = x^{ab}x^ay^{ab+2a} = x^{ab+a}y^{ab+2a}$$

Scientific notation is a method of writing a number as the product of a number greater than or equal to 1 but less than 10 and a power of 10. Multiplying a number by a positive integral power of 10 moves the decimal point to the right. Multiplying a number by a negative integral power of 10 moves the decimal point to the left.

Number	The decimal moves	Scientific Notation
412.36	Two places left	4.1236×10^2
0.041236	Two places right	4.1236×10^{-2}

Writing very large or small numbers in scientific notation can be helpful. Rules of exponents can be used to simplify expressions written in scientific notation.

Hint

The Commutative and Associative Properties of Multiplication allow you to move the numbers and to group them to make calculations easier.

Example 4 Simplifying Expressions in Scientific Notation

Simplify $\dfrac{(0.0003 \times 10^{-6})(4000)}{(0.006 \times 10^{15})(2000 \times 10^4)}$.

SOLUTION

Begin by writing all four numbers in scientific notation.
Then multiply and divide.

$$\frac{(3 \times 10^{-4} \times 10^{-6})(4 \times 10^3)}{(6 \times 10^{-3} \times 10^{15})(2.0 \times 10^3 \times 10^4)} = \frac{(3 \times 10^{-10})(4 \times 10^3)}{(6 \times 10^{12})(2 \times 10^7)}$$

$$= \frac{3 \cdot 4}{6 \cdot 2} \times \frac{10^{-7}}{10^{19}} = 1 \times 10^{-7+(-19)} = 1 \times 10^{-26}$$

Example 5 Astronomy

The speed of light is 3×10^8 meters/second. If the moon is 3.844×10^8 meters from the earth, how many seconds does it take light to reach the moon from the earth? Give the answer to the nearest hundredth.

SOLUTION Divide the moon's distance from the earth by the speed of light.

$$\frac{3.844 \times 10^8}{3 \times 10^8} = \frac{3.844}{3} \times \frac{10^8}{10^8} = \frac{3.844}{3} \approx 1.28 \text{ seconds}$$

It takes light about 1.28 seconds to reach the moon from the earth.

Lesson Practice

Simplify.

(Ex 1) **a.** $\frac{1}{2^{-3}}$ **b.** 2^{-4} **c.** -2^{-2} **d.** $(-2)^{-3}$

(Ex 1) **e.** $-(-2)^{-4}$ (Ex 2) **f.** $x^9y^{-1}x^{-2}y^3y^{-7}$

(Ex 2) **g.** $\frac{xx^6y^8x^{-11}y^{-3}}{x^{-5}yy^2x^{-4}}$ (Ex 3) **h.** $\frac{(y^2)^{-3}x^4(xy^{-2})^{-2}}{xy^4}$

(Ex 3) **i.** $\frac{(x^b)^{a-1}(xy^{-a})^{-b}}{y^{-1}}$

(Ex 4) **j.** $\frac{(0.004)(600 \times 10^9)}{(30000 \times 10^{-12})(0.0001 \times 10^3)}$

(Ex 5) **k.** The United States population is approximately 2.99×10^8. In one year, the United States produced 4.78×10^{11} pounds of garbage. Approximately how much garbage did the average American produce that year?

Practice Distributed and Integrated

Evaluate each expression for the given values of *a* and *b*.

(2) **1.** $-b^2 - b(a - b^2)$ if $a = 4$ and $b = -3$

(2) **2.** $a^2 - b^3(a - b)$ if $a = -2$ and $b = -3$

State the properties illustrated below.

(1) **3.** $a + b + c = b + a + c$

(1) **4.** $a(b + c) = ab + ac$

Use the rules of exponents to simplify the expressions below.

(3) ***5.** $\frac{(2x^2)^{-3}(xy^0)^{-2}}{2xx^0x^1xy^2}$

(3) ***6.** $\frac{a^0bc^0(a^{-1}b^{-1})^2}{ab(ab^0)abc}$

(3) ***7.** $\frac{(2x^2y^3)^{-3}y}{(4xy)^{-2}(x^{-2}y)^3y}$

(3) ***8.** $\frac{xx^{-2}y(x^{-3})^2xy^0}{(2xy)^{-2}x^2(y^{-3})^2}$

(SB) **9.** What is the area of a circle with radius 3 meters? Give your answer in terms of π.

10. *(SB)* Angles a and b are vertical angles. If angle b measures 74°, what is the measure of angle a?

***11.** *(3)* **Write** In 2002, the average height of a 10-year old girl was 1.4×10^{-3} km. Approximately 3×10^{7} 10-year old girls lying head-to-toe would equal the circumference of the earth at the equator. Explain how you could use properties of real numbers to mentally estimate the circumference.

12. *(1)* **Consumer** Your dinner total at your favorite restaurant is $18.

a. Use the Distributive Property to mentally calculate 10% of your dinner bill.

b. Explain how the result in part **a** would be helpful to mentally calculate a 15% tip.

13. *(1)* **Error Analysis** A student is asked to simplify an expression and identify which property was used for each step. His answer is shown below. Identify and explain the error(s).

$(8 + 4) \cdot 9$	
$(8 \cdot 9) + (4 \cdot 9)$	Associative Property of Addition
$72 + 36$	Distributive Property
108	Add.

14. *(2)* **a.** Evaluate the expression $1 - (3x^2 - 2y)$ if $x = 2$ and $y = 3$.

b. Explain how the result would be affected if y was doubled.

c. Show that this is true by evaluating the expression if $x = 2$ and $y = 6$.

15. *(2)* **Nutrition** The U.S. recommended daily amount of calcium for most adults is 1000 mg per day. Milana's breakfast consists of cereal that contains 33 mg calcium in $\frac{1}{4}$-cup and milk that contains 75 mg calcium per $\frac{1}{4}$-cup. Calculate the total amount of calcium in milligrams in Milana's breakfast of $\frac{3}{4}$-cup cereal with $\frac{1}{2}$-cup milk.

16. *(2)* **Justify** A square drawn on a coordinate grid has adjacent vertices at points $(u, 1)$ and $(5u, 1)$. Another square has adjacent vertices at points $(2u, 1)$ and $(4u, 1)$. Write a simplified expression of their combined area. Explain how you obtained your answer.

17. *(2)* **Coordinate Geometry** A figure on a coordinate graph is translated 2 units to the right and 5 units up. Its new coordinates can be written with the expressions $(x + 2, y + 5)$. A point is translated to (19, 19), what are the coordinates of the original point?

18. *(2)* **Verify** Evaluate the expression $3g - 2f + fg^2 - 5g + 4fg^2 + 6f$ if $f = 1$ and $g = -1$ by inserting the variable values into the expression as written. Verify your answer by simplifying the expression first, then inserting the variable values.

***19.** *(3)* **Multiple Choice** Simplify $\dfrac{y^{-3}xy^2x^{-4}x^5}{x^3y^{-1}}$.

A $\frac{1}{x}$ **B** $\frac{1}{x^2y^5}$ **C** $\frac{x^5}{y^2}$ **D** $\frac{x^3}{y}$

20. *(SB)* **Write** Why is $1^{10000000} = 1$?

***21.** *(3)* **Error Analysis** Two students incorrectly simplified the expression $\frac{2^3(2^{-2})^3}{2(2^2+2)}$. Their steps are shown below. Explain the error of each student.

Student A	Student B
$\frac{2^3(2^{-2})^3}{2(2^2+2)} = \frac{2^3(2^{-6})}{2(4+2)}$ $= \frac{2^{-3}}{12} = \frac{1}{12(-2)^3} = -\frac{1}{96}$	$\frac{2^3(2^{-2})^3}{2(2^2+2)} = \frac{2^3(2^{-6})}{2(2^3)}$ $= \frac{2^{-3}}{2^4} = \frac{1}{2^7} = \frac{1}{128}$

22. *(SB)* **Volcanos** Vailulu'u is an underwater volcano in the Pacific Ocean. The volcano has a height of about 4,200 m off the sea floor. If the top of the volcano is at a depth of 590 m below sea level, what is the depth of the sea floor in this area?

***23.** *(3)* **Verify** The Atlantic Ocean is 3.93×10^3 m deep and has 8.24×10^{13} m^2 of surface area. Calculate the total volume of the Atlantic Ocean. Write your answer in scientific notation. Verify your answer if you know that the Pacific Ocean has approximately two times the volume and approximately the same depth of the Atlantic Ocean, and a surface area of about 1.66×10^{14} m^2.

24. *(SB)* **Geography** The Danube River is Europe's second longest river. It flows through 9 countries and has a length of 1,770 miles. About how many feet long is the Danube?

***25.** *(3)* **Geometry** A box has a volume of $\frac{x^3y^2x^{-2}}{y^{-3}x^2y}$. The area resulting from length times width is $\frac{x^{-2}y^3x^{-1}}{y^{-4}x^{-2}y^2}$. Find an expression for the height of the box.

26. *(2)* **Analyze** Rita can choose to rent movies from one of two movie stores. The first store charges \$5 per movie, and she can rent her 8th movie for free. The second store charges \$7 per movie, and she can rent her 4th movie for free. Rita assumes that the second shop has a better deal because she must only rent 3 movies before receiving a free movie rental. Explain why she should not make this assumption.

***27.** *(3)* **Graphing Calculator** Simplify $\frac{2^{-3}}{8^{-2}}$. Use a calculator to check your answer.

28. *(2)* **Error Analysis** The EPA lists the greenest hybrid car as getting 61 miles per gallon on the highway and 50 miles per gallon in the city. Assuming the fuel tank holds 15 gallons of gas, one owner decided to calculate the total miles that he could drive one tank of gas if 4 gallons were used in the city. Explain and correct his error.

$$61(4) + 50(15 - 4) = 244 + 75 - 200 = 119 \text{ miles}$$

29. *(1)* **Analyze** Identify which property of real numbers is being demonstrated. Explain why this property might be helpful in solving this problem without a calculator.

$$(9 \cdot 3) \cdot 3 = 9 \cdot (3 \cdot 3) = 9 \cdot 9 = 81$$

***30.** *(2)* **Multi-Step** Shane spent \$2.24 on breakfast and x dollars on lunch.

a. Write an expression for the amount of money Shane spent on breakfast and lunch.

b. If Shane spent \$5.97 on lunch, how much money did he spend in all?

LAB
1

Graphing a Function and Building a Table

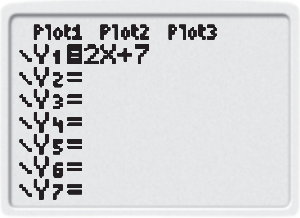
Graphing Calculator Lab (*Use with Lesson 4*)

A graphing calculator can be used to analyze functions. The calculator can be used to graph the function as well as build tables of data. Use the graphing calculator to graph the line $y = 2x + 7$.

Graphing a Function in the Standard Window

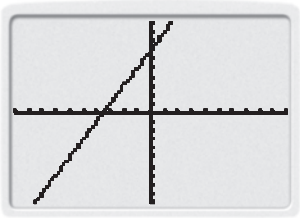
1. Enter the equation $y = 2x + 7$.

   Open the Y= equation editor by pressing Y=.
   Enter the equation in the first row next to \Y1= by pressing 2 X,T,θ,n + 7.



2. Graph the line.

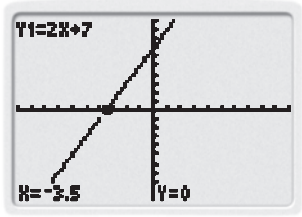
   Press GRAPH, and the line will be displayed on a coordinate plane. If the graph does not appear identical to the screen to the right, press ZOOM and select [**6: ZStandard**].



Tracing a Line to Find Points

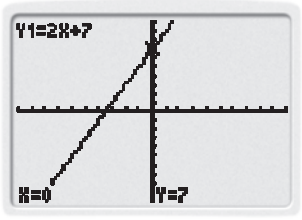
1. Trace the Line.

   Press TRACE. A blinking cursor will appear on the graph as well as an x-and y-value displayed at the bottom of the screen. The values at the bottom are the coordinates of the cursor. By pressing the arrow buttons ◀ ▶, you can move the cursor along the line to find the x-and y-intercepts.



2. Find the x-and y-intercepts.

   Use the coordinates displayed at the bottom to find the place where $y = 0$ (x-intercept) and $x = 0$ (y-intercept). While in the TRACE feature, you can also move the cursor by entering the x-value of the coordinate you are looking for and pressing ENTER.



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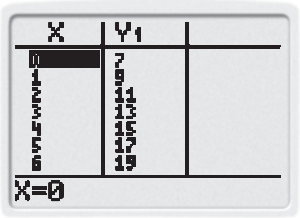
Building a Table From a Function

The calculator can also produce a table of values for an equation. Generate a table of values for the equation $y = 2x + 7$, graphed above.

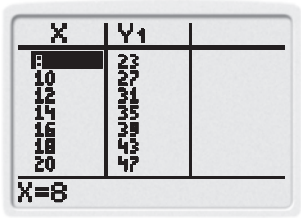
1. Display a table.

   Press 2nd GRAPH to open the [**TABLE**] feature. The x-values will be displayed in the first column, and the corresponding y-values will be in the second column. These points correspond to points on the graph of the line. Use the arrow keys to scroll through the data.



2. Adjust the table.

   Press 2nd WINDOW to open the [**TBLSET**] menu. The line [**TblStart=**] allows you to change the first x-value listed in the table, or where you want the table to start. The line [Δ **Tbl=**] defines the change between each subsequent x-value in the table. To change the table in the [**TBLSET**] menu, press 8 ENTER after [**TblStart=**] and press 2 ENTER after [Δ **Tbl=**]. Press 2nd GRAPH to reopen the table feature. The table's x-values will now be the even numbers between 8 and 20. The arrow keys can still be used to scroll through the table's data.



Lab Practice

a. Use the graphing calculator to graph the line $y = -5x + 3$.

b. Use the graphing calculator to find the x-and y-intercepts of $y = -2x + 4$.

c. Use the graphing calculator to create a table of the odd numbers between 1 and 13, for the equation $y = -3x + 8$.

d. Use the graphing calculator to fill in the table for the graph of $y = 4x - 6$.

x	y
0	
0.5	
1	
1.5	
2	

LESSON 4

Identifying Functions and Using Function Notation

Warm Up

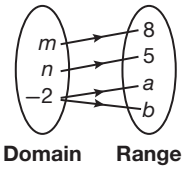
1. (SB) **Vocabulary** A number used to locate a point on a graph is called a
   a. coordinate. **b.** integer. **c.** line.

2. (2) Evaluate $5x^3 - 3x$ for $x = 2$.

3. (SB) List the x-coordinates from the ordered pairs: $(-2, 4)$, $(0, 7)$, $(2, 10)$.

New Concepts

A **relation** is a pairing of input and output values.



The **domain** contains the input values: m, n, and -2. The **range of a function or relation** contains the output values: 8, 5, a, and b.

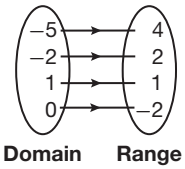
The input value, normally x, is the **independent variable.** The output, normally y, is called the **dependent variable.**

Caution

In a function, the y values may repeat. The x values may not.

A **function** is a mapping between two sets that associates with each element of the first set, the domain, a unique (one and only one) element of the second set, the range.

x	y
−5	4
−2	2
1	0
0	−2



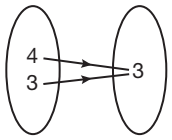
Example 1 Identifying Functions, Domain, and Range

Which of the following depict functions? Identify the domain and the range.

Math Reasoning

Formulate Give three ordered pairs for the function in Example 1b.

a.



SOLUTION Function.
Domain: 4, 3; Range: 3

b.

x	y
7	3
5	3
5	4

SOLUTION Not a function.
Domain: 5, 7; Range: 3, 4

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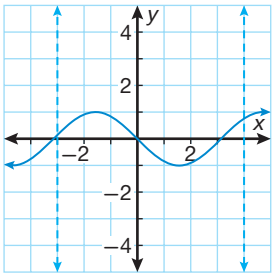
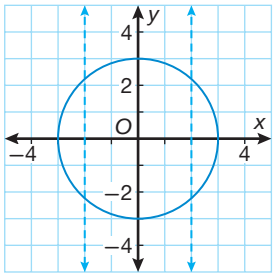
When a relation is represented with a graph, the **vertical line test** is used to determine if the relation is a function.

A graph on the coordinate plane represents the graph of a function provided that any vertical line intersects the graph no more than once.



Graph of a Function



Graph of a Relation but Not a Function

Math Reasoning

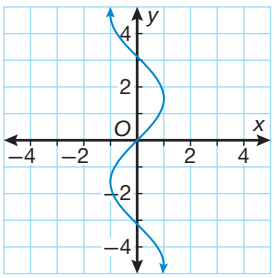
Model Draw a line that does not pass the vertical line test.

The graph on the left represents the graph of a function since any vertical line intersects the graph no more than once. The graph on the right is not the graph of a function as there are vertical lines that intersect the graph at more than one point.

Example 2 Using the Vertical Line Test

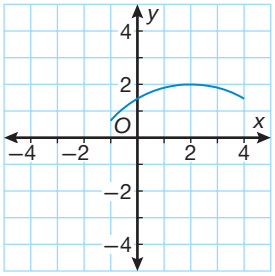
Determine if each graph represent the graph of a function.

a.



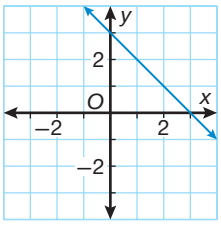
SOLUTION Not a function. The graph does not pass the vertical line test. A vertical line can be drawn so that it intersects the graph at more than one point. For example, the y-axis is such a line.

b.



SOLUTION Function. Every vertical line drawn intersects the graph no more than once.

c.



SOLUTION Function. Every vertical line drawn intersects the graph no more than once.

In mathematics it is sometimes necessary to work with more than one equation at a time. If given the two equations

$$y = x + 2 \text{ and } y = x - 5$$

to evaluate for $x = 2$, it may be confusing to know which equation to evaluate. **Function notation** uses parentheses and letters to distinguish between equations.

$$f(x) = x + 2 \text{ and } g(x) = x - 5$$

The notation $g(2) = (x) - 5$ means to find the value of the g equation when $x = 2$.

$g(2) = (x) - 5$

$g(2) = (2) - 5$ Replace x with 2.

$g(2) = -3$ Simplify.

Since the answer is -3 when x equals 2, the answer is read "g of 2 equals -3."

Reading Math

The equation on the left is read "f of x equals x plus 2" and the equation on the right "g of x equals x minus 5."

Math Reasoning

Verify Have students use a graphing calculator to check the solution.

Example 3 Using Function Notation to Evaluate Functions

If $h(x) = 4x - 3$ and $p(x) = x^2 - 3x$, find $p(-3)$.

SOLUTION Substitute -3 for x in $p(x)$ not $h(x)$.

$p(-3) = (-3)^2 - 3(-3)$ Replace x with -3.

$p(-3) = 9 + 9$ Simplify.

$p(-3) = 18$ Simplify.

$p(x)$ is 18 when x is -3.

Example 4 Consumer Math

a. A company charges \$0.25 per minute for a cell phone call. This can be expressed as the ordered pair (1, 0.25). Find the cost of a 2-minute, 3-minute, and 4-minute call. Express the answers as ordered pairs in set notation. Identify the domain and range. Determine if the set of ordered pairs represents a function. Explain.

SOLUTION {(2, 0.5), (3, 0.75), (4, 1)} Domain: 2, 3, 4. Range: 0.5, 0.75, 1. The set of ordered pairs represents a function since for every value of x there is exactly one y value.

b. Suppose that the same company charges \$1.00 for any call up to and including 4 minutes in length. Find the cost of a 2-minute, 3-minute, and 4-minute call. Express the answers as ordered pairs in set notation. Identify the domain and range. Determine if the set of ordered pairs represents a function. Explain.

SOLUTION {(2, 1), (3, 1), (4, 1)} Domain: 2, 3, 4. Range: 1. The set of ordered pairs represents a function since for every value of x there is exactly one y value.

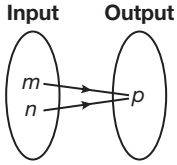
Lesson Practice

Which of the following depict functions? Identify the domain and the range.
(Ex 1)

a.

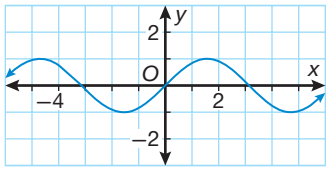
x	3	6	2	3
y	9	7	5	4

b.

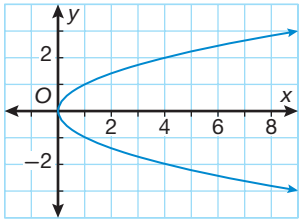


Determine if the graph represents a function.
(Ex 2)

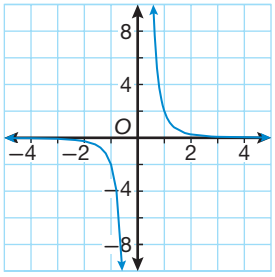
c.



d.



e.



f. *(Ex 3)* If $a(x) = 9 + 6x$ and $v(x) = 9x + 3x^2$, find $v(2)$.

g. *(Ex 4)* If you buy one ticket to a local baseball game, the cost is \$25. This can be expressed as the ordered pair (1, 25). There is a one-day special if you buy one ticket at regular price, each additional ticket is \$20. Find the total cost if you buy 2, 3, and 4 tickets. Express the answers as ordered pairs in set notation. Identify the domain and range. Determine if the set of ordered pairs represents a function. Explain.

h. *(Ex 4)* If you buy one t-shirt from your favorite store, the cost is \$15.50. This can be expressed as the ordered pair (1, 15.5). The store is having a buy-one-get-one-free sale on t-shirts. For each t-shirt that you purchase at regular price you get the next one free. Find the total cost if you buy 2, 3, 4, 5, and 6 t-shirts. Express the answers as ordered pairs in set notation. Identify the domain and range. Determine if the set of ordered pairs represents a function. Explain.

Practice Distributed and Integrated

Find the additive inverse for each of the following.

1. *(1)* $-3p$

2. *(1)* 50

Evaluate each expression for the given values of m and n.

3. *(2)* $2m - 4n(3m - 4)$ if $m = 1$ and $n = -3$

4. (2) $-m^3 + mn^2(n - m)$ if $m = -2$ and $n = -3$

Use the rules of exponents to simplify the expressions below.

5. (3) $\dfrac{(m^3n^{-1})^{-4}}{(3mn)^{-2}}$

6. (3) $\dfrac{-a^3(a^3b^{-2})^3}{ab(ab)^{-4}}$

Identify whether each of the following represent a function. If so, state the domain and range.

***7.** (4) $y^2 = x$

***8.** (4) $y = 3x^2 - 2$

9. (3) **Error Analysis** Two students simplified $\dfrac{x^{-2}(y^2)^3}{y(x^{-1})^2}$. Their steps are shown below. Who is correct? Explain the error in the other student's work.

Student A	Student B
$\dfrac{x^{-2}(y^2)^3}{y(x^{-1})^2} = \dfrac{x^{-2}y^6}{yx^{-2}} = y^5$	$\dfrac{x^{-2}(y^2)^3}{y(x^{-1})^2} = \dfrac{x^{-2}y^5}{yx} = \dfrac{y^4}{x^3}$

10. (SB) What is the radius of a circle with circumference 16π feet?

***11.** (4) Use the table to answer the following questions.

a. What is the domain?

b. What is the range?

c. Does this data represent a function? Explain.

x	y
2	9
5	7
3	4
2	11
7	7

***12.** (4) **Multiple Choice** Which set of x- and y-values makes the equation $y = 3x^2 + 4$ true?

A $x = 1, y = 10$ **B** $x = 3, y = 31$ **C** $x = 5, y = 229$ **D** $x = 2, y = 24$

13. (1) **Write** Alyson saves \$75 each month. Use the distributive property to mentally calculate the amount of money that she will save in 12 months. Explain.

***14.** (4) **Geometry** Explain whether the relationship from a circle's radius to its area is a function.

15. (3) Hawaii Mauna Kea in Hawaii is about 1.02×10^4 meters from its base to its peak. The speed of light is about 3×10^8 meters/second. About how many seconds would it take light to reach the base from the peak? Write your answers in scientific notation.

16. (2) **Verify** Evaluate the expression $ab + 3b - a - 5ab + 2b$ if $a = -3$ and $b = 2$ by inserting the variable values into the expression as written. Verify your answer by simplifying the expression first, then inserting the variable values.

17. (3) **Multiple Choice** Simplify $\dfrac{a^3b^{-2}c^{-3}b^6c^{-1}a^{-1}}{b^2c^4b^{-1}a^{-5}}$.

A $\dfrac{b^5c^8}{a^3}$ **B** a^3b^5 **C** $\dfrac{a^7b^3}{c^8}$ **D** $\dfrac{b^6}{a^2c}$

18. (1) **Baseball** The table shows the number of homeruns hit by Ryan Howard of the Philadelphia Phillies over a three-year period. Which property of real numbers would be useful to mentally calculate as quickly as possible the total homeruns that he hit over this period? Explain.

	2004	2005	2006
Homeruns	2	22	58

19. (2) **Multi-Step** A computer sales representative makes a base salary of \$9/hr. She can also earn an additional \$30 for every computer that she sells and \$20 for every printer that she sells.

a. Write an expression to represent the total amount the representative will make in an 8-hour day.

b. How much will she make if she sells 10 computers and 4 printers in an 8-hour day?

20. (1) **Generalize** The product of a number and its multiplicative inverse is one. What is the product of a number and its additive inverse?

21. (2) **Hybrids** The EPA lists the greenest hybrid car as getting 61 miles per gallon on the highway and 50 miles per gallon in the city. Write and simplify an expression for the total number of miles the car will travel if the fuel tank holds 15 gallons of gas.

***22.** (4) **Error Analysis** PC Haven sells computers at a 15% discount on the original price plus a \$200 rebate. Which function correctly represents the final price of a computer at PC Haven? Explain the error.

$$f(r) = 0.85r - 200 \qquad g(r) = 0.15r + 200$$

***23.** (4) **Analyze** What values of a and b would make the relation not a function? Explain your reasoning.

x	2	3	a
y	5	b	8

24. (3) **Precipitation** New York City gets an average of 1.14×10^2 cm of rainfall each year. The area of New York City is 8.02×10^8 m^2. Calculate the average annual volume in cubic meters of precipitation in New York City. Write your answer in scientific notation.

For the function $f(x) = 3x^2 + x$, evaluate:

***25.** (4) $f(-1)$ ***26.** (4) $f(0)$ ***27.** (4) $f\left(\frac{3}{2}\right)$

***28.** (4) **Justify** Simplify the expression, $-36 + (36 + 17)$, using properties of real numbers. Justify your answer by identifying which property you used for each step.

***29.** (4) **Graphing Calculator** For the function $f(x) = x^2$, use a calculator to find $f(-2.8)$ and $f(2.8)$. Did you use the same key strokes to evaluate both? Explain.

30. (3) **Probability** In a year, lightning strikes one person out of every 2.8×10^5 people. Calculate the number of people in the United States that will likely be struck by lightning this year if the population is 2.8×10^8. Write your answer in standard notation.

LAB
2

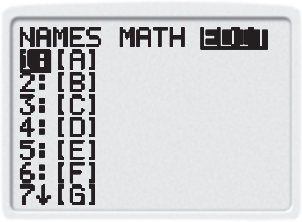
Storing and Recalling Data in a Matrix

Graphing Calculator Lab (*Use with Lessons 5, 9, and 14*)

A graphing calculator can be used to work with matrices. Use the graphing calculator to store and manipulate data in 2 × 2 matrices.

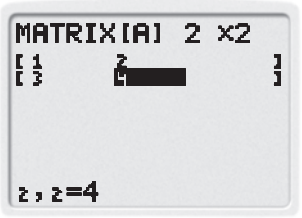
Storing Data in a Matrix

1. Input the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.



   a. Press 2nd x^{-1} to get to the matrix menu. Use the arrow button ▶ to highlight [**EDIT**] and select [**1:** [**A**]] to open matrix A.

   b. The calculator will prompt you for the dimensions of the matrix you want to create. Enter the number of rows in the matrix and press ENTER, and then enter the number of columns and press ENTER.



   c. Input each element of the matrix as prompted by the calculator and press ENTER after each.

2. Input the matrix $B = \begin{bmatrix} 5 & 0 \\ -2 & 1 \end{bmatrix}$.

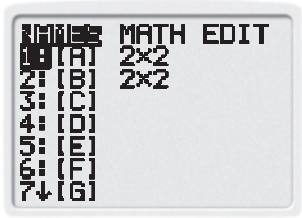
   Press 2nd x^{-1} to return to the matrix menu. Use the arrow button ▶ to highlight [**EDIT**] and use ▼ to select [**2:** [**B**]] to open matrix B. Repeat the same process for inputting matrix B.

Graphing Calculator

To start with a blank home screen press 2nd MODE CLEAR.

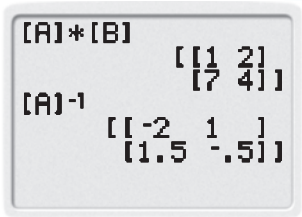
Recalling Matrices to Use in Calculations

1. Calculate $A \times B$.



   From the home screen, press 2nd x^{-1} to open the matrix menu and while the [**NAMES**] heading is still highlighted select [**1:** [**A**]] and press ENTER. Press ×, and then 2nd x^{-1} to return to the [**NAMES**] menu and use ▼ to select [**2:** [**B**]] and press ENTER. Press ENTER again to calculate.

2. Find the inverse of matrix A.



   Press 2nd x^{-1} to open the matrix menu and while the [**NAMES**] menu is still highlighted use ▼ to select [**1:** [**A**]] and press ENTER. Press x^{-1}, and then press ENTER.

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3. Find the determinant of matrix A.

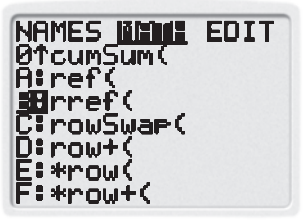
Press 2nd x^{-1} and press ▶ to highlight the [**MATH**] menu, use ▼ to select [**1: det(**]. Press ENTER. Press 2nd x^{-1} to return to the matrix menu and select [**1:** [**A**]] and press ENTER. Press ENTER. The determinant of A = −2.

Reducing a Matrix into Reduced-Row-Echelon Form

1. Enter the system of equations into the calculator as an augmented matrix.

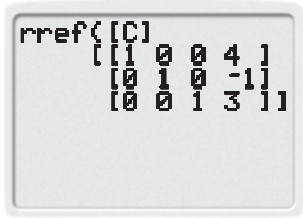
Use the method shown on the previous page to input a 3 × 4 matrix,

$$C = \begin{bmatrix} 3 & -2 & -3 & 5 \\ -1 & 4 & 5 & 7 \\ 2 & 5 & -1 & 0 \end{bmatrix}.$$



2. Reducing a matrix to echelon form.

Within the home screen, press 2nd x^{-1} and press ▶ to highlight the [**MATH**] menu, then use ▼ to select [**B: rref(**] and press ENTER. Press 2nd x^{-1} to return to the matrix menu and use ▼ to select [**3:** [**C**]]. Press ENTER. The row reduced matrix will be displayed.



Lab Practice

a. Use the graphing calculator to input matrices $A = \begin{bmatrix} -2 & 0 \\ 7 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -5 \\ -3 & 0 \end{bmatrix}$ and calculate $A \times B$.

b. Use the graphing calculator to calculate B^{-1}.

c. Use the graphing calculator to find the determinant of matrix A.

d. Use the graphing calculator to find the reduced-row-echelon form of the matrix, $C = \begin{bmatrix} -2 & -10 & 9 & 1 \\ 1 & 7 & -5 & -6 \\ 5 & 10 & 13 & 7 \end{bmatrix}$.

LESSON 5

Using Matrices to Organize Data and to Solve Problems

Warm Up

1. (1) **Vocabulary** The Additive _____ Property states that $a + (-a) = 0$.
2. (SB) Add $16.5 - (-24.8)$.
3. (SB) Solve $4 - 3y = 16$.
4. (1) True or False: By the Commutative Property: $j - k = k - j$.
5. (2) Simplify $-4(x + 1) + 3(2x - 7)$.

New Concepts

A **matrix** is a rectangular array of numbers. The number of rows and columns in a matrix gives the **dimensions of a matrix.** A matrix with r rows and c columns is called an $r \times c$, r by c, matrix. Look at the matrices shown below. Matrix A is a 3×2 matrix and B is a 4×1 matrix.

Reading Math

Matrix A can be written as $A_{3\times2}$ to show both its name and dimensions.

$$A = \begin{bmatrix} -1 & 0 \\ 4 & -5 \\ 3 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ -6 \\ 2 \\ 11 \end{bmatrix}$$

Each number in the matrix is an **element** and has a unique address which tells its location in the matrix. The **address** of an element is formed by the lowercase letter of the matrix name, followed by its row number and column number, in subscripts. Therefore, the element located at a_{12} is 0 and the element at b_{31} is 2.

Matrices can be used to store data that would otherwise be presented in a table. This can make it easier to perform calculations on the data, such as **Matrix Addition.**

Matrix Addition

To add two matrices, A and B, of the same dimensions, add each element in the first matrix to the element that is in the same location in the second matrix.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

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The Commutative Property holds true for Matrix Addition. That is, $A + B$ gives the same matrix as $B + A$.

Example 1 Creating and Adding Matrices

Create two matrices, one for male students, M, and one for female students, F. Then find $M + F$.

Male Students

	9th	10th	11th	12th
JV Sports	82	54	8	0
Varsity Sports	44	62	71	124
No Sports	50	93	85	43

Female Students

	9th	10th	11th	12th
JV Sports	91	46	39	5
Varsity Sports	22	45	112	137
No Sports	86	95	30	66

SOLUTION Enter the data in the order shown in the table.

$$M = \begin{bmatrix} 82 & 54 & 8 & 0 \\ 44 & 62 & 71 & 124 \\ 50 & 93 & 85 & 43 \end{bmatrix} \qquad F = \begin{bmatrix} 91 & 46 & 39 & 5 \\ 22 & 45 & 112 & 137 \\ 86 & 95 & 30 & 66 \end{bmatrix}$$

$$M + F = \begin{bmatrix} 173 & 100 & 47 & 5 \\ 66 & 107 & 183 & 261 \\ 136 & 188 & 115 & 109 \end{bmatrix}$$

Math Reasoning

Connect The zero matrix is the additive identity matrix. Explain.

A matrix in which every element is zero is a **zero matrix.** A zero matrix is formed when a matrix is added to its additive inverse matrix. The elements in an **additive inverse matrix** are the opposite of every element in the original matrix. Here, $-A$ is the additive inverse of A.

$$A = \begin{bmatrix} 4 & -15 \\ 9 & -1 \end{bmatrix} \qquad -A = \begin{bmatrix} -4 & 15 \\ -9 & 1 \end{bmatrix} \qquad A + (-A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix Subtraction

To subtract two matrices of the same dimensions, $A - B$, take the opposite, or additive inverse, of B and add it to A.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + (-b_{11}) & a_{12} + (-b_{12}) \\ a_{21} + (-b_{21}) & a_{22} + (-b_{22}) \end{bmatrix}$$

Example 2 Using the Additive Inverse and Subtracting Matrices

a. Find the additive inverse matrix of $\begin{bmatrix} 10 & -6.5 \\ 0 & 3 \\ -3 & 4 \end{bmatrix}$.

SOLUTION Make a matrix so that every element is the opposite. The additive inverse matrix is

$$(-)\begin{bmatrix} 10 & -6.5 \\ 0 & 3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} (-)10 & (-)-6.5 \\ (-)0 & (-)3 \\ (-)-3 & (-)4 \end{bmatrix} = \begin{bmatrix} -10 & 6.5 \\ 0 & -3 \\ 3 & -4 \end{bmatrix}.$$

Caution

To subtract 2 matrices, A–B, by adding the opposite, be sure to take the opposite of each element in B.

b. Find $\begin{bmatrix} 3 & 16 \\ -23 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 9 \\ -18 & 14 \end{bmatrix}$.

SOLUTION Subtract by adding the opposite.

$$\begin{bmatrix} 3 & 16 \\ -23 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 9 \\ -18 & 14 \end{bmatrix} = \begin{bmatrix} 3 & 16 \\ -23 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -9 \\ 18 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 3+1 & 16+(-9) \\ -23+18 & 0+(-14) \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ -5 & -14 \end{bmatrix}$$

In a **matrix equation,** an entire matrix can be a variable. To solve a matrix equation, begin by isolating the variable using inverse operations.

Example 3 Solving a Matrix Equation

Solve for X: $\begin{bmatrix} 3 & 7 \\ 9 & 1 \end{bmatrix} + X = \begin{bmatrix} 5 & 9 \\ 2 & -6 \end{bmatrix}$.

SOLUTION Subtract $\begin{bmatrix} 3 & 7 \\ 9 & 1 \end{bmatrix}$ from both sides of the equation.

$$\begin{bmatrix} 3 & 7 \\ 9 & 1 \end{bmatrix} + X = \begin{bmatrix} 5 & 9 \\ 2 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 7 \\ 9 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 7 \\ 9 & 1 \end{bmatrix} + X = \begin{bmatrix} 5 & 9 \\ 2 & -6 \end{bmatrix} - \begin{bmatrix} 3 & 7 \\ 9 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 2 \\ -7 & -7 \end{bmatrix}$$

Variables may also appear as elements of a matrix or within elements of a matrix.

Example 4 Solving for Variables in Matrices

Solve for a, b, c, and d: $\begin{bmatrix} a+12 & 2b \\ 23 & d \end{bmatrix} = \begin{bmatrix} 18 & -14 \\ a+c & 3 \end{bmatrix}$.

SOLUTION Equal matrices have equal elements in matching locations. Therefore, $a + 12 = 18$ and $a = 6$. Also, $2b = -14$, so $b = -7$. Continuing to match elements, $23 = a + c$, and after substituting 6 for a, $23 = 6 + c$, and $c = 17$. Last, $d = 3$.

A **scalar** is a constant by which a matrix is multiplied. Scalar multiplication is analogous to repeated matrix addition.

Scalar Multiplication

To multiply matrix A by scalar n, multiply every element of A by n.

$$n\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} n \cdot a_{11} & n \cdot a_{12} \\ n \cdot a_{21} & n \cdot a_{22} \end{bmatrix}$$

Math Reasoning

Analyze Why are we finding $0.75B$ and not $0.25B$? What extra step would we need to take if we did find $0.25B$?

Example 5 Application: Transportation

The tables show proposed fees for a new toll road. No one is happy with either plan. Some believe the fees in Plan A should be doubled; others believe that Plan B is appropriate if the fees are reduced by 25%. Create matrices A and B and find $2A - 0.75B$ to discuss differences in costs between these two ideas.

Plan A

	Class 1	Class 2	Class 3
Exit 1	\$1.25	\$1.75	\$2.00
Exit 2	\$2.50	\$3.00	\$3.50

Plan B

	Class 1	Class 2	Class 3
Exit 1	\$3.00	\$4.60	\$5.80
Exit 2	\$4.00	\$6.20	\$8.00

SOLUTION

$$2A = \begin{bmatrix} 2(1.25) & 2(1.75) & 2(2.00) \\ 2(2.50) & 2(3.00) & 2(3.50) \end{bmatrix} = \begin{bmatrix} 2.50 & 3.50 & 4.00 \\ 5.00 & 6.00 & 7.00 \end{bmatrix}$$

$$0.75B = \begin{bmatrix} 0.75(3.00) & 0.75(4.60) & 0.75(5.80) \\ 0.75(4.00) & 0.75(6.20) & 0.75(8.00) \end{bmatrix} = \begin{bmatrix} 2.25 & 3.45 & 4.35 \\ 3.00 & 4.65 & 6.00 \end{bmatrix}$$

$$2A - 0.75B = \begin{bmatrix} 2.50 & 3.50 & 4.00 \\ 5.00 & 6.00 & 7.00 \end{bmatrix} - \begin{bmatrix} 2.25 & 3.45 & 4.35 \\ 3.00 & 4.65 & 6.00 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & 0.05 & -0.35 \\ 2.00 & 1.35 & 1.00 \end{bmatrix}$$

Differences range from a nickel to \$2.00. In all but one situation, doubling Plan A is more expensive than reducing Plan B.

Lesson Practice

(Ex 1) **a.** Use the data in the tables to create two matrices, one for the West-Side store, W, and one for the East-Side store, E. Then find $W + E$.

West-Side Store

	Small	Medium	Large
Vests	13	4	8
Jackets	1	6	2

East-Side Store

	Small	Medium	Large
Vests	0	11	3
Jackets	4	2	5

(Ex 1) **b.** Find the additive inverse matrix of $\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.

c. (Ex 2) Find $\begin{bmatrix} -1 & 0 \\ 6 & -8 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 2 & -1 \\ 5 & -3 \end{bmatrix}$.

d. (Ex 3) Solve for Y: $Y - \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -10 & 2 \end{bmatrix}$.

e. (Ex 4) Solve for x, y, and z: $\begin{bmatrix} 10 & x & 1 \\ 20 & -5 & x-y \end{bmatrix} = \begin{bmatrix} 10 & 25 & 1 \\ x+z & z & 18 \end{bmatrix}$.

f. (Ex 5) The tables show the cost of tickets at two competing theaters. Next Saturday, all categories of tickets at both theaters will be on sale. Theater A will sell their tickets at a discount of 20% of the original price and Theater B will sell theirs at 30% of the original price. Find the matrix that represents the difference of the sale prices and discuss the differences in price.

Theater A

Ticket Category	Cost
Child 0–3	Free
Child 4–14	$4.50
Adult 15–65	$8.50
Senior Citizen 66+	$5.50

Theater B

Ticket Category	Cost
Child 0–3	Free
Child 4–14	$5.00
Adult 15–65	$9.00
Senior Citizen 66+	$6.00

Practice Distributed and Integrated

Find the multiplicative inverse for each of the following.

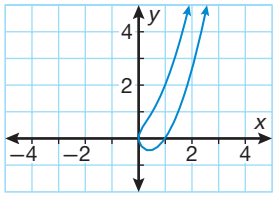
1. (1) $-\frac{2}{3}$

2. (1) 6

3. (2) Evaluate $x^2y(x + 3y)$ for $x = 0$ and $y = -6$.

4. (3) Simplify: $4x^2(x^{-2}y^{-4})^{-1}$

5. (4) Determine whether the graph represents a function. Explain.



Graphing Calculator **Simplify the following matrices. Check using a graphing calculator.**

***6.** (5) $4\begin{bmatrix} 2 & 1 \\ -6 & 3.5 \end{bmatrix} - 3\begin{bmatrix} 0 & 0.4 \\ 8 & -2 \end{bmatrix}$

***7.** (5) $\frac{1}{4}\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -\frac{1}{2} & 5 \end{bmatrix}$

Find the area of each triangle.

8. (SB) $h = 2$ cm, $b = 10$ cm

9. (SB) $h = \frac{1}{3}$ feet, $b = 18$ feet

10. (SB) Angles x and y are alternate interior angles formed by a transversal and 2 parallel lines. What is the measure of x if y measures 32°?

11. (1) **Golf** Tiger Woods obtained an average score of 72 during the four rounds of golf at the Player's Championship in 2007. Explain how you can use the distributive property to mentally calculate his total score for the four rounds of golf.

12. (1) **Multi-Step** Classify each statement as sometimes, always, or never true. Use properties of real numbers to support your answer.

a. $a \cdot \left(-\frac{1}{b}\right) = b \cdot \left(-\frac{1}{a}\right)$

b. if a and b are rational numbers, then $a + b$ is a real number.

13. (2) **Museums** The Bob Bullock Texas State History Museum offers student tickets for 3 dollars to view the exhibits. Tickets to view the IMAX theatre cost 5 dollars. The table below shows expressions for the number of students that visited the exhibits and theater over a 2-day period. Write an expression for the total cost of student tickets to view the exhibits during the 2 days.

Number of Students

	Exhibits	IMAX Theater
Day 1	$9v$	$7v$
Day 2	$7v$	$6v$

14. (3) **Solar System** The average distance between Jupiter and the Sun is approximately 7.786×10^{11} m.

a. Estimate the distance in kilometers that Jupiter travels around the Sun in one rotation. Write your answer in scientific notation. Round to the nearest hundredth.

b. Estimate the average speed at which Jupiter rotates around the Sun in km/s. Jupiter makes one rotation in 4,333 Earth days.

15. (4) Tell whether the given ordered pairs satisfy a function.

$$\{(2, 5), (2, 4), (2, 3), (2, 2), (2, 1)\}$$

***16.** (5) **Geometry** The matrix $R = \begin{pmatrix} 3 & 3.5 \\ 4 & 4.5 \end{pmatrix}$ shows the radii of four circles.

a. Write the matrix operation that gives the related circumferences.

b. Is there an addition or scalar-multiplication matrix operation that could show the related areas of the circles? Explain.

17. (4) **Precipitation** The mass of a large raindrop is equivalent to 1×10^{22} water molecules. The mass of a snowflake is equivalent to 1×10^{20} water molecules. Write a function to represent the mass of a raindrop in snowflakes.

***18.** (5) **Error Analysis** Find and explain the error the student made below.

$$\begin{bmatrix} 14 \\ 0 \\ 7 \end{bmatrix} + [23 \quad -8 \quad -6] = [37 \quad -8 \quad 1]$$

*19. (4) **Write** Explain why the vertical line test works.

20. (SB) The length of the side of a square is given by the expression $2x + 5$. Use the Distributive Property to write an expression for the perimeter of the square.

*21. (5) **Estimate** Estimate $4.8P$ given that $P = \begin{bmatrix} 2\frac{2}{3} & \pi \\ -\frac{1}{8} & 0.4\overline{5} \end{bmatrix}$.

22. (2) **Multiple Choice** Evaluate $x(xy + 2y^2)$ if $x = 4$ and $y = -1$.

A -24 **B** -12 **C** -8 **D** -2

*23. (5) **a.** Create matrix A such that $a_{11} = 4$, $a_{12} = 6$, $a_{21} = 0$, and $a_{22} = -1$.

b. What is the additive inverse of A?

24. (1) **Error Analysis** Explain the error in the statement that follows and correct the statement. "$\sqrt{48}$ is a real number and a rational number."

25. (2) **Verify** Evaluate the expression $uv + 8v - u - 4uv + 7u$ if $u = -1$ and $v = -1$ by inserting the variable values into the expression as written. Verify your answer by simplifying the expression first, then inserting the variable values.

*26. (4) $f(x) = x + 2$. Find $f(16)$.

*27. (4) **Formulate** Write a function that represents the data set $(-4, 9)$, $(0, 1)$, $(1, -1)$, $(3, -5)$, $(4, -7)$.

28. (4) **Model** One day the value of one US dollar was equivalent to 0.49 British pounds. On the same day, one US dollar was equivalent to 115.84 Japanese yen. Write a function to represent the value of yen in pounds.

*29. (5) **Justify** Does the Commutative Property hold true for Matrix Subtraction? Justify your answer.

30. (3) **Skyscrapers** One cubic foot of pennies contains approximately 4.9×10^4 pennies. If the Sears Tower in Chicago were filled with pennies, it would hold about 2.6×10^{12} pennies. What is the approximate volume of the Sears Tower? Write your answer in scientific notation rounded to the nearest tenth.

LESSON 6

Finding Percent of Change

Warm Up

1. (SB) **Vocabulary** Percent means per _____ (hundred/thousand).
2. (SB) Find 30% of 40.
3. (SB) Write $\frac{5}{8}$ as a decimal.
4. (SB) Write 25% as a fraction.
5. (SB) Find 20% of 60.

New Concepts

Any fraction or decimal can be written as a percent.

To write a decimal as a percent, multiply the decimal by 100.

To write a fraction as a percent, divide the numerator by the denominator to get a decimal, and then multiply the decimal by 100.

Example 1 **Changing Fractions and Decimals to Percents**

Hint

To multiply a number by 100, move the decimal point two places to the right.

a. Change 0.035 to a percent.

SOLUTION $0.035 \times 100 = 3.5$, so $0.035 = 3.5\%$.

b. Change 4 to a percent.

SOLUTION $4 \times 100 = 400$, so $4 = 400\%$.

c. Change $\frac{1}{8}$ to a percent.

SOLUTION $1 \div 8 = 0.125 = 12.5\%$

d. Change $\frac{12}{5}$ to a percent.

SOLUTION $12 \div 5 = 2.4 = 240\%$

The **percent of change** is the increase or decrease given as a percent of the original amount.

$$\text{percent of change} = \frac{\text{amount of increase or decrease}}{\text{original amount}}$$

When the new amount is greater than the original amount, the change is a percent increase. When the new amount is less than the original amount, the change is a percent decrease.

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Example 2 Finding the Percent Change

Calculate the percent change and tell if the change is a percent increase or percent decrease.

a. from 120 to 168

SOLUTION Use the formula.

$$\text{percent change} = \frac{\text{amount of increase}}{\text{original amount}}$$ The change is an increase.

$$= \frac{48}{120}$$ $168 - 120 = 48$

$$= 0.4 = 40\%$$

The percent change is a 40% increase.

b. from 6 to 5.1

SOLUTION Use the formula.

$$\text{percent change} = \frac{\text{amount of decrease}}{\text{original amount}}$$ The change is a decrease.

$$= \frac{0.9}{6}$$ $6 - 5.1 = 0.9$

$$= 0.15 = 15\%$$

The percent change is a 15% decrease.

Example 3 Determining the New Amount

Math Language

In Example 3a, the **percent decrease** is 70% and the **amount of decrease** is 40.6.

a. What is the new amount when 58 is decreased by 70%?

SOLUTION Find the amount of decrease.

70% of 58

0.7×58 Write the percent as a decimal and multiply.

40.6

Subtract the amount of decrease from the original amount to find the new amount: $58 - 40.6 = 17.4$.

b. What is the new amount when 620 is increased by 350%?

SOLUTION Find the amount of increase.

350% of 620

3.5×620 Write the percent as a decimal and multiply.

2170

Add the amount of increase to the original amount to find the new amount: $620 + 2170 = 2790$.

When an item in a store is on sale, the new price reflects a discount. The amount of discount is the difference between the old and new price and the percent of discount is a percent decrease.

On the other hand, when the price of an item increases, the increase is a markup and the percent change is a percent increase. A markup can refer to an increase in the retail price of an item, or it can refer to the amount by which the wholesale cost of the item is increased.

$$\text{sale price} = \text{original price} - \text{discount}$$

$$\text{marked up price} = \text{original price (or wholesale cost)} + \text{markup}$$

Example 4 Determining a Sale Price

A hiking backpack that regularly costs $98 is being offered at a 15% discount. Find the sale price of the backpack.

SOLUTION

1. Understand The sale price is the original price, $98, minus the discount, which is 15% of $98.

2. Plan Translate the sentence into algebra.

$$\text{sale price is} \quad \text{original price} - \text{discount}$$

$$\text{sale price is} \quad 98 \quad - 15\% \text{ of } 98$$

$$s = 98 - 0.15(98)$$

3. Solve $s = 98 - 14.7$

$$s = 83.3$$

The sale price is $83.30.

4. Check Find the percent decrease: $\frac{98 - 83.3}{98} = \frac{14.7}{98} = 0.15 = 15\%$.

Math Language

When translating from words to algebra, **is** changes to = and **of** changes to ×.

Example 5 Determining a Marked Up Price

A store owner buys candles from a local candle maker at $3.15 each and sells them at a 400% markup. How much does the store owner sell each candle for?

SOLUTION

1. Understand The marked up price is the wholesale cost, $3.15, plus the markup, which is 400% of $3.15.

2. Plan Translate the sentence into algebra.

$$\text{marked up price is} \quad \text{wholesale cost} + \text{markup}$$

$$\text{marked up price is} \quad 3.15 \quad + 400\% \text{ of } 3.15$$

$$m = 3.15 + 4(3.15)$$

3. Solve $m = 3.15 + 12.6$

$$m = 15.75$$

The store owner sells each candle for $15.75.

4. Check Find the percent increase: $\frac{15.75 - 3.15}{3.15} = \frac{12.6}{3.15} = 4 = 400\%$.

Lesson Practice

a. (1) Change 1.65 to a percent.

b. (1) Change 0.2 to a percent.

c. (1) Change $\frac{7}{4}$ to a percent.

d. (1) Change $\frac{3}{5}$ to a percent.

Calculate the percent change and tell if the change is a percent increase or percent decrease.

e. (2) from 10 to 11

f. (2) from \$50 to \$12

g. (3) What is the new amount when 700 is increased by 3%?

h. (3) What is the new amount when 6.2 is decreased by 90%?

i. (4) Ethan receives a 10% employee discount on any item in the store. How much will he pay for a printer that costs \$230?

j. (5) The price of a baseball card is marked up by 45%. What is the new price of the card if the price before the markup was \$60?

Practice Distributed and Integrated

Determine whether the following numbers are rational numbers.

1. (1) $\sqrt{225}$

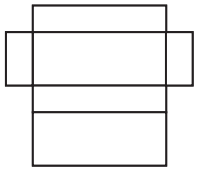
2. (1) π

3. (2) Evaluate $\frac{xy(x^2)}{2x}$ for $x = -1$ and $y = 4$.

4 (3) Simplify $x^5y(2x^{-5}y^4)^3$.

5. (5) **Graphing Calculator** Find $\begin{bmatrix} 1 & 5 \\ 2 & -3.2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$. Check using the calculator.

6. (SB) What solid figure is described by the net?



Find the percent change.

***7.** (6) from 3 to 4.5

***8.** (6) from 0.8 to 2.74

Identify whether each of the following represent a function. If so, state the domain and range.

9. (4) $y^2 = x + 1$

10. (4) $y = x^2 - 2$

11. (1) **Error Analysis** Two students are asked to simplify an expression and identify which property was used for each step. Their answers are shown below. Identify which student has an error in the procedure and correct the error. Simplify $(4 + 2) \cdot 10$.

Student A	
$(4 \cdot 10) + (2 \cdot 10)$	Associative Property
$40 + 20$	Distributive Property
60	Add.

Student B	
$6 \cdot 10$	Add
60	Multiply

***12.** (6) A student bought a \$3.20 notebook for 30% off and a \$78 calculator for 10% off.

a. How much did the student spend?

b. **Verify** Show that it is not correct to add the prices of the items and then take 40% off their total.

13. (2) **Multiple Choice** Evaluate $-x^2 + 6y$ if $x = 2$ and $y = 0$.

A -10 **B** -4 **C** 4 **D** 10

14. (4) Ty uses the function $g(x) = 0.5 + 0.2(x - 1)$ to calculate the cost in dollars of using a calling card to make a long distance call lasting x minutes. The variable x must be a whole number. Determine the cost of a 10-minute call.

***15.** (6) **Population** The population of Indiana in 1990 was reported to be 5,544,159 and in 2000 was reported to be 6,080,485. Find the percent increase. Round to the nearest hundredth of a percent.

16. (1) **Multiple Choice** Identify the subset of real numbers of which -5 is not a member.

A integer **B** real number **C** whole number **D** rational number

17. (5) **Sports** The table shows results from the first three track meets last year and this year.

Last Year

	1st	2nd	3rd
Meet 1	27	33	51
Meet 2	29	42	41
Meet 3	25	29	60

This Year

	1st	2nd	3rd
Meet 1	18	45	47
Meet 2	34	33	62
Meet 3	35	31	54

a. Create a matrix for each year.

b. Use matrix subtraction to subtract last year's matrix from this year's matrix.

18. (2) **Multi-Step** **a.** Evaluate the expression $-5vw(-2 + w)$ if $v = 3$ and $w = -4$.

b. Explain how the result would be affected if v was tripled.

c. Show that this is true by evaluating the expression if $v = 9$ and $w = -4$.

***19.** (6) **Analyze** What is the percent decrease when the original amount is 75 and the new amount is 0? Explain.

20. (2) **Planets** The speed of light is about 3×10^8 meters/second.

a. About how many seconds would it take light to reach Pluto from the Sun at their minimum distance apart of about 4.437×10^{12} meters?

b. About how many seconds would it take light to reach Pluto from the Sun at their maximum distance apart of about 7.376×10^{12} meters? Round your answers to the nearest second.

***21.** (6) **Write** Compare and contrast markups and discounts.

***22.** (6) **Geometry** Every side of a 2 by 7 rectangle is tripled to form a new rectangle. By what percent did the area of the rectangle increase?

23. (1) **Verify** Simplify the expression $(2 + 3) + 1$ using only one property of real numbers. Identify the property that you used. Verify your result using two properties of real numbers.

24. (4) **Currency** On the first day of 2007, the value of one US dollar was equivalent to 119 Japanese yen. On the same day, one euro was equivalent to 157 Japanese yen.

a. Write a function to represent the value of US dollars in euros.

b. What is the value of the function for an input of 5 rounded to the nearest cent, and what does it represent?

***25.** (2) Evaluate $a^2b^3 - (5a - b^2)$ if $a = 1$ and $b = 3$.

26. (1) For the expression $\frac{-4x}{7}$

a. find the additive inverse.

b. find the multiplicative inverse.

27. (5) **Coordinate Geometry** The coordinates of points in the coordinate plane can be written in matrices where the x-coordinates are listed in the first row and the corresponding y-coordinates are listed in the second row.

a. Graph $A = \begin{bmatrix} -1 & 0 & 5 \\ 4 & -2 & 2 \end{bmatrix}$.

b. Graph $A + B$ where $B = \begin{bmatrix} 3 & 4 & 9 \\ 6 & 0 & 4 \end{bmatrix}$. What happened?

***28.** (5) Find the additive inverse matrix of $\begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix}$.

29. (3) **Estimate** One cubic foot of pennies contains approximately 4.9×10^4 pennies. The Giants Stadium in New Jersey has an approximate volume of 6.45×10^7 cubic feet. About how many pennies would fit in the Giants Stadium?

30. (5) **Write** Explain how Scalar Multiplication is similar to the Distributive Property and how it is different.

LESSON 7

Solving Linear Equations

Warm Up

1. (1) **Vocabulary** $x + 3 = 3 + x$ demonstrates the _____ Property of Addition.

2. (1) Simplify $5(x - 3)$.

3. (2) Simplify $3x + 4y - 8x - 4y$.

New Concepts

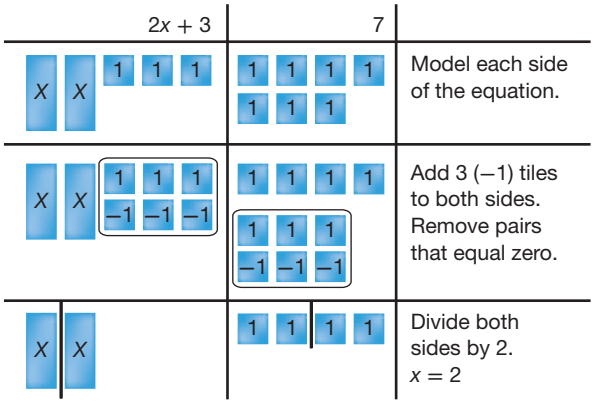
An **equation** is a statement which indicates that two expressions are equal.

$2 + 3 = 5$ $\quad x = 3$ $\quad 5a - 12 = 13$ are all examples of equations.

You can use algebra tiles to help you solve equations.

Exploration **Solving Equations Using Algebra Tiles**

Use algebra tiles to model and solve $2x + 3 = 7$.



Use algebra tiles to model and solve $3x + 1 = 10$.

A number is a **solution of an equation** in one variable if substituting the number for the variable results in a true statement. An equation can be solved using the properties of equality to transform the equation into an equivalent equation.

Math Language

A **solution set of an equation** is the set of values that makes an equation true.

Math Reasoning

Write Why is there a restriction of $c \neq 0$ in the Multiplication and Division Properties?

Properties of Equality		
Addition Property of Equality	Add the same number to both sides.	If $a = b$ then $a + c = b + c$.
Subtraction Property of Equality	Subtract the same number from both sides.	If $a = b$ then $a - c = b - c$.
Multiplication Property of Equality	Multiply both sides by the same number.	If $a = b$ and $c \neq 0$ then $ac = bc$.
Division Property of Equality	Divide both sides by the same number.	If $a = b$ and $c \neq 0$ then $a \div c = b \div c$.

Hint

Use the Addition or Subtraction Property of Equality to solve before the Multiplication or Division property.

Example 1 Solving Equations with a Variable on One Side

Solve $-5x + 12 = -23$.

SOLUTION

$$-5x + 12 = -23$$

$$\underline{\quad -12 = -12 \quad}$$ Subtract 12 from both sides.

$$-5x \quad = -35$$

$$\frac{-5x}{-5} = \frac{-35}{-5}$$ Divide both sides by -5.

$$x = 7$$

Check Check 7 in the original equation.

$$-5x + 12 = -23$$
$$-5(7) + 12 \stackrel{?}{=} -23$$
$$-5(7) + 12 \stackrel{?}{=} -23$$
$$-35 + 12 \stackrel{?}{=} -23$$
$$-23 = -23 \quad \checkmark$$

Equations can be transformed using the properties of equality on terms with variables. Variable terms can be added to or subtracted from both sides of an equation.

Example 2 Solving Equations with Variables on Both Sides

Solve $2n - 3 = 6n + 25$.

SOLUTION

$$2n - 3 = 6n + 25$$

$$\underline{+3 \qquad +3}$$ Add 3 to the both sides.

$$2n \quad = 6n + 28$$

$$\underline{-6n \quad = -6n}$$ Subtract $6n$ from both sides.

$$\frac{-4n}{-4} = \frac{28}{-4}$$ Divide both sides by 4.

$$n = -7$$

Check $2(-7) - 3 \stackrel{?}{=} 6(-7) + 25$

$$-14 - 3 \stackrel{?}{=} -42 + 25$$
$$-17 = -17 \quad \checkmark$$

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Example 3 Solving Equations Using the Distributive Property

Solve $12(r + 3) = 2(r + 5) - 3r$.

SOLUTION

$$12(r + 3) = 2(r + 5) - 3r$$

$12r + 36 = 2r + 10 - 3r$ Distribute.

$12r + 36 = -r + 10$ Combine like terms.

$-36 = -36$ Subtract 36.

$12r = -r - 26$

$+r = +r$ Add r.

$13r = -26$

$\frac{13r}{13} = \frac{-26}{13}$ Divide by 13.

$r = -2$

Check $12(r + 3) = 2(r + 5) - 3r$

$12(-2 + 3) \stackrel{?}{=} 2(-2 + 5) - 3(-2)$

$12(1) \stackrel{?}{=} 2(3) - (-6)$

$12 \stackrel{?}{=} 6 - (-6)$

$12 = 12$ ✓

Example 4 Solving Equations with Fractions

Solve $\frac{1}{2}x + 4 = -\frac{2}{3}x + \frac{1}{2}$.

SOLUTION

$$\frac{1}{2}x + 4 = -\frac{2}{3}x + \frac{1}{2}$$

$6\left(\frac{1}{2}x + 4\right) = 6\left(-\frac{2}{3}x + \frac{1}{2}\right)$ Multiply both sides by 6.

$3x + 24 = -4x + 3$

$-24 = -24$ Subtract 24.

$3x = -4x - 21$

$+4x = +4x$ Add $4x$.

$7x = -21$

$\frac{7x}{7} = \frac{-21}{7}$ Divide by 7.

$x = -3$

Math Reasoning

Generalize For any equation containing fractions how do you eliminate the fractions?

Math Reasoning

Estimate Work the problem by estimating the populations to the nearest thousands first.

Example 5 Application: Population

According to the 2000 census, the three smallest states in the United States by population are North Dakota, Vermont, and Wyoming. The total population of these three states is 1,774,779. If Vermont has a population of 623,908 and North Dakota has a population of 635,867, write and solve an equation to determine the population of Wyoming.

SOLUTION Let x equal the population of Wyoming.

$$\begin{aligned} x + 623{,}908 + 635{,}867 &= 1{,}774{,}779 \\ x + 1{,}259{,}775 &= 1{,}774{,}779 \\ -1{,}259{,}775 &= -1{,}259{,}775 \\ \hline x &= 515{,}004 \end{aligned}$$

The population of Wyoming is 515,004 people.

Check $515{,}004 + 623{,}908 + 635{,}867 = 1{,}774{,}779$ ✓

Lesson Practice

Solve.

a. (Ex 1) $8t - 21 = 11$

b. (Ex 2) $5r - 12 = -4r - 30$

c. (Ex 3) $6(y - 5) + 4y = 5(y + 2)$

d. (Ex 4) $-\frac{3}{4}s - 3 = \frac{1}{3}s + \frac{1}{4}$

e. (Ex 5) Aaron is traveling 955 miles from Phoenix, AZ to Jackson, WY. He leaves Phoenix, driving 410 miles before stopping for gas in Cortez, Co. He makes one last stop in Rock Springs, WY before traveling the final 180 miles to Jackson. Write and solve an equation to find the number of miles between Cortez and Rock Springs.

Practice Distributed and Integrated

Find the measure of the angles that are supplementary to the angles given.

1. (SB) 65°

2. (SB) 138°

3. (2) Evaluate $3xy^3 + x - y$ for $x = -2$ and $y = -2$.

***4.** (5) **Graphing Calculator** Find $\frac{3}{4}\begin{bmatrix} 8 & -4 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -6 \\ -7 & -2 \end{bmatrix}$. Check using a graphing calculator.

Find the area of each rectangle given the dimensions below.

5. (SB) $l = 2$ inches, $w = 1.3$ inches

6. (SB) $l = 4$ cm, $w = 3$ cm

7. (6) Find the percent change from 5 to 6.1.

Solve each equation.

***8.** (7) $3(x + 2) = 2x - 1$

***9.** (7) $\frac{1}{2}x + 8 = 24$

10. (4) Use the table to answer the following questions.

x	0	1	2	3
y	9	7	5	3

a. What is the domain?

b. What is the range?

c. Does this data represent a function? Explain.

11. (3) **Astronomy** Saturn's smallest moon, Tethys, has a diameter of about 6.5×10^2 miles. The diameter of Jupiter's largest moon, Ganymede, is 5 times that of Tethys. Determine the diameter of Ganymede. Write your answer in standard form.

12. (1) **Justify** Simplify the expression using properties of real numbers. Justify your answer by identifying which property you used for each step.

$$3 \cdot 5 \cdot \frac{1}{3}$$

13. (4) **Diving** The pressure on a diver is given by $P(d) = \frac{1}{33}d + 1$ where the pressure is in atmospheres and d is the diver's depth in feet. What is the pressure of the diver at 50 feet under water?

14. (6) Which is less expensive: an \$85 pair of shoes with a 40% discount or a \$68 pair of shoes with a 15% discount? How much less expensive is it?

***15.** (7) Sandra's brother is 6 years older than her, her sister is 2 years younger than her, and her mother is 3 times her age. Her brother, sister, and mother's combined age is 64.

a. Write an equation to solve for Sandra's age.

b. Solve your equation.

16. (3) **Speed of Sound** Mauna Kea in Hawaii is about 1.02×10^4 meters from its base to its peak. Assume that the speed of sound is about 3.44×10^2 meters/second. Use this information to solve the following questions. Write your answers in scientific notation.

a. About how many seconds would it take sound to reach the base from the peak? Round your answer to the nearest second.

b. Approximately 5.96×10^3 meters of Mauna Kea is below sea level. About how many seconds would it take sound to reach sea level from the peak? Write your answers in scientific notation.

***17.** (7) **Verify** Is $x = \frac{1}{14}$ the solution to the equation $9x - \frac{2}{7} = 8x + \frac{11}{14}$?

18. (4) **Coordinate Geometry** Explain whether a circle drawn on a coordinate grid represents a function.

19. *(6)* **Economics** The median income for US females with a doctorate degree rose from $40,172 in 1990 to $60,733 in 2000. The median income for US males with a doctorate degree rose from $54,626 to in 1990 to $78,092 in 2000. Which group had the greater percent increase? How much greater? (Round to the nearest percent.)

20. *(5)* **Error Analysis** Find and explain the error the student made below.

$$6\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix}=\begin{bmatrix}8 & 11\\ 15 & 4\end{bmatrix}$$

21. *(5)* Given that $A=\begin{bmatrix}4 & 5 & -6\\ -9 & 0 & 3\\ 8 & 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 7\\ 8 & -14\\ 5 & -9\end{bmatrix}$, what is $a_{13}+b_{21}$?

***22.** *(6)* **Error Analysis** A student said that a change from 15 to 30 is a 50% increase because 15 is 50% of 30. Explain the error in the student's thinking.

***23.** *(7)* **Coffee** On Sunday a coffee shop sold 115 lb of coffee. The amount of regular coffee sold was 5 lb more than the amount of decaffeinated coffee sold. There were twice as many pounds of flavored coffee sold than there was regular coffee sold.

a. Write an equation to solve for the amount of regular coffee sold.

b. Solve the equation from part a.

***24.** *(6)* **Multiple Choice** What is the discount when a $14 shirt is on sale for 20% off?

A $2.80 **B** $7.00 **C** $11.20 **D** $16.80

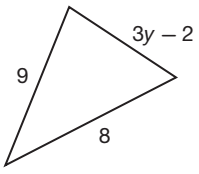
25. *(2)* **Multiple Choice** Evaluate $-5x^2+2y$ if $x=1$ and $y=-1$.

A -7 **B** -3 **C** 3 **D** 7

26. *(4)* **Formulate** Write a function that represents the data set $(-19, -12)$, $(-12, -5)$, $(-5, 2)$, $(2, 9)$, $(9, 16)$.

27. *(6)* **Verify** Show how to check that 28 increased by 80% is 50.4.

***28.** *(7)* **Geometry** Janice wants to find the value of y for the triangle shown. The perimeter of the triangle is $8y$. Write and solve an equation to find the value of y.



29. *(2)* **Multi-Step** **a.** Evaluate the expression $6st(6+t)$ if $s=6$ and $t=-2$.

b. Explain how the result would be affected if s was doubled.

c. Show that this is true by evaluating the expression if $s=12$ and $t=-2$.

***30.** *(7)* **Sales Tax** The state sales tax for Colorado, Louisiana, and Nevada total to 13.4%. The sales tax in Colorado is 1.1% less than that of Louisiana and 3.6% less than that of Nevada. Find the state sales tax rate for Colorado.

LESSON 8

Finding Direct Variation

Warm Up

1. (2) **Vocabulary** The value of a _____ (***constant, variable***) never changes.

2. (SB) Solve: $42 = 6k$

3. (SB) Solve: $\frac{3}{x} = \frac{4}{12}$

New Concepts

When the statement of a problem says that A ***varies directly as*** B or that A ***is directly proportional to*** B, the equation

$$A = kB$$

is implied. This relationship is called **direct variation.** The constant k in the equation is called the **constant of variation.** The following are examples of direct variation.

Statement	Implied Equation
The circumference of a circle varies directly as the radius.	$C = kR$
The resistance is directly proportional to the length.	$R = kL$
The water produced varied directly as the amount of hydrogen burned.	$W = kH_B$

Example 1 Solving a Direct Variation Problem

The number of seconds varies directly as the number of minutes. When 120 seconds have passed, 2 minutes have passed. If 300 seconds have passed, how many minutes have passed?

SOLUTION

$S = kM$	Write a direct variation.
$(120) = k(2)$	Substitute the given values.
$k = 60$	Solve for k.
$S = 60M$	Replace k with its value.
$(300) = 60M$	Substitute for S.
$\frac{300}{60} = \frac{60M}{60}$	Divide.
$M = 5$ minutes	

Hint

Choosing appropriate variable names helps avoid confusion when substituting values.

Example 2 Solving with a Fractional Constant

Distance traveled is directly proportional to the time spent traveling. A motorized scooter can travel 10 kilometers in 30 minutes. How far can this scooter travel in 45 minutes?

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SOLUTION

$D = kT$ Write a direct variation.

$10 = k(30)$ Substitute the given values.

$k = \frac{1}{3}$ Solve for k.

$D = \frac{1}{3}T$ Replace k with its value.

$D = \frac{1}{3}(45)$ Substitute for T.

$D = 15$ Multiply.

It can travel 15 kilometers in 45 minutes.

Hint

The cross products of a proportion are equal. For $\frac{A_1}{A_2} = \frac{B_1}{B_2}$, $A_1 \cdot B_2 = B_1 \cdot A_2$.

There is another way to work direct variation problems. In addition to the equation discussed above, the statement *A* ***varies directly as*** *B* also implies the proportion.

$$\frac{A_1}{A_2} = \frac{B_1}{B_2}$$

Example 3 Direct Variation as a Ratio

a. Cost varies directly as the number purchased. If 12 items can be purchased for \$78, how much would 42 items cost?

SOLUTION

$\frac{C_1}{C_2} = \frac{N_1}{N_2}$ Write a proportion.

$\frac{78}{C_2} = \frac{12}{42}$ Substitute the given values.

$78 \cdot 42 = 12C_2$ Cross multiply.

$\frac{78 \cdot 42}{12} = \frac{12C_2}{12}$ Divide both sides by 12.

$273 = C_2$

42 items would cost \$273.

b. The number of railroad ties used in a section of track is directly proportional to the length of the section. If 15 ties are used in a 30 foot section of track, how many ties are used in a 50 foot section?

SOLUTION Write the equation and substitute the given values.

$$\frac{T_1}{T_2} = \frac{L_1}{L_2} \rightarrow \frac{(15)}{T_2} = \frac{(30)}{(50)}$$

Cross multiply and solve.

$$15 \cdot 50 = 30T_2 \rightarrow \frac{15 \cdot 50}{30} = \frac{30T_2}{30} \rightarrow T_2 = 25 \text{ ties}$$

Example 4 Application: Chemistry

The volume of a certain gas varies directly as its temperature in kelvins. The table below shows the volume of this gas at two temperatures.

Volume	224 mL	236 mL	?
Temperature	280 K	295 K	315 K

Math Language

A **kelvin** (K) is a unit for measuring temperature. Temperature in kelvins is equal to temperature in degrees Celsius plus 273.15.

a. Find the volume of the gas at the third temperature in the table.

SOLUTION The implied equation is $V = kT$. Use either of the two given volume/temperature pairs to find the constant of variation.

$$(236) = k(295)$$

$$k = \frac{4}{5}$$

So the equation that represents the relationship between the volume and temperature of this gas is

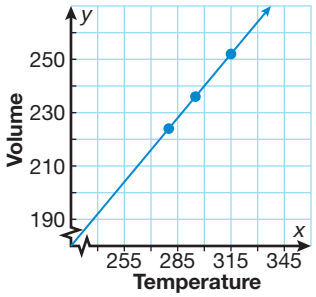
$$V = \frac{4}{5}T$$

Substitute the temperature that corresponds to the unknown volume to find V.

$$V = \frac{4}{5}(315)$$

$$V = 252 \text{ mL}$$

b. Plot the three volume/temperature pairs on a graph and draw the line that connects the three points.



This is a visual representation of the equation $V = \frac{4}{5}T$. Like the equation, this graph also represents the relationship between the volume and temperature of this gas.

c. **Predict** Use the equation to predict what volume the gas would have at 400 K.

$$V = \frac{4}{5}(400)$$

$$V = 320 \text{ mL}$$

Lesson Practice

a. (Ex 1) The number of inches varies directly with the number of feet. If a board measures 36 inches, it measures 3 feet. If a board measures 60 inches, how many feet does it measure?

b. (Ex 2) The number of cups varies directly with the number of fluid ounces. In 5 cups, there are 40 fluid ounces. How many cups are there in 64 fluid ounces?

c. (Ex 3) The cost of gas varies directly with the number of gallons purchased. If 10 gallons can be purchased for $25, how much would 15 gallons cost?

d. (Ex 3) The amount of dirt need to cover a garden varies directly with the area of the garden. If 2 bags of dirt are needed to cover a garden with an area of 54 square feet, how many bags are needed to cover a garden with an area of 135 square feet?

e. (Ex 4) The volume of a certain pipe varies directly with its length. The table below shows the volume of the pipe at two lengths.

Volume	30 gallons	45 gallons	?
Length	20 feet	30 feet	40 feet

Find the volume of the pipe at the third length in the table.

f. (Ex 4) Plot the three volume/length pairs on a graph and draw the line that connects the three points.

g. (Ex 4) Write the equation that represents the relationship between volume and length, and use it to predict the volume of the pipe at a length of 100 feet.

Practice Distributed and Integrated

1. (1) Simplify the expression using the properties of real numbers.

$$(-7 + 3) - |-2 + 9| - 2[1 - (-5)]$$

2. (2) Simplify $\frac{3x}{y} - 7x^2x^{-1}y^{-1} + 2y^2y^{-1}x^{-1}$.

3. (3) Simplify $\frac{xy^2}{x^0x^{-3}}\left(\frac{xy^{-2}}{x(y^2)^0} - \frac{3y^{-2}}{x^4}\right)$.

4. (SB) Find the area of a circle with radius 1.5 inches.

5. (6) Find the percent change from 10 to 12.8.

Solve each equation.

***6.** (7) $15(4 - 5b) = 16(4 - 6b) + 10$

***7.** (7) $-2(2x - 3) - 2^3 - 3 = -x - (-4)$

For the following problems, x varies directly with y. Find the constant of variation.

***8.** (8) $x = 12, y = 3$

***9.** (8) $x = 4, y = 16$

10. (2) Simplify the expression $5x^2 - 2(x - 3x^2)$.

*11. (1) **Graphing Calculator** The average credit card annual interest rate is around 19%. Use the distributive property to mentally calculate the amount of interest paid annually on a $600 balance. Use a graphing calculator to verify your answer.

12. (6) **Multi-Step** Admission to the museum is $12. Students receive a 15% discount.

a. How much is the discount?

b. How much do students pay?

13. (5) **Education** The table shows the number of men and women who attended a university and if they were attending on a full or part-time basis.

University Enrollment

	2000		2005	
	Men	**Women**	**Men**	**Women**
Full-time	315	484	511	562
Part-time	631	709	826	914

a. Create a matrix to show the number of men attending the university in both years.

b. Create another matrix to show the number of women attending the university in a way that it could be added to the matrix in part a.

c. Add the matrices.

*14. (8) **Multiple Choice** In a peanut butter cookie recipe, the amount of sugar used varies directly with the amount of peanut butter used. For 2 c of peanut butter, $1\frac{1}{2}$ c of sugar is needed. Circle the letter that best represents the constant of variation.

A 1 **B** $\frac{1}{4}$ **C** $\frac{3}{4}$ **D** 2

15. (3) **Analyze** A quarter is about 6.89×10^{-2} inches thick. A quarter has a diameter that is about 13.9 times its thickness. Estimate the diameter of a quarter in inches. Write your answer in scientific notation. Explain whether you think the magnitude of your answer seems reasonable.

16. (6) **Statistics** The first time a teacher tested her students, the scores were 75, 82, 93, 81, 64, 78, and 80. The second time she tested them, the scores were 65, 72, 100, 83, 68, 55, and 40. What was the percent change in the average score from the first to the second testing? Round to the nearest whole percent.

*17. (8) **Bowling** The number of pounds varies directly with the number of ounces. A bowling ball that weighs 12 pounds weighs 192 ounces. What is the constant of variation? Find the weight, in pounds, of a bowling ball that weighs 240 ounces.

18. (5) **Multi-Step** Solve for x, y, and z: $\begin{bmatrix} 1 & x^2 \\ y+4 & \frac{y}{2} \end{bmatrix} = \begin{bmatrix} z+2 & 25 \\ 16 & x+1 \end{bmatrix}$.

*19. (8) **Geometry** The area of a triangle varies directly with the product of the base and height of the triangle. If the area of a triangle is 6 cm^2 then the product of the base and height is 12 cm^2, find the area of a triangle whose product of base and height is 27 cm^2.

20. (5) **Consumer** A ticket service marks up prices on concerts and other events by 150%. Use a scalar product to find the marked up prices.

Concert Ticket Prices

Day	Floor	Balcony
Friday	$60	$35
Saturday	$80	$45

*21. (7) **Verify** Is $x = 2$ the solution to the equation $-10x + 6 = 4x + 34$?

22. (7) Solve the equation $6x + 3 - 8x = 13$.

23. (4) **Marketing** If you buy one quarter page ad, the cost is $400 per week. This can be expressed as the ordered pair (1, 400). There is a special if you buy one ad at the regular price, each additional week is $300. Find the total cost if you buy 2, 3, and 4 ads. Express the answer as ordered pairs in set notation.

24. (3) **Write** Does $(x^2)^3 = (x^3)^2$? Explain.

25. (6) **Analyze** When will a percent change equal 0%? Give an example.

*26. (8) **Verify** The volume of a sphere varies directly with the cube of the radius. The table shows the volume and radius of three different spheres. Verify the constant of variation is 4.189.

Volume	Radius
2,144.768 in^3	8 in.
20,580.557 in^3	17 in.
65,453.125 in^3	25 in.

27. (5) **Multiple Choice** What are the dimensions of matrix $C = [\,9\ 1\ 7\ 0\,]$?

A 1×3 **B** 1×4 **C** 3×1 **D** 4×1

*28. (7) **Exports** The total exports of goods and services, in millions of dollars, for the U.S. in 2004, 2005, and 2006 was 3,886,023. The exports in 2005 were 125,820 more than in 2004 and 162,633 less than in 2006. What were the exports in millions of dollars in 2005?

29. (4) If $a(x) = 12 + 9x$ and $v(x) = 5x + 2x^2$, find $v(4)$.

30. (1) **Error Analysis** Explain the error in the statement that follows and correct the statement. "3 is an irrational number and an integer."

LESSON 9

Multiplying Matrices

Warm Up

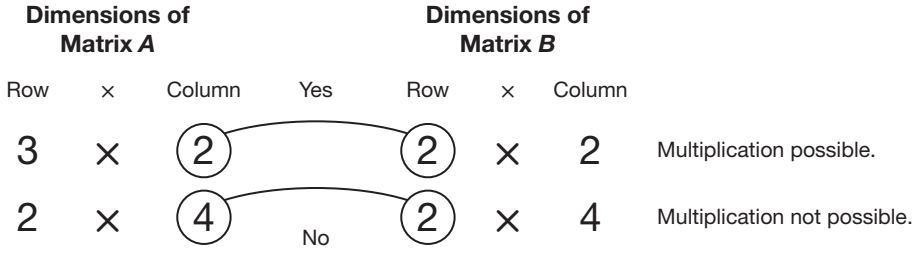
1. (7) **Vocabulary** The individual symbols in a matrix are the _____ of the matrix.

2. (1) Simplify $2(-5) + 2(6) + (-1)(-5) + (-1)(6) + 3(-5) + 3(-6)$.

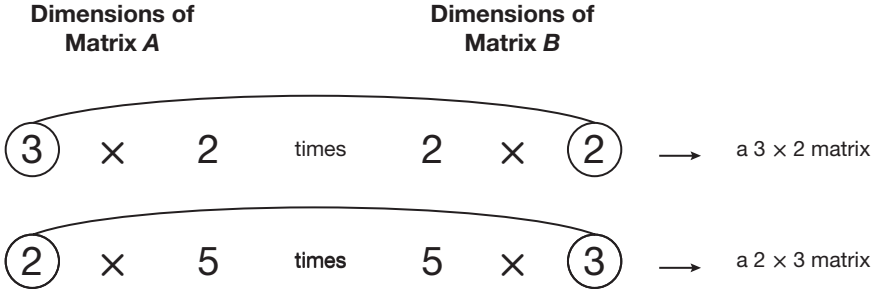
3. (5) Give the dimensions of matrix $A = [-2 \quad 3 \quad 0 \quad -1]$.

New Concepts

A matrix A can be multiplied by another matrix B if the number of columns of matrix A is the same as the number of rows of matrix B. The "inside" numbers, that is the last number of the dimensions of matrix A and the first number of the dimensions of matrix B, must be the same.



The product of two matrices is a matrix whose dimensions are determined by the two "outside" numbers.



When two matrices can be multiplied, the product is said to be defined. The matrix that results is called a **product matrix.**

Caution

Pay attention to the order in which matrices are multiplied. To find the product AB, keep matrix A on the left and matrix B on the right.

Example 1 Using Dimensions of Matrices

Determine if the product of the matrices is defined. If so, give the dimensions of the product matrix.

a. Matrix A: 4×3, Matrix B: 3×2

SOLUTION

Yes, the "inside" numbers are equal. The number of columns of A = the number of rows of B. AB is a 4×2 matrix.

b. Matrix A: 3×1, Matrix B: 2×1

SOLUTION

No, the "inside" numbers are not equal. The number of columns of $A \neq$ the number of rows of B.

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c. Matrix A: 1×5, Matrix B: 5×1

SOLUTION

Yes, the "inside" numbers are equal. AB is a 1×1 matrix.

Matrix Multiplication
Any element in a particular row and column of the product matrix is the sum of the product of the corresponding elements in the corresponding row and column of the matrices being multiplied.

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \cdot \underset{B}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} = \underset{AB}{\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}}$$

The row one elements of matrix A are multiplied by the corresponding column one elements of matrix B. Then these products are added to find the corresponding element in the product matrix AB. Continue this pattern row by column to find all of the elements in the product matrix.

Example 2 Multiplying Matrices

Hint

Find the dimensions of the product matrix before multiplying the matrices.

a. Find AB if $A = \begin{bmatrix} 2 & -3 \\ 3 & 1 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & -5 \end{bmatrix}$

SOLUTION

Since matrix A is a 3×2 matrix and matrix B is a 2×2, the number of columns of A = the number of rows of B. The product AB is defined.

$$AB = \begin{bmatrix} 2 \cdot 1 + (-3) \cdot 4 & 2 \cdot 2 + (-3) \cdot (-5) \\ 3 \cdot 1 + 1 \cdot 4 & 3 \cdot 2 + 1 \cdot (-5) \\ 4 \cdot 1 + 5 \cdot 4 & 4 \cdot 2 + 5 \cdot (-5) \end{bmatrix} = \begin{bmatrix} -10 & 19 \\ 7 & 1 \\ 24 & -17 \end{bmatrix}$$

b. Find BA if $A = [-3 \quad 0 \quad 4]$ and $B = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$

SOLUTION

Since matrix B is a 3×1 matrix and matrix A is a 1×3, the number of columns of B = the number of rows of A. The product BA is defined.

$$BA = \begin{bmatrix} 2 \cdot (-3) & 2 \cdot 0 & 2 \cdot 4 \\ (-5) \cdot (-3) & (-5) \cdot 0 & (-5) \cdot 4 \\ 1 \cdot (-3) & 1 \cdot 0 & 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 8 \\ 15 & 0 & -20 \\ -3 & 0 & 4 \end{bmatrix}$$

c. Determine if $AB = BA$. $A = \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 \\ 0 & 1 \end{bmatrix}$

SOLUTION

$$AB = \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 15 & -11 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -14 & 6 \\ 3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 3 \\ 15 & 11 \end{bmatrix} \neq \begin{bmatrix} -14 & 6 \\ 3 & -2 \end{bmatrix}$$

$$AB \neq BA$$

Refer to page 27 to check the answers using a graphing calculator.

The multiplication of real numbers is a commutative operation because either number can be first: $ab = ba$. This property does not generally hold for matrix multiplication.

A **square matrix** is a matrix with the same number of rows and columns. It has a **main diagonal** of elements from the top left hand corner to the bottom right hand corner of the matrix. The **multiplicative identity matrix,** called I, is a square matrix in which all of the elements are zero except the main diagonal. All of the elements in the main diagonal are ones.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is a } 2 \times 2 \text{ multiplicative identity matrix.}$$

The product of any matrix A and the multiplicative identity matrix I is matrix A.

$$AI = IA = A$$

Example 3 Working with Multiplicative Identity Matrices

Find AI. $A = \begin{bmatrix} 4 & 3 & -1 \\ 6 & -2 & -3 \\ -5 & 1 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

SOLUTION

$$\begin{bmatrix} 4 & 3 & -1 \\ 6 & -2 & -3 \\ -5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \cdot 1 + 3 \cdot 0 + (-1)0 & 4 \cdot 0 + 3 \cdot 1 + (-1) \cdot 0 & 4 \cdot 0 + 3 \cdot 0 + (-1) \cdot 1 \\ 6 \cdot 1 + (-2) \cdot 0 + (-3) \cdot 0 & 6 \cdot 0 + (-2) \cdot 1 + (-3) \cdot 0 & 6 \cdot 0 + (-2) \cdot 0 + (-3) \cdot 1 \\ (-5) \cdot 1 + 1 \cdot 0 + 0 \cdot 0 & (-5) \cdot 0 + 1 \cdot 1 + 0 \cdot 0 & (-5) \cdot 0 + 1 \cdot 0 + 0 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & -1 \\ 6 & -2 & -3 \\ -5 & 1 & 0 \end{bmatrix}$$

Math Reasoning

Verify In Example 3, show that $AI = IA$.

Example 4 Consumer Math Application

John, My-Lieng, and Rochelle are purchasing supplies for a camping trip. Use matrix multiplication to find the amount each spent on supplies.

Camping Gear

	Day pack(s)	Tent	Sleeping bag(s)
John	1	1	3
My-Lieng	1	1	2
Rochelle	2	0	1

Cost of Gear

Day pack	$16
Tent	$50
Sleeping bag	$85

Hint

To multiply $A \times B$, multiply rows of A with columns of B.

SOLUTION

$$\begin{bmatrix}1 & 1 & 3\\1 & 1 & 2\\2 & 0 & 1\end{bmatrix}\begin{bmatrix}16\\50\\85\end{bmatrix}=\begin{bmatrix}1\cdot 16+1\cdot 50+3\cdot 85\\1\cdot 16+1\cdot 50+2\cdot 85\\2\cdot 16+0\cdot 50+1\cdot 85\end{bmatrix}=\begin{bmatrix}321\\236\\117\end{bmatrix}$$

John spent $321, My-Lieng spent $236, and Rochelle spent $117 on camping gear.

Refer to page 27 to check the answer using a graphing calculator.

Lesson Practice

a. *(Ex 1)* Determine if the product of matrices A and B is defined. If so, give the dimensions of the product matrix.

Matrix A: 5×1, Matrix B: 1×4

b. *(Ex 1)* Determine if the product of matrices A and B is defined. If so, give the dimensions of the product matrix.

Matrix A: 4×7, Matrix B: 4×7

c. *(Ex 2)* Determine if the product of matrices A and B is defined. If so, give the dimensions of the product matrix.

Matrix A: 3×9, Matrix B: 9×3

d. *(Ex 2)* Find AB if $A=\begin{bmatrix}4 & 2\\1 & 6\\3 & 9\end{bmatrix}$ and $B=\begin{bmatrix}0 & 3\\5 & 8\end{bmatrix}$.

e. *(Ex 3)* Find BA if $A=\begin{bmatrix}4 & -2 & 10\end{bmatrix}$ and $B=\begin{bmatrix}-1\\3\\-6\end{bmatrix}$.

f. *(Ex 3)* Determine if $AB = BA$. $A=\begin{bmatrix}4 & -2\\2 & 0\end{bmatrix}$ $B=\begin{bmatrix}1 & 0\\4 & -2\end{bmatrix}$.

g. *(Ex 4)* Find AI. $A=\begin{bmatrix}-1 & 4 & 2\\0 & 5 & -3\\1 & -2 & 10\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$.

h. (Ex 5) Alana, Maurice, and Jon are each ordering lunch. Use matrix multiplication to find the amount each spends on lunch.

Lunch

	Sandwich	Side order	Drink
Alana	1	2	1
Maurice	2	0	1
Jon	1	2	2

Cost of Lunch

Sandwich	$3
Side order	$2
Drink	$1

Practice Distributed and Integrated

Write each decimal as a fraction in lowest terms.

1. (SB) 0.24 **2.** (SB) 0.82 **3.** (SB) 0.32

4. (SB) What is the measure of an interior angle of a regular pentagon?

5. (4) Determine whether $y^2 = 8x$ is a function. If so, state the domain and range.

6. (5) Find the sum: $2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}-3 & -2\\-4 & 0\end{bmatrix}$

***7.** (8) If m varies directly with n, and $m = 8$ when $n = 2$, what is the value of the constant of variation?

Graphing Calculator **Find each product. Check using the graphing calculator.**

***8.** (9) $\begin{bmatrix}2 & -1\\3 & 0\\1 & 4\end{bmatrix} \times \begin{bmatrix}1 & 1 & -3\\6 & -5 & 0\end{bmatrix}$

***9.** (9) $\begin{bmatrix}-1 & -1 & -1\\2 & 5 & 3\\-5 & 4 & -2\end{bmatrix} \times \begin{bmatrix}0\\1\\0\end{bmatrix}$

10. (5) **Verify** Use the matrices A and B below to verify that $A + B = B + A$.

$$A = \begin{bmatrix}-3 & 7\\12 & -1\end{bmatrix} \quad B = \begin{bmatrix}9 & 17\\3 & -1\end{bmatrix}$$

11. (1) **Baseball** The table shows the number of home runs hit by Orlando Hernandez of the New York Yankees over a three-year period. Which property of real numbers would be useful to mentally calculate as quickly as possible the total home runs that he hit over this period? Explain.

	2004	2005	2006
Home Runs	9	18	22

12. (3) **Earth** The surface area of the land on Earth is approximately 1.488×10^{14} m^2. The state of New York has a surface area of approximately 1.272×10^{11} m^2. Approximately how many times could the state of New York fit on the land of the Earth? Write your answer in scientific notation.

*13. (9) **Multiple Choice** Circle the letter that best represents the product matrix AB.

$$A = \begin{bmatrix} -3 & 1 & -1 \\ -4 & 6 & 2 \\ -1 & -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -7 & 0 & 1 \\ -8 & -2 & -1 \\ 0 & 3 & -3 \end{bmatrix}$$

a. $\begin{bmatrix} 20 & -8 & 11 \\ 33 & -19 & 0 \\ -9 & 21 & -6 \end{bmatrix}$ **b.** $\begin{bmatrix} 13 & -5 & -1 \\ -20 & -6 & -16 \\ 15 & 14 & -12 \end{bmatrix}$

c. $\begin{bmatrix} 21 & 0 & -1 \\ 32 & -12 & -2 \\ 0 & -3 & -12 \end{bmatrix}$ **d.** $\begin{bmatrix} -10 & 1 & 0 \\ -12 & 4 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

14. (5) **Scores** Create matrices to represent the following: Juan's high scores on three arcade games in one year are 34,950, 7800, and 12,300. His low scores on the same games are 6400, 2100, and 970. Tatia's high scores on the same games are 28,900, 10,450, and 11,950. Her low scores are 7200, 3600, and 4860.

15. (3) **Write** Is $a^{-1} < 1$, for all $a > 1$? Explain why or why not.

16. (6) **Population** The population of Alaska in 1990 was reported to be 550,043 and in 2000 was reported to be 626,932. Find the percent increase. Round to the nearest hundredth of a percent.

17. (5) **Property Management** The table shows the cost of renting three different apartment styles in an apartment complex.

	Monthly Rental Cost
One bedroom	$650
Two bedrooms	$890
Three bedrooms	$940

a. Convert the table into a 3×1 matrix.

b. All rents at an apartment complex will be increased by 10%. What scalar can be used to make a matrix with the updated prices? Create this matrix.

18. (5) **Verify** Verify that $D = \begin{bmatrix} -14 & -6 \\ -6 & -1 \end{bmatrix}$ if $\begin{bmatrix} 1 & -6 \\ -6 & 4 \end{bmatrix} = D + \begin{bmatrix} 15 & 0 \\ 0 & 5 \end{bmatrix}$.

19. (1) Identify which property of real numbers is being demonstrated.

$$5(2 + 9) = 5 \cdot 2 + 5 \cdot 9$$

*20. (8) **Planets** The weight of an object on Saturn varies directly with the weight of that object on Earth. An object that weighs 2 kg on Earth, weighs 1.86 kg on Saturn. Find the weight of an object on Saturn that weighs 85 kg on Earth.

21. (4) **Error Analysis** Two students were asked to determine which of the following sets of ordered pairs are functions. Set 1: (9, 4), (7, 9), (8, 2) and Set 2: (3, −1), (4, −1), (7, 9). Their answers are below. What is the correct answer? Explain the error(s).

Britan's Answer: Set 2 because the 9's are paired with different values.

Sara's Answer: Set 1 because the −1's are paired with different values.

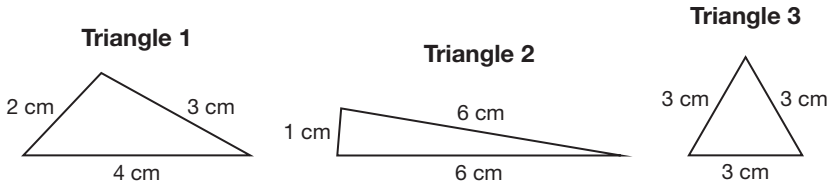
22. (6) **Economics** In 2002, San Jose, California topped the list for the US metropolitan areas with the highest average annual salaries with an average salary of $76,252. Find the approximate average annual salary in 2001 if the percent change from 2002 to 2001 was a percent decrease of about 13.5%.

23. (1) Classify each statement as sometimes, always, or never true. Give examples to support your answer.

a. $a \cdot \left(\frac{1}{b}\right) = b \cdot \left(\frac{1}{a}\right)$

b. if a and b are irrational numbers, then $a \cdot b$ is an irrational number.

***24.** (9) **Geometry** Create a matrix representing the three triangles below.



Use matrix multiplication to double side 1 (left), triple side 2 (bottom) and half side 3 (right) of each triangle and find the new perimeter of each triangle. Which triangle has the greatest perimeter?

25. (6) **Multi-Step** A mechanic bought a part for $4.25 and marked the price up by 200% to sell to a customer. How much did the customer pay for the part? How much would the customer have paid if she had used a 5% off coupon?

Solve for x.

26. (7) $3x - 5 = 2 - 4x$

***27.** (7) $\frac{3}{4}x - \frac{1}{5}x = 2\frac{3}{4}$

28. (7) $5x - 2 = 7 + 2x$

29. (5) **Coordinate Geometry** The coordinates for a triangle are $A\ (-3, 7)$, $B\ (6, 8)$, and $C\ (2, -3)$.

a. Write the coordinates as a matrix such that row one is x-values, row two is y-values, and each column is a coordinate.

b. Add the transformation matrix $\begin{bmatrix} -2 & -2 & -2 \\ 3 & 3 & 3 \end{bmatrix}$.

c. What are the coordinates of the transformation?

***30.** (8) Juanita is conducting an experiment where the temperature of a liquid varies directly with time. Her observations are recorded in the table below.

Temperature	80 degrees	120 degrees	? degrees
Time	120 seconds	180 seconds	240 seconds

a. Using the values in the table, identify the constant of variation.

b. Find the missing temperature in the table.

c. Determine an accurate prediction for the temperature of the liquid after 360 seconds.

LESSON
10

Solving and Graphing Inequalities

Warm Up

1. (7) **Vocabulary** A mathematical statement comparing quantities that are equal is called an _____.

2. (7) For what value of x is the sentence $3x - 4 = -12$ true?

3. (7) True or False: -5 is a solution to the equation $-x - 2 = -3$.

4. (SB) Compare 0.25 and $\frac{1}{4}$. Use $<$, $>$, or $=$.

5. (SB) Compare 6.5 and 6.38. Use $<$, $>$, or $=$.

New Concepts

An **inequality** is a mathematical statement comparing quantities that are not equal. There are four inequality symbols:

$<$	less than	$>$	greater than
$\leq$	less than or equal to	$\geq$	greater than or equal to

Solving an inequality is similar to solving an equation. All the values that satisfy the original inequality make up the solution set. The properties used to solve inequalities are similar to those used when solving equations.

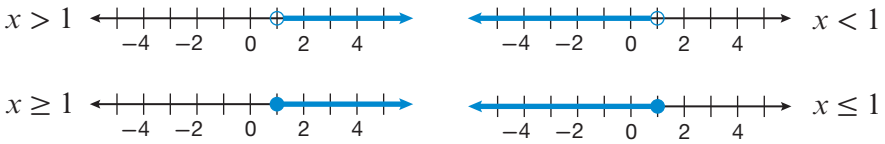
Math Reasoning

Analyze Why does the inequality symbol change direction when multiplying or dividing by a negative number?

Properties of Inequalities

The following properties hold true for real numbers a, b, and c.
Addition and Subtraction Properties for Inequalities
If $a < b$ then $a + c < b + c$ and $a - c < b - c$.
Multiplication and Division Properties for Inequalities
If $c < 0$ and $a < b$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.
If $c > 0$ and $a < b$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
Transitive Property for Inequalities
If $a < b$ and $b < c$, then $a < c$.
Note: All the properties above hold true for the inequality symbols $\leq$, $\geq$, and $>$.

The solution of an inequality can be graphed on a number line. Look at the graphs below.



Notice that the symbols $>$ and $<$ have open dots. Since 1 is not part of the solution set, the open dot is used to symbolize that all numbers greater than 1 but not including 1 are in the solution. The solid dot is used for the symbols $\leq$ and $\geq$ to show that 1 is included in the solution set.

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Example 1 Solving Inequalities Using Properties

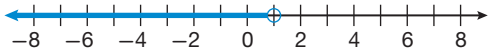
Solve each inequality and graph the solution set.

a. $t + 6 < 7$

SOLUTION Solve the inequality.

$t + 6 - 6 < 7 - 6$	Subtraction Property of Equality
$t < 1$	Simplify.

Graph the solution set.



Hint

< or >	open circle
≤ or ≥	closed circle

The arrow drawn shows that all values less than 1 are included in the solution set.

The solution is all values of t less than 1.

Check your solution. To check the solution, substitute 1 in the inequality and a number less than 1 to verify that it is included in the solution set.

Let $t = 1$	Let $t = -2$
$(1) + 6 \stackrel{?}{<} 7$	$(-2) + 6 \stackrel{?}{<} 7$
$7 \not< 7$	$4 < 7$ ✓
1 is not in the solution set.	−2 is in the solution set.

b. $-2(4 + b) \leq 4$

SOLUTION

Solve the inequality.

$-2(4 + b) \leq 4$	
$-8 - 2b \leq 4$	Distributive Property
$-8 + 8 - 2b \leq 4 + 8$	Addition Property of Equality
$-2b \leq 12$	Simplify.
$\frac{-2b}{-2} \geq \frac{12}{-2}$	Division Property
$b \geq -6$	Simplify.

Graph the solution set.

Check your solution.

Let $b = -6$	Let $b = 0$
$-2(4 + (-6)) \leq 4$	$-2(4 + (0)) \leq 4$
$4 \leq 4$ ✓	$-8 \leq 4$ ✓
−6 is in the solution set.	0 is in the solution set.

Example 2 Solving Inequalities that are Always True or Always False

Solve each inequality.

a. $-2(x + 3) < -2x + 1$

SOLUTION Solve the inequality.

$$-2(x + 3) < -2x + 1$$

$$-2x - 6 < -2x + 1 \qquad \text{Distributive Property}$$

$$-2x + 2x - 6 < -2x + 2x + 1 \qquad \text{Addition Property of Equality}$$

$$-6 < 1$$

Notice that the inequality will always be true, as -6 is always less than 1. Any value of x will make the inequality true. The solution is all real numbers.

The graph includes all values of x.

b. $4b < 3 + 4(b - 1)$

Solve the inequality.

$$4b < 3 + 4(b - 1)$$

$$4b < 3 + 4b - 4 \qquad \text{Distributive Property}$$

$$4b - 4b < 3 + 4b - 4b - 4 \qquad \text{Subtraction Property of Equality}$$

$$0 < 3 - 4 \qquad \text{Simplify.}$$

$$0 < -1$$

This solution is always false since 0 will never be less than -1. There are no values of x that will make this statement true.

Math Reasoning

Verify Check your answer by substituting different values for x. The resulting inequality will always be $-6 < 13$.

A compound inequality is composed of two inequalities combined with the word *and* or *or*. Examples of compound inequalities include:

a. $-5 < -2$ and $x + 4 > 1$

b. $y > y - 4$ or $2y < y + 5$

c. $2 \leq 3x + 2 < 14$

In example **a,** the solution set includes the *intersection* of the solutions of the two inequalities given.

In example **b,** the solution set will be the *union* of the solutions to each of the inequalities given.

In example **c,** the *and* is not written, but understood. This inequality can also be written as two inequalities: $2 \leq 3x + 2$ and $3x + 2 < 14$. The solution set is the *intersection* of the solutions to each of the inequalities that compose the compound inequality.

A compound inequality such as $-2 < x < 6$ can be read as "-2 is less than x, which is less than 6."

Example 3 Solving and Graphing a Compound Inequality

Solve each compound inequality and graph the solution set.

a. $4x - 10 \geq -2$ *and* $4x - 10 \leq 10$

SOLUTION There are two methods for solving a compound inequality involving *and*.

Method 1: Solve each inequality separately.

$$\begin{aligned} 4x - 10 &\geq -2 & \qquad 4x - 10 &\leq 10 \\ 4x &\geq 8 & \qquad 4x &\leq 20 \\ x &\geq 2 & \qquad x &\leq 5 \end{aligned}$$

Write the solution as a compound inequality: $2 \leq x \leq 5$.

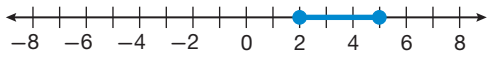
Method 2: Write the inequalities as a compound inequality and apply the properties for solving inequalities.

$$\begin{aligned} -2 \leq 4x - 10 &\leq 10 \\ -2 + 10 \leq 4x - 10 + 10 &\leq 10 + 10 \\ 8 \leq 4x &\leq 20 \\ 2 \leq x &\leq 5 \end{aligned}$$

Reading Math

The compound inequality $2 \leq x \leq 5$ may also be written as $\{x | \ 2 \leq x \leq 5\}$.

The graph of the solution set of this inequality includes all values of x such that $2 \leq x \leq 5$.



b. Solve and graph the compound inequality.

$$2x + 1 < 3 \text{ or } 3x - 6 > 6$$

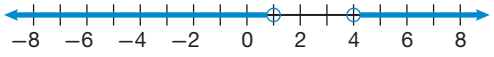
SOLUTION The compound inequality includes two inequalities. Solve each inequality separately.

$$\begin{aligned} 2x + 1 &< 3 & \qquad 3x - 6 &> 6 \\ 2x &< 2 & \qquad 3x &> 12 \\ x &< 1 & \qquad x &> 4 \end{aligned}$$

Reading Math

The compound inequality $x < 1$ *or* $x > 4$ may be written as $\{x | \ x < 1 \text{ or } x > 4\}$.

The graph of the solution is the set of all values of x such that $x < 1$ *or* $x > 4$.



Notice that the inequality includes all values greater than 4, not including four, and all values less than 1, not including 1. Any values between 1 and 4, inclusive, are not solutions to the compound inequality.

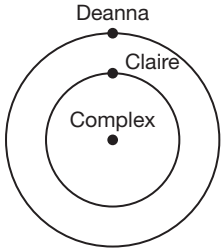
Example 4 Application: Distances

Claire and Deanna are soccer players. In the off-season they practice together at an indoor complex. Claire lives 12 miles from the complex and Deanna lives 18 miles from the complex. Write an inequality to represent the possible distances between their homes.

SOLUTION To solve this problem, there are three things to consider:

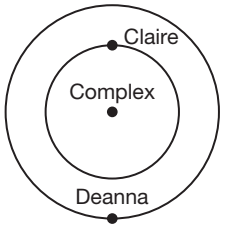
1. The girls may live in the same direction, or along the same path, from the complex.

Look at the picture below. The distance between Deanna's and Claire's homes is exactly 6 miles if they live in the same direction, or along the same path, from the complex.



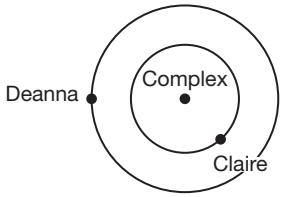
2. The girls may live in opposite directions from the complex.

The illustration shows the distance if Claire and Deanna live in opposite directions from the complex. The distance between Claire's home and the complex is 12 miles and the distance from Deanna's home to the complex is 18 miles, so $12 + 18 = 30$ miles.



3. The girls may live in directions other than those described above.

The illustration shows one possible scenario where Deanna and Claire live in directions that are not directly opposite or along the same path as the complex.



The largest distance will happen when Deanna and Claire live in directly opposite directions. The shortest distance will happen when they live along the same path as the complex. Based on this information, the distance between the girl's houses may be written as $6 \leq x \leq 30$.

Hint

Because three points describe a triangle, we can use the **Triangle Inequality Theorem** to solve.

Lesson Practice

Solve each inequality and graph the solution set when possible.

a. *(Ex 1)* $3y + 2 > 8$.

b. *(Ex 1)* $2(5 + a) \leq 5 - a$.

c. *(Ex 2)* $4(x - 1) > 4x + 3$.

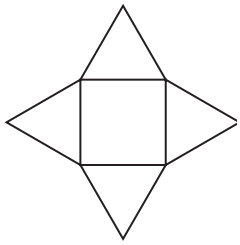
d. *(Ex 2)* $2w < -1 + 2(w + 3)$.

e. *(Ex 3)* $8 \leq 5x + 3 \leq 23$.

f. *(Ex 4)* David kicked his soccer ball 6 feet from a tree. Alannah kicked her soccer ball 10 feet from the same tree. Write an inequality to determine the distance David's ball is from Alannah's ball.

Practice Distributed and Integrated

1. (SB) What solid figure is described by the net shown?



2. (4) Determine whether $y = -2x^2$ is a function. If so, state the domain and range.

Simplify.

3. (3) $\dfrac{(2x^{-2}y^0)^{-2}yx^{-2}}{xxxy^2(y^{-2})^2}$

4. (3) $\dfrac{(m^2n^{-5})^{-2}m(n^0)^2}{(m^2n^{-2})^{-3}m^2}$

Solve each equation.

5. (7) $0.003x + 0.02x - 0.03 = 0.177$

6. (7) $4\frac{1}{3}x - \frac{1}{2} = 3\frac{2}{5}$

Solve and graph each inequality.

***7.** (10) $3(x - 4) > 6$

***8.** (10) $2x + 3 \geq 4x - 5$

9. (8) **Multi-Step** The amount of water added to a dry mortar mix varies directly with the amount of dry mortar. 100 lb of dry mortar requires 11.25 pt of water.

a. Find the constant of variation for the dry mortar mix.

b. Find how many pints of water are needed for 240 lb of dry mortar mix.

10. (2) Simplify $4[3(x + 9) + 2]$.

Graphing Calculator **Find each product. Check using the graphing calculator.**

***11.** (9) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$

***12.** (9) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 4 & -1 \\ -2 & -6 \end{bmatrix}$

Find each percent change. Tell whether it is a percent increase or decrease.

13. (6) 265 to 318

14. (6) 17 to 14.45

15. (3) **Estimate** One cubic foot of pennies contains approximately 4.9×10^4 pennies. Think about covering the entire earth with two layers of pennies. The total number of pennies needed could be stacked in a cube that measures 2.73×10^4 ft on each side. Approximately how many pennies are in this cube?

***16.** (10) **Error Analysis** Look at the explanation given for graphing the solution of two inequalities. To draw the graph of the solution $x \leq 2$ or $x > 5$ you draw a number line. Draw an open circle around the point 2 and a line through all the values less than 2. Next draw an open circle around the point 5 and a line through all values greater than 5. Is this explanation correct?

17. (4) **Currency** On the first day of 2007, the value of one US dollar was equivalent to 0.76 euro. On the same day, one US dollar was equivalent to 1.22 Swiss francs.

a. Write a function to represent the value of francs in euros.

b. What is the value of the function for an input of 5 rounded to the nearest euro, and what does it represent?

c. How can you tell from the given data that your answer makes sense?

18. (6) **a.** Which is the greater percent change: from 50 to 60 or from 60 to 50?

b. Analyze How can the answer to part a be found without actually finding the percents?

19. (7) **Error Analysis** Find and explain the error in solving the equation below. Correct the error and solve:

$$\begin{aligned} 7x - 14 &= -9 + 12x \\ 7x &= -23 + 12x \\ -5x &= -23 \\ x &= \frac{23}{5} \end{aligned}$$

20. (1) **Geometry** The length of a rectangle equals $x + 4$ and the width equals 5 inches. The perimeter of the rectangle can be calculated as shown below. Which properties of real numbers are demonstrated?

$$P = 2(x + 4) + 2(5) = 2(x + 4 + 5) = 2(x + 9) = 2x + 18.$$

21. (7) **Statistics** The mean for the list of numbers below is 11.8. Use the formula for mean to create a linear equation to solve for the missing data point.

12, 15, 8, ?, 20

22. (8) **Predict** The weight of an object on Neptune varies directly with the weight of that object on Earth. An object that weighs 5 g on Earth, weighs 6 g on Neptune. Without doing any calculations, predict the weight of an object on Neptune that weighs 10 g on Earth. Explain your prediction.

23. (3) In 2002, the average height of a 15-year old boy was 1.737×10^3 mm. About how many 15-year old boys lying head-to-toe would it take to equal the circumference of the earth at the equator, if its circumference measures about 4.0076×10^4 km? Write the answer in scientific notation rounded to the nearest ten thousandth.

***24.** (8) **Verify** The circumference of a circle varies directly with the radius. The table below shows the circumference and radius of three different circles. Verify that the constant of variation is 6.283.

Circumference	Radius
12.566 cm	2 cm
94.245 cm	15 cm
232.471 cm	37 cm

*25. (9) **Football** The first table below show the scoring plays made in three games by the Philadelphia Eagles. The second table shows the point value of each play. Use matrix multiplication to find out in which game the team scored the most points. Use your graphing calculator to verify your answer.

Plays Made

Game	Touchdown	Field Goal	Extra Point
# 4	4	1	4
# 9	3	2	3
# 12	5	0	5

Point Value

Touchdown	6
Field Goal	3
Extra Point	1

*26. (10) **Multiple Choice** A school is holding an International dinner as a fundraiser to raise money for a local charity. The budget for the food is $300. It will cost $2.25 to prepare the food and beverages for one meal. Which inequality represents the number of meals that can be prepared?

A $x \leq 134$ **B** $x \geq 134$ **C** $x \leq 133$ **D** $x \geq 133$

27. (5) **Bulk Foods** The table shows the cost of three sizes of bottles of water.

Cost of Water Bottles

16 oz	24 oz	32 oz
$0.75	$1.20	$1.60

a. Create two different matrices with different dimensions for the table.

b. Choose one of the matrices in part **a** and multiply it by a scalar to show the cost of purchasing 200 bottles of each size.

c. When a total of 500 bottles or more are purchased, a discount of 5% off each bottle is given. Use scalar multiplication to show the cost of each bottle when 500 or more of each size are purchased.

28. (6) **Multiple Choice** What is the percent change from 35 to 31.5?

A 10% **B** 90% **C** 100% **D** 900%

*29. (9) $A = \begin{bmatrix} 6 & -3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -2 & 6 \end{bmatrix}$.

a. Find AB. **b.** Find BA. **c.** Is $AB = BA$?

*30. (9) **Write** If the number of columns of matrix A is not the same as the number of rows of matrix B, explain why the product AB is not defined.

INVESTIGATION

1

Logic and Truth Tables

Biology Field biologists are constantly discovering new organisms. One of their tasks is to classify any new organism based on existing taxonomies. For example, biologists know that all organisms with feathers are birds. So if they encounter a new organism with feathers, it can be classified as a bird.

Let p be the statement "an organism has feathers" and q be the statement "it is a bird." The logic statement "If an organism has feathers, than it is a bird" can be written in symbolic form as $p \rightarrow q$, "p implies q." This is an example of logical implication.

You can use a truth table to determine the conditions under which $p \rightarrow q$ is true. The statement $p \rightarrow q$ is false only when p is true and q is false.

1. Generalize Complete the truth table for $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	
T	F	
F	T	
F	F	

Math Reasoning

Formulate Suppose a logic statement includes n different statements p_1 to p_n. How many total entries would there be in the truth table?

The converse of $p \rightarrow q$ is $q \rightarrow p$. The statement $q \rightarrow p$ is false only when q is true and p is false.

2. The statement "If an organism is an eagle, then it eats fish." is true. Write the converse. Is it true? Explain.

3. Justify If $p \rightarrow q$ is true, is $q \rightarrow p$ also true? Construct a truth table to justify your answer.

The negation of p is written $\neg p$, read "not p." If p is true, then $\neg p$ is false. The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

4. Justify If $p \rightarrow q$ is true, is the contrapositive also true? Construct a truth table to justify your answer.

Some logic statements are *always* true. Such a statement is known as a tautology. Take the statement,

"An organism is either a bird (1) OR not a bird (2)"

Let p be statement 1 and $\neg p$ be statement 2. The logic statement above becomes $p \vee \neg p$. The symbol $\vee$ is "or." An "or" statement, also called a **disjunction,** is true as long as *at least* one of the statements is true.

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5. Create a truth table for the logic statement $p \vee \neg p$ to show that it is a tautology.

On the other hand, a logical contradiction is a statement that is always false. Take the statement,

"An organism is both a bird (1) AND not a bird (2)"

Let p be statement 1 and $\neg p$ be statement 2. The logic statement above becomes $p \wedge \neg p$. The symbol $\wedge$ is "and." An "and" statement, also called a **conjunction,** is true when *both* statements are true.

6. Create a truth table for the logic statement $p \wedge \neg p$ to show that it is a contradiction.

Although the converse of any statement isn't always true, there are cases when it is. For example, since *all* birds have feathers and all animals that have feathers are birds, the following statement and its converse are true.

$p \rightarrow q$	$q \rightarrow p$
"If an organism has feathers, it is a bird."	"If an organism is a bird, then it has feathers."

A logic statement whose converse is true is bi-directional and is written this way, $p \leftrightarrow q$, "p is true if and only if q is true."

Math Reasoning

Generalize Describe the conjunction of a tautology and a contradiction. Describe the disjunction.

7. Construct the truth table for $p \leftrightarrow q$.

8. **Multi-Step** All birds have feathers, but only those that fly have flight feathers. Consider the following statement, "If an organism has flight feathers, then it can fly."
   - **a.** Write a logic statement.
   - **b.** Is the converse of the statement true? Explain.
   - **c.** Is the contrapositive true? Explain.
   - **d.** Is it correct to conclude $q \leftrightarrow p$? Explain.

9. **Multi-Step** All birds have contour feathers, but only birds that fly have flight feathers. So, consider the following statement, "If a bird has contour feathers AND has flight feathers, then it can fly."
   - **a.** Write a logic statement.
   - **b.** Create a truth table.
   - **c.** Is the statement a tautology, a contradiction, or neither?

10. Consider the following statement, "If an organism has contour feathers OR flight feathers, then it is a bird."
    - **a.** Write a logic statement.
    - **b.** Create a truth table.
    - **c.** Is the statement a tautology, a contradiction, or neither?

Some logic statements can appear different but have the same truth table. In such a case the statements are said to be logically equivalent.

11. Verify Show that the logic statements $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Investigation Practice

a. Create the truth table for $\neg p \rightarrow \neg q$.

1. If $\neg p \rightarrow \neg q$ is true, what can you conclude about $q \rightarrow p$?
2. Why?

b. Identify if each statement is a tautology, a contradiction, or neither.

1. $(\neg p \wedge \text{p}) \rightarrow \neg q$
2. $(\neg p \wedge p) \rightarrow p$
3. $(\neg p \wedge p) \rightarrow (\neg q \wedge q)$

c. Determine if each set of statements is logically equivalent.

1. $(\neg p \wedge p) \rightarrow (\neg q \wedge q)$ and $(\neg p \vee p) \rightarrow (\neg q \vee q)$
2. $(p \wedge q) \rightarrow r$ and $r \rightarrow (p \wedge q)$

LESSON 11

Understanding Polynomials

Warm Up

1. (2) **Vocabulary** A _____ is a letter that is used to represent a number.
2. (1) Distribute. $-5(6x - 2)$
3. (2) Collect like terms. $6 + 3x - 5x + 4y + 17 - 2y$
4. (6) Let $g(x) = 2x + 7$ and $f(x) = x - 5$. Find $f(g(x))$.
5. (4) True/False The function $f(x) = 3x^2 + 4$ is linear.

New Concepts

A **monomial** is a number, a variable, or the product of a number and one or more variables with whole-number exponents. A **polynomial** is a monomial or sum of monomials. The terms of a polynomial are its monomials. The **degree of a monomial** is the sum of the exponents of its variable factors. The **degree of a polynomial** is the degree of its monomial with the greatest degree.

Math Language

The suffix *nomial* means a name or a term.

The monomial $5ab^3$ has degree 4.

$$5a^1b^3 \leftarrow 1 + 3 = 4$$

The polynomial $-xy^3 - 3x^2y^2 + x^2y^3$ has degree 5.

$$-x^1y^3 - 3x^2y^2 + x^2y^3 \leftarrow 2 + 3 = 5$$

A one-variable polynomial is in **standard form** when all like terms have been combined and its terms are in descending order by degree.

The polynomial $-x^4 + x^3 - 3x + 2$ is in standard form.

$$-x^4 + x^3 - 3x^1 + 2x^0 \leftarrow \text{4, 3, 1, 0 are in descending order.}$$

A **polynomial function** is a function of the form

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0,$$

where n is a nonnegative integer and the coefficients $a_n, \ldots, a_0$ are real numbers. The **leading coefficient** is a_n, the **constant term** is a_0, and the degree is n.

Example 1 Writing a Polynomial in Standard Form

Write the polynomial $x - 5 + 4x^2 + 2x^5 - x^2$ in standard form. Then identify the leading coefficient and the constant term.

SOLUTION

$$x - 5 + 4x^2 + 2x^5 - x^2 = 2x^5 + 4x^2 - x^2 + x - 5$$
$$= 2x^5 + 3x^2 + x - 5$$

Write terms in descending order by degree. Then combine like terms.

The leading coefficient is 2 and the constant term is -5.

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Polynomials can be classified by degree and by number of terms. Some examples are given below.

Hint

The prefix *quart-* indicates 4, as in *quarter* and *quartet*. The prefix *quint-* indicates 5, as in *quintuplets*.

Polynomial	Degree	Classification by Degree	Number of Terms	Classification by Number of Terms
-4	0	constant	1	monomial
$x + 2$	1	linear	2	binomial
$x^2 + x - 6$	2	quadratic	3	trinomial
x^3	3	cubic	1	monomial
$-3x^4 + 2x^3 - x^2$	4	quartic	3	trinomial
$x^5 - 4x^2$	5	quintic	2	binomial

Example 2 Classifying Polynomials

Classify each polynomial by degree and by number of terms.

a. $x^2 - 9$

SOLUTION The degree is 2, so it is quadratic. It has 2 terms, so it is a binomial.

b. $x^3 + x - 2x^3 - \frac{1}{2}x^5$

SOLUTION Combine like terms and write in descending order by degree to write the polynomial in standard form.

$$x^3 + x - 2x^3 - \frac{1}{2}x^5 = -\frac{1}{2}x^5 - x^3 + x$$

The degree is 5, so it is quintic. The standard form has 3 terms, so it is a trinomial.

Example 3 Adding Polynomials

Add: $(2x^3 - 3x^2 + x - 5) + (5 - x^2)$.

SOLUTION To add polynomials, combine like terms.

$$\begin{aligned}(2x^3 - 3x^2 + x - 5) + (5 - x^2) &= 2x^3 - 3x^2 + x - 5 + 5 - x^2 \\ &= 2x^3 - 3x^2 - x^2 + x - 5 + 5 \\ &= 2x^3 - 4x^2 + x\end{aligned}$$

Example 4 Subtracting Polynomials

Subtract: $(5x^3 - 3x^2) - (2x^3 - x^2 + 4)$.

SOLUTION To subtract a polynomial, add the opposite of each term.

$$\begin{aligned}(5x^3 - 3x^2) - (2x^3 - x^2 + 4) &= 5x^3 - 3x^2 - 2x^3 + x^2 - 4 \\ &= 5x^3 - 2x^3 - 3x^2 + x^2 - 4 \\ &= 3x^3 - 2x^2 - 4\end{aligned}$$

Example 5 Application: Revenue from Sales of Electricity

Revenue (in billions of dollars) from retail sales of electricity in the U.S. for the years 2000 through 2005, is modeled by these polynomial functions. The variable x represents the number of years since 2000.

$$\text{Residential } R = 0.75x^2 + 1.73x + 99.32$$

$$\text{Commercial } C = 0.44x^2 + 3.91x + 79.26$$

$$\text{Industrial } I = 0.63x^2 - 1.49x + 49.89$$

Write a model for total revenue from sales in all three sectors combined.

SOLUTION

1. **Understand** Sales in each sector is modeled by a polynomial function.

2. **Plan** To find a model for all three sectors combined, add the polynomials.

3. **Solve**

$$\begin{array}{r} 0.75x^2 + 1.73x + 99.32 \\ 0.44x^2 + 3.91x + 79.26 \\ +\ 0.63x^2 - 1.49x + 49.89 \\ \hline 1.82x^2 + 4.15x + 228.47 \end{array}$$

$f(x) = 1.82x^2 + 4.15x + 228.47$ is a model for all three sectors combined.

4. **Check** The answer makes sense because the polynomial $1.82x^2 + 4.15x + 228.47$ is the sum of the three given polynomials.

Math Reasoning

Estimate Use the model in Step **3** to estimate the revenue for the year 2002.

Lesson Practice

a. *(Ex 1)* Write the polynomial $3x - 1 + 4x^4 + x$ in standard form. Then identify the leading coefficient and the constant term.

b. *(Ex 2)* Classify $-x^3$ by degree and by number of terms.

c. *(Ex 2)* Classify $x^2 + 40x + 10 - 9x^2 - 1$ by degree and by number of terms.

d. *(Ex 3)* Add: $(2x^2 - 3x^5) + (2x^5 - x^2 + 1)$.

e. *(Ex 4)* Subtract: $(1 - x - x^3) - (x^3 + 2x^2 - x)$.

f. *(Ex 5)* The average retail price (in cents per kilowatt hour) of electricity for the years 2000 through 2005, is modeled by these polynomial functions. The variable x represents the number of years since 2000.

$$\text{Residential } R = 0.041x^2 + 0.009x + 8.334 \text{ (highest price)}$$

$$\text{Commercial } C = 0.013x^2 + 0.137x + 7.555$$

$$\text{Industrial } I = 0.028x^2 + 0.037x + 4.756 \text{ (lowest price)}$$

What polynomial function is a model that describes how much higher the average retail price was in the residential sector than in the industrial sector?

Practice Distributed and Integrated

*1. (7) **Geography** The table shows the relationship of the annual estimated population for three cities in the year 2005. The population in Oklahoma City, Oklahoma, is represented by the variable x.

2005 City Population

City	Population
Oklahoma City	x
Chicago	$5.354x$
Arlington	$0.683x$
Total of 3 cities	3,736,275

a. Formulate Write a linear equation to approximate the population in Oklahoma City in 2005.

b. Estimate Find the approximate population in Oklahoma City in 2005. Round to the nearest hundred.

2. (7) **Multi-Step** Twice a number is decreased by 9, and then multiplied by 4. The result is 8 less than 10 times the number. What is the number?

Simplify the expressions.

3. (3) $x^6(2x^2)$

4. (3) $2(2x^2)^6$

5. (3) $\frac{x^2}{2x^6}$

6. (7) **Geometry** Let x represent the first angle in a triangle. The second angle is twice the measure of the first. The third angle is triple the measure of the first. Write and solve a linear equation to find the measure of all three angles.

Solve the inequalities.

*7. (10) $-7 \le -2x + 3 < 9$

*8. (10) $\frac{3}{2}x + 2 < \frac{1}{2}x + 5$

Evaluate.

9. (2) $(a - b) - a(-b)$ if $a = -5$ and $b = 3$

10. (2) $(a - x)(x - a)$ if $a = -2$ and $x = 4$

11. (2) $a^2 - y^3(a - y^2)y$ if $a = -2$ and $y = -3$

Write each fraction or decimal as a percent.

12. (6) 0.05

13. (6) $\frac{3}{10}$

14. (6) $\frac{4}{5}$

15. (6) 0.37

*16. (8) **Error Analysis** Two scientists solved the following problem:

Sulfur is mixed with other chemicals to make sulfuric acid. If it takes 16 tons of sulfur to make 49 tons of sulfuric acid, how many tons of other constituents are needed to make 294 tons of acid? (Hint: Sulfuric acid is the total.) Which scientist solved the problem using the correct steps? Explain your reasoning.

Scientist A

$$\frac{16}{49} = \frac{x}{294}$$
$$(16)(294) = 49x$$
$$x = 96 \text{ tons}$$

Scientist B

$$\frac{33}{49} = \frac{x}{294}$$
$$(33)(294) = 49x$$
$$x = 198 \text{ tons}$$

17. (8) **Chemistry** A formula requires that 20 kilograms (kg) of carbon be used to produce 160 kg of a compound. How many kilograms of other components are needed to make 640 kg of the compound?

Justify Determine if these statements are a tautology, a contradiction, or neither. Justify your answer by constructing truth tables.

18. (Inv. 1) $\neg p \rightarrow q$

19. (Inv. 1) $\neg p \vee q$

20. (10) **Consumer** Nerissa's car can travel between 380 and 410 miles on a full tank of gas. She filled her gas tank and drove 45 miles. How many miles m can she drive without running out of gas?

21. (3) **Astronomy** The speed of light is about 3×10^8 meters/second. If Saturn's distance from Earth is about 1.321×10^{12} meters, about how many seconds would it take light to reach Saturn from Earth? Round the answer to the nearest second.

22. (3) **Earth Science** The garter snake is a small snake that is typically no more than 6×10^{-4} km in length. About how many garter snakes would it take to equal the circumference of the earth at the equator, if its circumference measures about 4.0076×10^4 km? Write the answer in scientific notation rounded to the nearest ten thousandth.

Analyze Determine if the expressions are polynomials. If the expression is a polynomial, rewrite it in standard form. If the expression is not a polynomial, explain why.

***23.** (11) $5x^2 + 3x^4 + 2$

***24.** (11) $\sqrt{3x + 5x^3}$

Classify the polynomial expressions by their degree and by the number of terms.

***25.** (11) $7x^3 + 3x^2 + 5x$

***26.** (11) $x^1 + 5x^0$

Simplify the expression.

***27.** (11) $(7x^3 + 3x^2 + 5x) - (3x^3 + x)$

28. (4) **Multiple Choice** Choose a letter that represents a set of x- and y-values that makes the equation $y = \frac{1}{2}x + 4$ true.

A $x = 2,\ y = 4$ **B** $x = 6, y = 6$

C $x = 4, y = 6$ **D** $x = 4, y = 8$

Graphing Calculator Multiply the matrices. Check using a graphing calculator.

***29.** (9) $\begin{bmatrix} 2 & -3 & 4 \\ 5 & 8 & -1 \\ 7 & 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} -4 & 7 \\ 0 & 1 \\ 3 & 6 \end{bmatrix}$

***30.** (9) $\begin{bmatrix} 4 & -1 \\ 6 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ -2 & 4 & -3 \end{bmatrix}$

LESSON 12

Solving Inverse Variation Problems

Warm Up

1. (8) **Vocabulary** Of the terms x, z, and 3, only the number 3 is a _____.

2. (7) Is $(3, -3)$ a solution to the equation $3x - 7y = 15$?

3. (5) Graph $5x + 4y = 20$.

New Concepts

If the product of two variables is a constant, then the equation is an **inverse variation.**

$$xy = k \text{ or } y = \frac{y}{x}$$

Both x and y are variables and k is a nonzero constant. The term k is also referred to as the constant of variation. You have previously learned about direct variations. Here are some differences between direct and inverse variations.

Math Reasoning

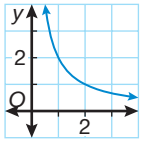
Generalize Compare the relationship of the variables in a direct and inverse variation.

Direct Variation	Inverse Variation
As x increases, y increases.	As x increases, y decreases.
As x decreases, y decreases.	As x decreases, y increases.
The ratio $\frac{y}{x}$ is a constant.	The product yx is a constant.

Example 1 Modeling Inverse Variation

The values in the table and the graph represent the equation $y = \frac{2}{x}$.

x	1	2	4
y	2	1	0.5



a. Is this equation an inverse variation? If so, find the constant of variation.

SOLUTION Multiply the x and y values in the data table.

x	1	2	4
y	2	1	0.5
xy	$(1)(2) = 2$	$(2)(1) = 2$	$(4)(0.5) = 2$

The equation is an inverse variation with a constant of variation equal to 2.

b. **Predict** What will the y value be when the x value is 8?

SOLUTION The product of x and y will be 2.

$$8y = 2$$

$$y = \frac{1}{4}$$

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Given a set of coordinates, it is possible to determine if they represent an inverse variation. By finding the constant of variation, it is possible to find the equation for the inverse variation.

Example 2 Testing for an Inverse Variation

a. Determine if the data set shown represents an inverse variation. If so, find the constant of variation and the equation.

x	1	2	3	4
y	3	1.5	1	0.75

SOLUTION Multiply the x and y coordinates to see if the products equal the same constant.

x	1	2	3	4
y	3	1.5	1	0.75
xy	$(1)(3) = 3$	$(2)(1.5) = 3$	$(3)(1) = 3$	$(4)(0.75) = 3$

Yes, each product is equal to 3, which is the constant of variation for the inverse variation $y = \frac{3}{x}$.

b. Determine if this data set represents a direct or an inverse variation. If so, find the constant of variation and the equation.

x	1	2	3	4
y	25	10	3	1

SOLUTION

Step 1: Before performing any calculations with the data, look at the basic relationship between x and y.

As x increases, does the value of y decrease? Yes. This could be an inverse variation and is definitely not a direct variation.

Step 2: Multiply the x and y coordinates to see if the products equal the same constant.

x	1	2	3	4
y	25	10	3	1
xy	25	20	9	4

The product xy is not the same from one set of coordinates to another, so the data set is not an inverse variation.

Caution

Check carefully to make sure that each product of the variables equals the same constant.

Reading Math

The expression $\frac{z}{xy} = k$ is read "z varies jointly as x and y." This means that z varies directly as the product of x and y.

A **joint variation** involves three variables, one of which depends on changes in the other two variables. A joint variation can be written as shown below.

$$\frac{z}{xy} = k$$

The variables are x, y, and z, and k is the nonzero constant of variation.

Example 3 Joint Variation

a. The force (F) on an object of mass m that causes it to move with acceleration a can be found using the equation $F = ma$. Is this an example of a joint variation? Explain.

SOLUTION In the equation shown, there are three variables, F, m, and a. The equation can be rewritten as $\frac{F}{ma} = 1$.

Since 1 is a constant, the equation $F = ma$ is a joint variation.

b. The kinetic energy (KE) of a moving object is based on its mass (m) and the square of its velocity (v^2). Use the data set below to see if this is a joint variation.

KE	2	9	24	50
m	1	2	3	4
v^2	4	9	16	25

SOLUTION Calculate the ratio $\frac{KE}{mv^2}$ for each set of data points to see if they equal a constant value.

KE	2	9	24	50
m	1	2	3	4
v^2	4	9	16	25
$\frac{KE}{mv^2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

In each case the ratio is the same value: $\frac{1}{2}$. So this is a joint variation.

c. In an electric circuit, the electric power (P) is the product of the square of the current (I) and the electrical resistance (R). Write an equation that represents this relationship. Is it a joint variation?

SOLUTION The equation can be written as $P = I^2R$.

It is a joint variation with a constant of variation equal to 1.

Many real-world phenomena can be explained through direct, inverse, or joint variation. To determine which, if any, variation applies, look for the constant of variation.

Example 4 Determining the Type of Variation

a. Two objects of different masses experience a gravitational force of attraction. Isaac Newton found that this force is proportional to the product of the two masses. What kind of variation is this? Write the equation.

SOLUTION Define the variables m_1 and m_2 to represent the masses, and let F_g be the gravitational force of attraction. If F_g is proportional to the product of m_1 and m_2, then the equation can be written as $F_g = km_1m_2$.

Since there are two variables for mass, this represents a joint variation.

b. Newton also found that the gravitational force of attraction, F_g, is inversely proportional to the square of the distance between the two masses. What kind of variation is this? Write the equation.

SOLUTION

Step 1: Define the variable d to represent the distance between the two masses.

Step 2: If F_g is inversely proportional to d^2, then the equation can be written as $F_g = \frac{k'}{d^2}$.

(The constant of variation k' is used to distinguish it from the constant of variation, k, in part **a.**) This equation represents an inverse variation with constant of variation k'.

c. Newton's Universal Law of Gravitation shows that the gravitational force of attraction is both a joint variation and an inverse variation. Use the results from parts **a** and **b** to show how this equation is derived.

SOLUTION

Step 1: Define the constant of variation g.

Step 2: F_g is inversely proportional to d^2 and directly proportional to m_1m_2. So the combined equation can be written as $F_g = g\frac{m_1m_2}{d^2}$.

This is Newton's Universal Law of Gravitation and is made up of both an inverse and a joint variation. The accepted numerical value for g is approximately 6.67×10^{-11}.

Math Reasoning

Connect Use your understanding of scientific notation to judge how strong the force of gravitation is.

Lesson Practice

(Ex 1) **a.** Look at the data set below. How can you tell if it represents an inverse variation?

x	1	2	3	4
y	4	2	$1\frac{1}{3}$	1

(Ex 1) **b.** Is $k = 3.5$ the constant of variation for this data set? Explain.

x	3	4	5	6
y	0.5	0.375	0.3	0.25

(Ex 2) **c.** Determine if this data set is an inverse variation. If so, find the constant of variation and the equation.

x	$-\frac{1}{2}$	3	5	9
y	$-\frac{1}{2}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{36}$

d. *(Ex 2)* Determine the kind of variation represented by this data. Find the constant of variation and the equation for the data.

x	–2	8	10	20
y	–9	36	45	90

e. *(Ex 2)* Determine whether this is a direct or an inverse variation. If so, find the constant of variation and the equation.

x	1	2	3	4
y	2	6	8	10

f. *(Ex 3)* Is the equation $V = CR$, where V is voltage, C is current, and R is resistance, a joint variation? How do you know?

g. *(Ex 3)* **Economics** In economics, the graph of the *supply curve* has this behavior: As the supply (S) item decreases, the price (P) for the item increases. Likewise, when the supply of the item increases, the price decreases. The product of P and S is a constant. Name the type of variation that describes this economic phenomenon and write a generic equation.

h. *(Ex 3)* The potential energy (PE) of an object of mass m depends on the height (h) of the mass above Earth's surface. Use the data set at right to see if this is a joint variation. If so, find the constant of variation and the equation.

PE	m	h
196	2	10
588	3	20
1176	4	30
1960	5	40

i. *(Ex 4)* The intensity (I) of light is inversely proportional to the distance (d) from the light squared. Write an equation for this phenomenon.

j. *(Ex 4)* Two electrically charged objects have a charge of q_1 and q_2. The electric force (F_e) between the two charges is directly proportional to the product of the two charges. Write an equation for this phenomenon.

k. *(Ex 4)* Coulomb's Law states that the electric force (F_e) between two objects of charge q_1 and q_2 is directly proportional to the product of the charges and inversely proportional to the square of the distance (d) between the charges. Write an equation for this phenomenon.

Practice Distributed and Integrated

Analyze **Determine if each expression is a polynomial. If the expression is a polynomial, write it in standard form. If the expression is not a polynomial, explain why not.**

***1.** (11) $\dfrac{3x + 7x^2}{10x^5}$

***2.** (11) $28x^2 - x^9 + 12x^7$

***3.** (8) **Data Analysis** Two scientists solved the following problem:

It takes 900 kg of acetylene to make 2400 kg of a compound. How many kilograms of other components will be required to make 3600 kg of the compound? Which scientist solved the problem using the correct steps? Explain your reasoning.

Scientist A	Scientist B
$\frac{1500}{2400} = \frac{x}{3600}$	$\frac{900}{2400} = \frac{x}{3600}$
$\frac{5}{8} = \frac{x}{3600}$	$\frac{3}{8} = \frac{x}{3600}$
$8x = 5(3600)$	$8x = 3(3600)$
$x = 2250$ kg	$x = 1350$ kg

***4.** (7) **Geometry** An isosceles triangle has two angles that each measure $x°$. The third angle is 1.75 times the measure of either angle. Write and solve a linear equation to find the measure of each angle.

Determine the constant *k*, then solve the problem.

***5.** (12) The number of revolutions per minute (RPM) varies inversely as the number of teeth in the gear. If 40 teeth result in 120 RPM, what would the RPM be if the gear had 30 teeth?

***6.** (12) The number of students in every class varies directly with the number of basketballs. For one class there are 8 students and 2 basketballs. If in another class there are 7 basketballs, how many students are in this class?

***7.** (12) The number of runners varies jointly as bicyclists and walkers. If 100 runners went with 4 bicyclists and 5 walkers, how many bicyclists would go with 20 runners and 2 walkers?

Simplify the expressions.

8. (3) $5x^7(3^3xy^2)$

9. (3) $\left(\dfrac{5xy^3}{2x^3y^2z^{-4}}\right)^2$

10. (3) $\dfrac{2^{-2}}{x^{-6}}$

11. (3) **Earth Science** The typical length of a green frog is about 5 to 10 cm. If a green frog is about 7.5×10^{-5} km, about how many green frogs would it take to equal the diameter of the Earth? The diameter is about 1.2753×10^4 km. Round the answer to the nearest ten thousandth.

Evaluate.

12. (1) $-8 - 3^2 - (-2)^2 - 3(-2) + 2$

13. (1) $-\{-[-5(-3 + 2)\ 7]\}$

14. (2) $-p^2 - p(a - p^2)$ if $a = 4$ and $p = -3$

Solve the inequalities.

15. (10) $-9 \le 2x - 5 < 9$

16. (10) $-\frac{1}{2}x + 7 \ge 15$

*17. (7) **Geography** The table shows the relationship of the annual estimated population for three cities in the year 2005. The population in Phoenix, Arizona, is represented by the variable x.

2005 City Population

City	Population
Albuquerque	$0.151x$
Madison	$0.336x$
Phoenix	x
Total of 3 cities	2,185,816

a. Formulate Write a linear equation to approximate the population in Phoenix in 2005.

b. Estimate Find the approximate population of Phoenix in 2005. Round to the nearest thousand.

18. (7) **Multi-Step** The number of ducks on the pond tripled when the new flock landed. Then, 11 more ducks came. The resulting number of ducks was 13 less than 4 times the original number. How many ducks were there to begin with?

Write each fraction or decimal as a percent.

19. (6) 1.3 **20.** (6) $\frac{5}{8}$ **21.** (6) $\frac{13}{20}$ **22.** (6) 0.4

23. (8) **Chemistry** A formula required that 500 kilograms (kg) of sulfur be used to produce 3000 kg of a compound. How many kilograms of other materials are needed to make 6000 kg of the new compound?

24. (4) **Multiple Choice** Which set of x- and y-values would make the equation $y = \frac{3}{4}x - 1$ true?

A $x = 8, y = -5$ **B** $x = -8, y = -7$

C $x = 4, y = -2$ **D** $x = -8, y = 7$

Graphing Calculator **Multiply the matrices. Check using a graphing calculator.**

*25. (9) $\begin{bmatrix} -2 & 3 & -4 \\ 0 & -3 & -1 \\ 5 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 7 \\ -6 & 1 \\ 3 & 6 \end{bmatrix}$

*26. (9) $\begin{bmatrix} -2 & -1 \\ 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

27. (3) **Astronomy** The speed of light is about 3×10^8 meters/second. If Saturn is about 1.35×10^{12} meters from the sun, how many seconds would it take light to reach Saturn from the Sun? Round the answer to the nearest hundredth of a second.

Justify **Determine if these statements are a tautology, a contradiction, or neither. Justify your answer by constructing truth tables.**

28. (Inv. 1) $(p \wedge q) \rightarrow \neg p$ **29.** (Inv. 1) $(p \vee q) \vee q$

30. (5) **Justify** Does the Commutative Property hold true for matrix subtraction? Justify your answer.

LAB
3

Calculating Points on a Graph

Graphing Calculator Lab (Use with Lessons 13, 15, and 30)

Calculating *y*-Values and Roots

A graphing calculator can be used to calculate y-values and zeros of equations. Use the equation $y = 2x - 4$ and find the y-values for $x = 3.7$ and $x = 0$. Then find the zero.

Graphing Calculator Tip

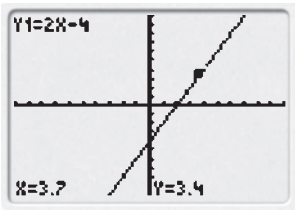
For help with graphing equations, see the graphing calculator keystrokes on page 19.

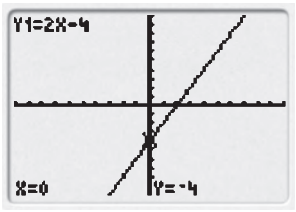
1. Enter the equation into the Y= editor.

2. Graph the equation.

3. Calculate the y-values.

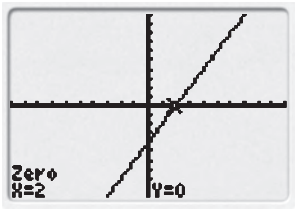
   Press 2nd TRACE to open the CALC menu, and select **1: value.** Enter the x-value and press ENTER. The x- and y-values will be displayed at the bottom of the screen and highlighted on the graph.





4. Calculate the zero.

   Press 2nd TRACE and select **2: zero.** Use the arrow keys to move the cursor to the left of the zero and press ENTER. Move the cursor to the right of the zero and press ENTER. Move the cursor to the approximate location of the zero and press ENTER. The zero will be displayed at the bottom of the screen and highlighted on the graph.



Calculating Minimums and Maximums

A graphing calculator can be used to find minimums and maximums.

Find the minimum of $y = x^2 + 3x - 4$.

1. Enter the equation into the Y= editor.

2. Graph the equation.

3. Calculate the minimum.

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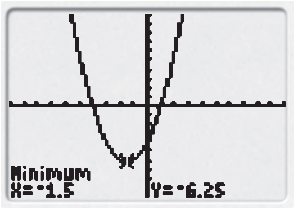
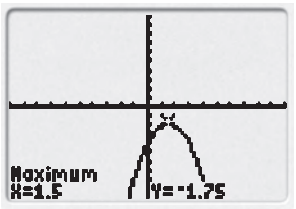
Press 2nd TRACE and select **3: minimum.**
Use the arrow keys ◀ ▲ ▼ ▶ to move the cursor to the left of the minimum and press ENTER. Move the cursor to the right of the minimum and press ENTER. Move the cursor to the approximate location of the minimum and press ENTER. The minimum will be displayed at the bottom of the screen and highlighted on the graph.



4. Calculate the maximum.

   The same process can be repeated to find the maximum of a parabola like $y = -x^2 + 3x - 4$. Simply select **4: maximum** from the CALC menu and repeat the same steps.



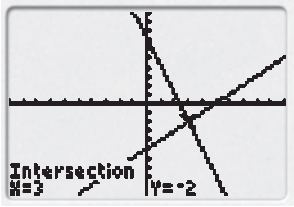
Calculating an Intersection Point

A graphing calculator can be used to find the intersection of two lines. Find the intersection of $y = x - 5$ and $y = -3x + 7$.

1. Enter the equations into the Y= editor.

2. Graph the equations.

3. Press 2nd TRACE and select **5: intersect.** Press ENTER to select the first line, and then ENTER to select the second line. Use the arrow keys ◀ ▲ ▼ ▶ to move the cursor to the approximate location of the intersection and press ENTER. The solution is displayed as a decimal at the bottom of the screen.



Graphing Calculator Practice

a. Use a graphing calculator to find the zero of $y = -17x + 51$.

b. Use a graphing calculator to find the maximum of the parabola $y = -x^2 - 4x + 5$.

c. Use a graphing calculator to find the intersection of $y = -x + 2$ and $y = -5x - 1$.

LESSON 13

Graphing Linear Equations I

Warm Up

1. (4) **Vocabulary** The _____ line test is used to determine if a graph is that of a function.

2. (SB) Subtract $-8 - (-6)$.

3. (2) Evaluate $-5x + 3y$ for $x = -2$ and $y = -3$.

New Concepts

An equation such as $y = 2x - 3$ is a linear equation in two variables. An ordered pair (x, y) is a solution to an equation in two variables if substituting the values of both x and y into the equation produces a true statement. For example, $(4, 5)$ is a solution to $y = 2x - 3$ since $5 = 2(4) - 3$. The graph of an equation in two variables is the set of all points (x, y) that are solutions to the equation.

Graphing Equations in Two Variables

Step 1: Construct a table of values. Choose a reasonable value for x and solve the equation for y. Repeat this step several times.

Step 2: Plot the points represented by the solutions to the equation.

Step 3: Connect the points to form a line. Extend the line and draw arrowheads on the ends to indicate that the line extends to infinity.

Example 1 Graphing a Linear Equation Using a Solution Table

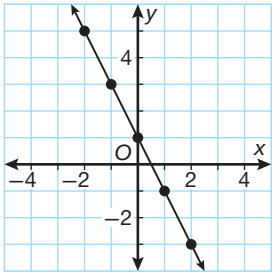
Graph the equation $y = -2x + 1$.

SOLUTION

Construct a table of values.

x	-2	-1	0	1	2
y	5	3	1	-1	-3

Plot and connect the points. The points should form a line.



Math Reasoning

Verify Have students use a graphing calculator to check the y values.

The points where the graph intersects the axes on the coordinate plane are called the **intercepts.** The point where the graph intersects the x-axis is called the ***x*-intercept,** and the point where the graph intersects the y-axis is called the ***y*-intercept.**

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Graphing a Linear Equation Using Intercepts

Find the x-intercept. Let $y = 0$. Solve the equation for x.

Find the y-intercept. Let $x = 0$. Solve the equation for y.

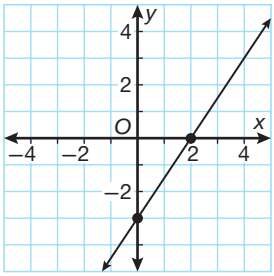
Plot and connect the intercepts.

Extend the line and add arrowheads to the ends.

Example 2 **Graphing a Linear Equation Using Intercepts**

Graph $3x - 2y = 6$ using the intercepts.

SOLUTION



Let $y = 0$. Solve for x. $3x - 2(0) = 6$

$$\frac{3x}{3} = \frac{6}{3}$$

$$x = 2$$

Let $x = 0$. Solve for y. $3(0) - 2y = 6$

$$\frac{-2y}{-2} = \frac{6}{-2}$$

$$y = -3$$

Plot and connect points $(2, 0)$ and $(0, -3)$. Extend the line through the points and draw arrows on the ends of the line.

Caution

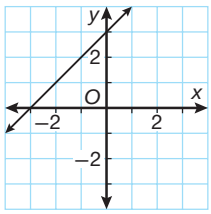
When finding the x intercept, do not make x equal 0. Make $y = 0$ then solve for x.

Slope of a Line

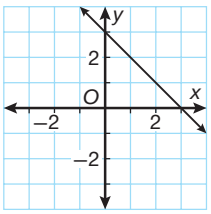
The **slope** of a line is the ratio of the vertical change (rise) to the horizontal change (run).

If (x_1, y_1) and (x_2, y_2) are the coordinates of any two points on the same line, the slope m of that line is calculated using the formula:

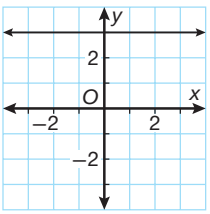
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run}$$



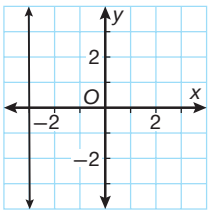
Rises



Falls



Horizontal



Vertical

Example 3 Classifying Lines Using Slope

Calculate the slope of the lines that contain the following pairs of points. Tell whether the line rises, falls, is horizontal, or is vertical.

a. $(-2, 3), (-1, 5)$

SOLUTION

$$m = \frac{5 - 3}{-1 - (-2)} = \frac{2}{1} = 2$$

Because m is positive, the line rises.

b. $(-4, -1), (2, -1)$

SOLUTION

$$m = \frac{-1 - (-1)}{2 - (-4)} = \frac{0}{6} = 0$$

Because $m = 0$, the line is horizontal.

c. $(6, -2), (6, -3)$

SOLUTION

$$m = \frac{-3 - (-2)}{6 - 6} = \frac{-1}{0}$$

Because m is undefined, the line is vertical.

d. $(3, 7), (-1, 10)$

SOLUTION

$$m = \frac{10 - 7}{-1 - 3} = \frac{3}{-4}$$

Because m is negative, the line falls.

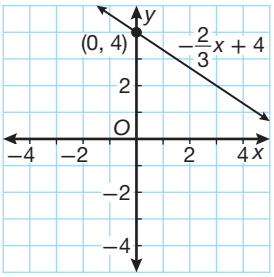
Math Reasoning

Verify If the order of the coordinates in the slope equation were reversed, would the value of the slope change?

The Slope-Intercept Form of a Linear Equation

A linear equation written in the form $y = mx + b$ is written in **slope-intercept form,** where m is the slope of the line and b is the y-intercept.

In the equation $y = -\frac{2}{3}x + 4$, the slope is $-\frac{2}{3}$ and the y-intercept is 4.



A line can be graphed using the slope-intercept form by plotting the y-intercept first, then using the slope to move to and plot another point. The slope is like a set of directions that indicate how many units to move and in which direction.

Moving from Point to Point Using the Slope

Positive slope	$\frac{up}{right}$	or	$\frac{down}{left}$
Negative slope	$\frac{up}{left}$	or	$\frac{down}{right}$

Example 4 Graphing a Line in Slope-Intercept Form

Identify the slope and y-intercept of the line with the given equation. Graph the line.

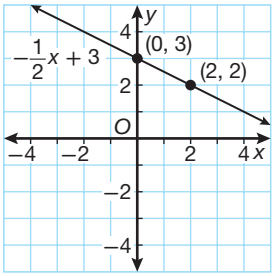
a. $y = -\frac{1}{2}x + 3$

SOLUTION

The equation is in the form $y = mx + b$. The slope of the line is $-\frac{1}{2}$ and the y-intercept is 3.

Plot the y-intercept (0, 3).

From (0, 3), move down 1 and 2 to the right to plot another point at (2, 2). Draw a line through (0, 3) and (2, 2).



b. $3x - 2y = 6$

SOLUTION

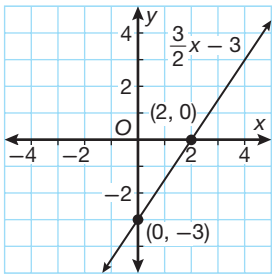
Rewrite the equation in slope-intercept form by solving for y.

$3x - 2y = 6$ Subtract $3x$ from both sides.

$-3x \quad = -3x$

$\frac{-2y}{-2} = \frac{-3x + 6}{-2}$ Divide each term by -2.

$y = \frac{3}{2}x - 3$ The line has a slope of $\frac{3}{2}$ and a y-intercept of -3.



Plot the y-intercept $(0, -3)$.

From $(0, -3)$, move up 3 and 2 to the right to plot another point at (2, 0). Draw a line through $(0, -3)$ and (2, 0).

Caution

When dividing by a coefficient to isolate y, divide every term in the equation, not just the term with x.

Example 5 Application: Railroads

A section of the Lookout Mountain Incline Railway in Tennessee has a vertical change of 1 mile upward for every 1.37 miles it moves along horizontally. What is the slope of the railroad for this section?

SOLUTION

The slope of the railroad can be calculated using $m = \frac{\text{rise}}{\text{run}}$.

The rise = 1 mi and the run = 1.37 mi.

$$\text{The slope} = \frac{1 \text{ mi}}{1.37 \text{ mi}}$$

$$= \frac{1}{1.37}$$

$$= 0.73.$$

Hint

Notice that the units must be the same when you are finding the slope so that they cancel out.

Lesson Practice

a. (Ex 1) Graph the equation $y = -\frac{1}{2}x + 3$ by constructing a table of values. (Hint: Choose x-values that are multiples of 2.)

b. (Ex 2) Graph $-x + 4y = 8$ using the intercepts.

Calculate the slope of the lines that contain the following pairs of points. Tell whether the line rises, falls, is horizontal, or is vertical. (Ex 3)

c. $(-3, 5), (5, -6)$

d. $(0, 4), (-2, 0)$

e. $(-4, 3), (8, 3)$

f. (Ex 4) Identify the slope and y-intercept of the line $y = \frac{1}{3}x + 2$. Graph the line.

g. (Ex 5) **Geography** A section of the Uncanoonuc Mountains, called the South Mountain USC Trail, is located in New Hampshire. The trail has a vertical change of about 24 feet vertically for every 50 feet it moves along horizontally. What is the slope of the trail?

Practice Distributed and Integrated

Determine the constant k, and solve the problem.

***1.** (12) **Multi-Step** The number of elk varied inversely as the number of deer and directly as the number of antelope. When there were 75 elk, there were 85 deer and 15 antelope. How many deer were there when there were 20 elk and 30 antelope? Round the answer down to the nearest whole number.

***2.** (12) **Formulate** The number of daisies varied inversely as the number of sunflowers and directly as the number of roses. When there were 65 daisies, there were 15 roses and 3 sunflowers. How many sunflowers were there when there were 5 daisies and 100 roses?

***3.** (13) Graph the equation $y = -\frac{3}{4}x + 1$ by constructing a table of values.

***4.** (13) Graph $-2x - y = 3$ using the intercepts.

***5.** (8) **Error Analysis** Two farmers solved the following problem:

Strawberries varied jointly as plums and tomatoes. If 500 strawberries went with 4 plums and 25 tomatoes, how many plums would go with 40 strawberries and 2 tomatoes?

Which farmer solved the problem using the correct steps? Explain your reasoning.

Farmer A	Farmer B
$500 = k(4)(25)$	$500 = k\frac{(25)}{(4)}$
$k = 5$	$k = 80$
$40 = 5(p)(2)$	$40 = 80\frac{(25)}{(p)}$
$p = 4$ plums	$p = 50$ plums

Calculate the slope of the lines that contain the following pairs of points. Tell whether the line rises, falls, is horizontal, or is vertical.

***6.** (13) $(-2, 0), (7, -5)$

7. (13) $(0, 4), (-2, 0)$

***8.** (13) Identify the slope and y-intercept of the line $y = -\frac{5}{4}x + 2$.

Analyze **Determine whether each expression is a polynomial. If the expression is a polynomial, rewrite it in standard form. If the expression is not a polynomial, explain why.**

***9.** (11) $25x^3 + x^{-7} + 2xy$

10. (11) $0.25x^4 + 5x^{\frac{4}{2}}$

11. (7) **Geography** The table shows the relationship of the annual estimated population for three cities in the year 2005. The population in Modesto, California, is represented by the variable x.

a. Formulate Write a linear equation to approximate the population in Modesto in 2005.

b. Estimate Find the approximate population of Modesto in 2005. Round to the nearest thousand.

2005 City Population

City	Population
Dallas	$5.879x$
St. Louis	$1.704x$
Modesto	x
Total of 3 cities	1,775,721

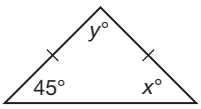
Evaluate.

12. (1) $-2 - 3\,(-2 - 2) - 5(-5 + 7)$

13. (1) $-[-2(-5 + 2) - (-2 - 3)]$

14. (2) $a^2 - y^3(a - y^2)y^2$ if $a = -2$ and $y = 3$

15. (7) **Geometry** Find x and y.



16. (10) **Justify** Explain why the inequality has no solution. $3x - 5 \geq 3(x + 7)$

17. (10) Solve the inequality, and graph its solution. $7x - 5 < 7(x + 5) + 2$

18. (10) **Civil Engineering** One cubic foot of pennies contains approximately 4.9×10^4 pennies. If approximately 1.8×10^{12} pennies would fit in the Empire State Building in New York City, what is the building's approximate volume in cubic feet? Write your answer in scientific notation. Round to the nearest tenth.

Solve for *p*.

19. (7) $(7p - 6) + 3p - (-2 - p) = 73$

20. (7) $\dfrac{7(p + 3)}{2} = 21$

21. (7) $4p + (p - 4)(2) = 8p - 7$

Use the matrices to find the products. Check the answer on a graphing calculator.

$$A = [-1 \quad 2],\ B = \begin{bmatrix} 4 & 0 & 2 \\ -3 & 5 & 1 \end{bmatrix},\ C = \begin{bmatrix} 7 & 1 \\ -2 & 0 \\ 0 & -3 \end{bmatrix}$$

22. (9) Find AB.

23. (9) Find ABC.

24. (11) **Citrus Production** The production of oranges and grapefruit (in millions of tons) in the U.S. for the years 2000 through 2006 is modeled by the polynomial functions below. The variable x represents the number of years since 2000.

Oranges $f(x) = -0.044x^3 + 0.258x^2 - 0.657x + 12.883$

Grapefruit $g(x) = 0.004x^3 - 0.065x^2 - 0.040x + 2.695$

What polynomial function is a model for the combined production of oranges and grapefruit in the United States for the years 2000–2006?

Write each fraction or decimal as a percent.

25. (6) 0.005

26. (6) $\frac{1}{6}$

27. (6) $\frac{5}{16}$

***28.** (12) **Meteorology** According to the Saffir-Simpson scale, hurricanes fall into five categories based on wind speed. The table below shows the relationship between category, wind speed, and storm surge (the height of waves). Analyze the data to determine the relationship between wind speed and storm surge. Is it a direct, joint, or inverse variation?

Category	Minimum Wind Speed (mph)	Minimum Storm Surge (ft)
1	74	4
2	96	6
3	111	9
4	131	13
5	155	19

29. (4) **Multiple Choice** Which of the following sets of ordered pairs are functions?

A $(-7, 6), (-7, 3), (4, 3)$ **B** $(-2, 5), (5, -2), (-2, 7)$

C $(1, -1), (-1, 1), (2, 1)$ **D** $(0, 1), (0, 2), (0, 3)$

***30.** (13) **Geography** A section of the Uncanoonuc Mountains, called the West Side Trail, is located in New Hampshire. The trail has a vertical change of about 268 feet vertically for every 1000 feet it changes horizontally. What is the slope of the trail?

LESSON 14

Finding Determinants

Warm Up

1. (5) **Vocabulary** A(n) _____ ***(element, matrix)*** is a rectangular array of numbers.

2. (7) Solve $4x + 6 = 22$.

3. (9) Simplify $\begin{bmatrix} 3 & 2 \\ -1 & 7 \end{bmatrix} \times \begin{bmatrix} 4 & -3 \\ 6 & 8 \end{bmatrix}$.

New Concepts

Every square matrix is associated with one real number called the **determinant** of the matrix. **If a matrix is not a square matrix, it does not have a determinant.** Enclosing the matrix within vertical lines designates the determinant of a matrix.

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \qquad \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}$$

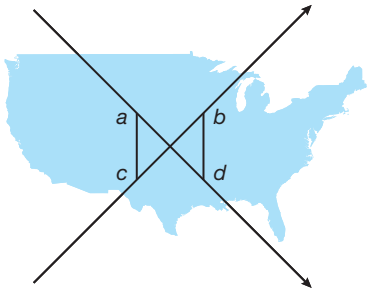
matrix determinant

The determinant of a 2×2 square matrix is found by subtracting the product of the entries in one diagonal from the product of the entries in the other diagonal. The order in which you subtract is important and is shown by this diagram.

$$\text{Determinant} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

(−cb, +ad)

This memory device can be used to help remember which product is positive and which is negative. The minus sign points to the cold state of Maine and the plus sign points to the warm state of Florida.



Caution

Subtracting the products in the incorrect order will give you the opposite of the value of the determinant.

Example 1 Finding the Determinant of a 2 × 2 Matrix

a. Evaluate $\begin{vmatrix} -4 & -3 \\ 2 & 7 \end{vmatrix}$.

SOLUTION Subtract cb from ad.

$$\begin{vmatrix} -4 & -3 \\ 2 & 7 \end{vmatrix} = (-4)(7) - (2)(-3) = -28 - (-6) = -22$$

b. Evaluate $\begin{vmatrix} -7 & 5 \\ 3 & -4 \end{vmatrix}$.

SOLUTION

$$\begin{vmatrix} -7 & 5 \\ 3 & -4 \end{vmatrix} = (-7)(-4) - (3)(5) = 28 - 15 = 13$$

Online Connection
www.SaxonMathResources.com

Example 2 Solving Determinant Equations

Find x. $\begin{vmatrix} x+4 & 3 \\ 2 & 4 \end{vmatrix} = 18$

SOLUTION Begin this type of problem the same as you would if you were finding the value of a determinant. Then solve for the missing value.

$(4)(x+4) - (2)(3) = 18$ Write the determinant as an equation.

$4x + 16 - 6 = 18$ Multiply.

$4x = 8$ Simplify.

$x = 2$ Divide.

There are two methods for finding the determinant of a 3×3 matrix with pencil and paper. The first method is known as expansion by minors. The second method is shown in Example 3 part b.

Expansion by Minors

If you pick an element of a matrix and cover its row and column, the determinant of the square matrix that remains is that element's **minor.**

$$\begin{bmatrix} \textcircled{a_1} & \cancel{b_1} & \cancel{c_1} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \begin{bmatrix} \cancel{a_1} & \textcircled{b_1} & \cancel{c_1} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \begin{bmatrix} \cancel{a_1} & \cancel{b_1} & \textcircled{c_1} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

To find the determinant, multiply each element in one row or column by its minor. Then subtract the middle product and add the final product.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Example 3 Finding the Determinant of a 3 × 3 Matrix

a. Find the determinant $\begin{vmatrix} 5 & 0 & 1 \\ -4 & 2 & -7 \\ 3 & -6 & 8 \end{vmatrix}$ using expansion by minors.

Hint

If you choose to expand a row or column that has a zero, it makes your calculations easier.

SOLUTION Find the minors for each element in one row or column.

$$\begin{vmatrix} 5 & 0 & 1 \\ \textcircled{-4} & \cancel{2} & \cancel{-7} \\ 3 & -6 & 8 \end{vmatrix} \quad \begin{vmatrix} 5 & 0 & 1 \\ \cancel{-4} & \textcircled{2} & \cancel{-7} \\ 3 & -6 & 8 \end{vmatrix} \quad \begin{vmatrix} 5 & 0 & 1 \\ \cancel{-4} & \cancel{2} & \textcircled{-7} \\ 3 & -6 & 8 \end{vmatrix}$$

Multiply each minor by its element and combine using the correct signs.

$$5\begin{vmatrix} 2 & -7 \\ -6 & 8 \end{vmatrix} - 0\begin{vmatrix} -4 & -7 \\ 3 & 8 \end{vmatrix} + 1\begin{vmatrix} -4 & 2 \\ 3 & -6 \end{vmatrix}$$

Evaluate the minors and simplify.

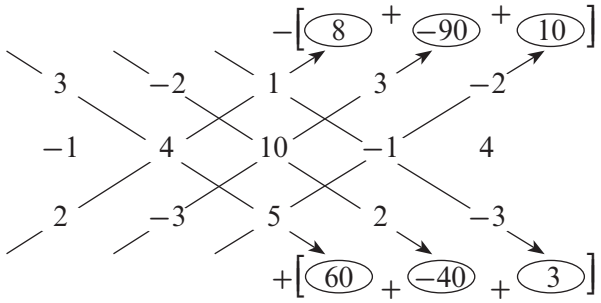
$$5[(2)(8) - (-6)(-7)] - 0[(-4)(8) - (3)(-7)] + 1[(-4)(-6) - (3)(2)]$$
$$= 5[16 - 42] - 0 + 1[24 - 6] = 5(-26) + 18 = -112$$

b. Find the determinant $\begin{vmatrix} 3 & -2 & 1 \\ -1 & 4 & 10 \\ 2 & -3 & 5 \end{vmatrix}$.

SOLUTION To begin the other method, copy the elements of the determinant and then repeat the first two columns to the right of the third column.

$$\begin{array}{ccccc} 3 & -2 & 1 & 3 & -2 \\ -1 & 4 & 10 & -1 & 4 \\ 2 & -3 & 5 & 2 & -3 \end{array}$$

Multiply on the diagonals as shown and add the products.



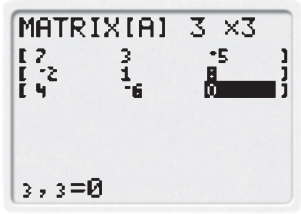
Subtract the sum of the upper products from the sum of the lower products.

$$\begin{vmatrix} 3 & -2 & 1 \\ -1 & 4 & 10 \\ 2 & -3 & 5 \end{vmatrix} = (60 - 40 + 3) - (8 - 90 + 10) = 95$$

Example 4 Using a Calculator to Find the Determinant

Find the determinant $\begin{vmatrix} 7 & 3 & -5 \\ -2 & 1 & 8 \\ 4 & -6 & 0 \end{vmatrix}$ using a calculator.

SOLUTION Store the data in a matrix.



2nd QUIT MODE; MATRX x^{-1}; Choose MATH; Choose **1:det(**; MATRX x^{-1}; Choose the matrix;); ENTER.

Verify Use expansion by minors to check the calculator answer.

$$7\begin{vmatrix} 1 & 8 \\ -6 & 0 \end{vmatrix} - 3\begin{vmatrix} -2 & 8 \\ 4 & 0 \end{vmatrix} + (-5)\begin{vmatrix} -2 & 1 \\ 4 & -6 \end{vmatrix} =$$

$$7[(1)(0) - (-6)(8)] - 3[(-2)(0) - (4)(8)] + (-5)[(-2)(-6) + (1)(4)] =$$
$$7[0 - (-48)] - 3[0 - 32] + (-5)[12 - 4] = 7(48) - 3(-32) + (-5)(8) = 392$$

Graphing Calculator

For help with storing data in a matrix, refer to page 27.

Example 5 Application: Geometry

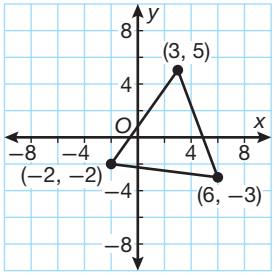
A triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) has an area equal to the absolute value of

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

Math Reasoning

Connect Why do you take the absolute value to find the area?

Find the area of the triangle below.



SOLUTION Substitute the coordinates of the vertices into the given matrix.

$$\frac{1}{2}\begin{vmatrix} 3 & 6 & -2 \\ 5 & -3 & -2 \\ 1 & 1 & 1 \end{vmatrix}$$

Expand by minors.

$$\begin{vmatrix} \cancel{3} & 6 & -2 \\ \cancel{5} & -3 & -2 \\ \textcircled{1} & \cancel{1} & \cancel{1} \end{vmatrix} \quad \begin{vmatrix} 3 & \cancel{6} & -2 \\ 5 & \cancel{-3} & -2 \\ \cancel{1} & \textcircled{1} & \cancel{1} \end{vmatrix} \quad \begin{vmatrix} 3 & 6 & \cancel{-2} \\ 5 & -3 & \cancel{-2} \\ \cancel{1} & \cancel{1} & \textcircled{1} \end{vmatrix}$$

$$\frac{1}{2}\begin{vmatrix} 3 & 6 & -2 \\ 5 & -3 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\left[1\begin{vmatrix} 6 & -2 \\ -3 & -2 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ 5 & -2 \end{vmatrix} + 1\begin{vmatrix} 3 & 6 \\ 5 & -3 \end{vmatrix}\right]$$

Simplify.

$$\frac{1}{2}[1[(6)(-2) - (-3)(-2)] - 1[(3)(-2) - (5)(-2)] + 1[(3)(-3) - (5)(6)]] =$$

$$\frac{1}{2}[(-12 - 6) - (-6 - (-10)) + (-9 - 30)] = \frac{1}{2}[-18 - 4 + (-39)] = -\frac{61}{2}$$

The absolute value of $-\frac{61}{2}$ is $\frac{61}{2}$ or $30\frac{1}{2}$.

The area of the triangle is $30\frac{1}{2}$ square units.

Lesson Practice

a. *(Ex 1)* Evaluate $\begin{vmatrix} 1 & -6 \\ 0 & 8 \end{vmatrix}$.

b. *(Ex 2)* Find x. $\begin{vmatrix} x-7 & 8 \\ -1 & -5 \end{vmatrix} = 23$

c. *(Ex 3)* Find the determinant $\begin{vmatrix} 5 & -9 & -5 \\ -5 & -3 & 8 \\ 2 & -8 & -1 \end{vmatrix}$ using expansion by minors.

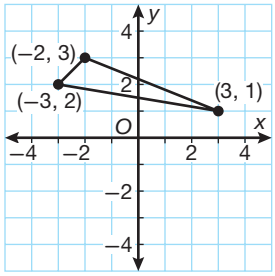
d. *(Ex 3)* Find the determinant $\begin{vmatrix} -4 & 2 & 7 \\ 9 & 3 & -4 \\ 7 & -1 & -2 \end{vmatrix}$.

e. *(Ex 4)* Find the determinant $\begin{vmatrix} -8 & -9 & -4 \\ -7 & 4 & 9 \\ 4 & 1 & 7 \end{vmatrix}$ using a calculator.

f. *(Ex 5)* A triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) has an area equal to the absolute value of

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

Find the area of the triangle.



Practice Distributed and Integrated

***1.** *(6)* **Multi-Step** The regular price of a cell phone was $89.99. The price was reduced by 25%.

a. Find the amount saved. Then find the new price of the cell phone. Round to the nearest hundredth.

b. If the new price is later reduced by $15, find the percent change from the original price to the lowest price at which the cell phone is sold. Round to the nearest percent.

***2.** *(12)* **Error Analysis** Two students analyzed the data table shown at right. One student determined that the relationship between the variables is an inverse variation, while the other determined that it is a direct variation. Which student is correct? What error did the other student make?

Beads

Quantity	Price
1000	$90
900	$100
750	$120
600	$150
500	$180

Student A

Quantity	Price	$Q \times P$
1000	$90	90,000
900	$100	90,000
750	$120	90,000
600	$150	90,000
500	$180	90,000

Student B

Quantity	Price	$Q \div P$
1000	$90	90,000
900	$100	90,000
750	$120	90,000
600	$150	90,000
500	$180	90,000

Identify which equations are direct variations and which are inverse variations.

3. (8, 12) $y = 3x$

4. (8, 12) $y = \frac{1}{2}x$

5. (8, 12) $y = \frac{20}{x}$

Model **Graph the equation by using the stated method.**

6. (13) Graph the equation $y = -\frac{2}{3}x + 4$ using a table of values.

***7.** (13) Graph the equation $y = \frac{5}{4}x + 1$ using its intercepts.

Calculate the slope of the lines that contain the following pairs of points. Tell whether the line rises, falls, is horizontal, or is vertical.

8. (13) $(-8, 2), (3, -1)$

9. (13) $(6, 4), (-5, -2)$

10. (13) $(6, 2), (-5, 2)$

***11.** (14) Evaluate $\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix}$.

***12.** (14) Find x. $\begin{vmatrix} 3 & 5 \\ -2x + 1 & -7 \end{vmatrix} = 24$

***13.** (14) **Verify** Evaluate $\begin{vmatrix} 7 & -1 & 2 \\ -3 & 0 & -2 \\ 4 & 3 & 5 \end{vmatrix}$. Show the steps of the method specified.

a. Use expansion by minors.

b. Set up diagonals. Find the difference of products.

***14.** (14) **Graphing Calculator** Use a graphing calculator to find $\begin{vmatrix} 7 & 3 & 8 \\ 0 & -5 & 6 \\ -1 & 2 & -4 \end{vmatrix}$.

***15.** (14) **Geometry** Find the area of the triangle with coordinates $(-2, 5), (7, 3), (-4, -3)$.

16. (5) **Meteorology** The average temperatures were calculated, in degrees Fahrenheit, for the city of Hilo, Hawaii, and for the city of Lihue, Hawaii, from the year 1971 to 2000.

City	Jan	Feb	Mar	Apr	May	Jun
Hilo	71.4	71.5	72.0	72.5	73.7	75.1
Lihue	71.7	71.1	72.7	73.9	75.4	77.7

a. **Model** Record this information in a matrix and name it matrix A.

b. Let $B = \begin{bmatrix} -32 & -32 & -32 & -32 & -32 & -32 \\ -32 & -32 & -32 & -32 & -32 & -32 \end{bmatrix}$. Convert the temperatures from Fahrenheit to Celsius by applying the operations on matrix A. Round each element to the nearest tenth.

$$C = \frac{5}{9}([A] + [B])$$

17. (8) **Measurement** The number of feet varies directly with the number of yards. If a rope measures 18 feet, it measures 6 yards. Josh is asked to write a proportion to find the number of yards in a rope that measures 33 feet. Josh wrote the proportion $\frac{18}{6} = \frac{x}{33}$. Identify Josh's error and solve the problem correctly.

18. (10) **Formulate** A friend is trying to earn at least \$500 this month by delivering newspapers. If she earns \$0.25 per delivery, how many papers will she have to deliver? Write an inequality and solve it.

Solve the inequalities.

19. (10) $-1.9 < 2(x - 0.25) \leq 2.5$

***20.** (10) $-2 < \frac{2(x-7)}{3} \leq 5$

Basketball **In basketball, points are determined by throwing the ball into the basket, also called making a shot. Use the tables to answer the problems.**

Player	Behind the 3-Point Line	Inside the 3-Point Line	Foul
Lan	2	1	1
Stacey	1	1	1
Robert	2	3	0

Type of Shot	Points
Behind the 3-Point Line	3
Inside the 3-Point Line	2
Foul	1

21. (5) Record Lan's, Stacey's, and Robert's points into a matrix. Name it matrix A. Record the point system into a matrix and name it matrix B.

***22.** (9) Find the total points for each player by finding the product $A \cdot B$.

Determine the kind of variation, if any, for each equation.

23. (8, 12) $y = mx + b$, for $m \neq 0$, $b \neq 0$

24. (8, 12) $y = mx + b$, for $m \neq 0$ and $b = 0$.

Identify the subsets of real numbers of which each number is a member.

25. (1) 0.33

26. (1) 0

27. (1) $\sqrt{5}$

28. (12) **Meteorology** According to the Saffir-Simpson scale, hurricanes fall into five categories based on wind speed. The table at right shows the relationship between category, wind speed, and storm surge (the height of waves). Analyze the data to determine the relationship between wind speed and storm surge. Is it a direct, joint, or inverse variation?

Category	Maximum Wind Speed (mph)	Maximum Storm Surge (ft)
1	95	5
2	100	8
3	130	12
4	155	18
5	155+	19+

29. (11) **Multiple Choice** Which polynomial is in standard form?

A $\frac{1}{2}x^3 - x$

B $\frac{1}{3} - \frac{1}{3}x^3$

C $4x^2 + 3x^3 + 2x^4$

D $x^4 + x^5$

30. (11) **Simple Interest** Al's grandfather deposited the amounts shown in Bank A as gifts on Al's birthday during the years 2002 through 2005. His aunt deposited the amounts shown in Bank B. The value of the accounts on 7/2/06 is represented by these polynomials, where $x = 1 + r$, and r is the interest rate for both accounts:

Bank A $500x^4 + 1000x^3 + 600x^2 + 500x$

Bank B $1000x^4 + 200x^3 + 200x^2 + 200x$

Date	Bank A	Bank B
7/2/02	\$500	\$1000
7/2/03	\$1000	\$200
7/2/04	\$600	\$200
7/2/05	\$500	\$200

a. Write the polynomial that represents the total value of both accounts on 7/2/06.

b. What is the total value of both accounts on 7/2/06 if the interest rate is 5%?

LESSON 15

Solving Systems of Equations by Graphing

Warm Up

1. (13) **Vocabulary** The _____ of a line is the change in the y-coordinates divided by the change in the x-coordinates.

2. (13) Give the slope and the y-intercept of the line with equation $y = -3x + 5$.

3. (2) Determine if the point $(2, 1)$ satisfies $y = 3x - 5$ and $y = x + 3$.

New Concepts

A **system of equations** is a collection of two or more equations containing two or more of the same variables. A **linear system** contains two linear equations in two like unknowns. Linear systems can be solved by graphing. Both of the equations are graphed on the same coordinate grid. The coordinates of the point where the lines **intersect,** or cross, give the solution.

Example 1 Solving Systems Using Tables and Graphs

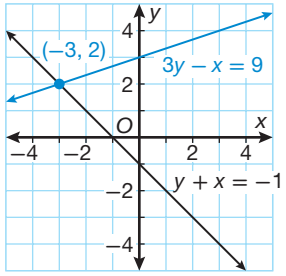
a. Solve this system by graphing.

$$3y - x = 9$$
$$y + x = -1$$

SOLUTION Solve each equation for y to get the slope-intercept form. Then graph the equations.

$$y = \frac{1}{3}x + 3$$

$$y = -x - 1$$



The solution of the system is $(-3, 2)$.

b. Make a table of values for each equation to solve the system in part **a.**

SOLUTION

$3y - x = 9$

x	y
−6	1
−3	2
0	3
3	4

$y = -x - 1$

x	y
−6	5
−3	2
0	−1
3	−4

When $x = -3$ the y values are the same for both equations. The solution to the system is $(-3, 2)$.

Hint

When choosing values of x to graph an equation, include negative values and zero.

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Not all solutions to a linear system will intersect at integer values of x and y. A graphing calculator enables more exact solutions to be found.

Math Reasoning

Verify Use an algebraic method to verify that the solution (1.6, −0.6) is correct.

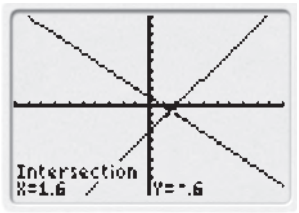
Example 2 Using a Graphing Calculator to Solve Systems

Solve this system by graphing.

$$3x - 2y = 6$$
$$y + x = 1$$

SOLUTION Solve each equation for y to get the slope-intercept form. Then graph the equations and find the point of intersection using a graphing calculator.

$$y = \frac{3}{2}x - 3$$
$$y = -x + 1$$



The lines intersect at (1.6, −0.6), so (1.6, −0.6) is the solution of the system.

Solutions of systems are ordered pairs. There are three possibilities for the number of solutions to a linear system: one ordered pair, an infinite number of ordered pairs, or no ordered pairs.

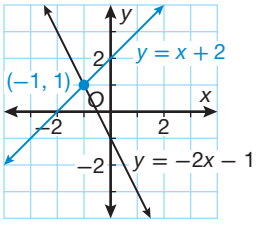
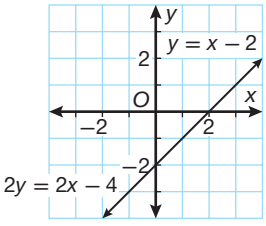
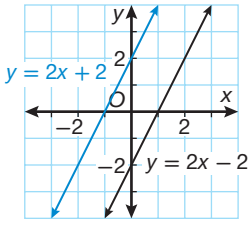
A linear system that has at least one solution is called a **consistent system.**

A system with no solution is called an **inconsistent system.**

A linear system is a **dependent system** when one of the equations in the system contains all solutions common to the other equation in the system.

Otherwise the system is an **independent system.**

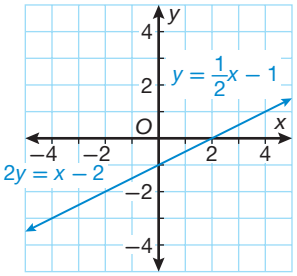
Classifying Systems of Equations

		
Lines intersect at one point. One solution: (−1, 1) System is consistent and independent.	Lines coincide. Infinite number of solutions. System is consistent and dependent.	Lines are parallel. No points in common. System is inconsistent.

Example 3 Classifying Linear Systems

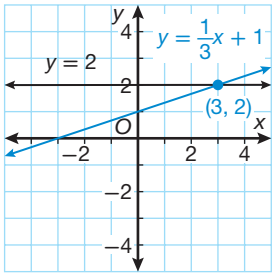
Determine if the following linear systems are consistent and independent, consistent and dependent, or inconsistent. If the system is consistent, give the solution.

a.



SOLUTION consistent and dependent; infinite set of ordered pairs

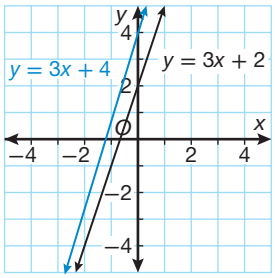
b.



SOLUTION consistent and independent; (3, 2)

c. $3x - y = -2$
$2y = 6x + 8$

SOLUTION Inconsistent



Math Reasoning

Generalize How can it be determined that a system is inconsistent without graphing?

Example 4 Application: Consumer Math

Music Masters sells MP3s for \$0.95 each to nonmembers. Members pay \$5 per year and \$0.90 for each MP3 purchase. After how many MP3 purchases will the total cost be the same for members and nonmembers?

SOLUTION Write two equations in two variables to represent the cost of each type of membership. Solve using a graphing calculator and check by substituting the solution into the equations.

Let x represent the number of MP3s purchased.
Let y represent the cost of the purchase.

Members: $y = 5 + 0.9x$

Nonmembers: $y = 0.95x$

When $x = 100$, the y values are the same for both equations. The cost for members and nonmembers is \$95 when 100 MP3s have been purchased.

Graphing Calculator Tip

The graphs of the lines $y = 0.95x$ and $y = 5 + 0.9x$ appear to be parallel in a standard window. Use the Zoom Out feature to see that the lines do intersect.

Check

$y = 0.95x$
$95 = 0.95(100)$
$95 = 95$

$y = 5 + 0.9x$
$95 = 5 + 0.9(100)$
$95 = 5 + 90$
$95 = 95$

Lesson Practice

a. *(Ex 1)* Solve this system by graphing.

$$2x - y = 6$$
$$y + 2x = -10$$

b. *(Ex 1)* Make a table of values for each equation to solve the system in part **a.**

$2x - y = 6$

x	y

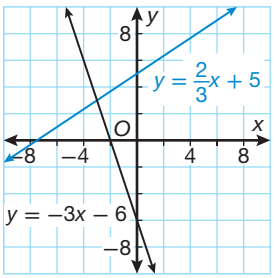
$y + 2x = -10$

x	y

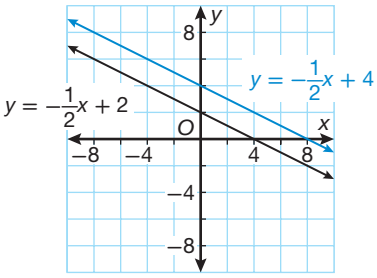
c. *(Ex 2)* Solve this system by graphing.

$$5x - 6y = 36$$
$$2y - 3x = -4$$

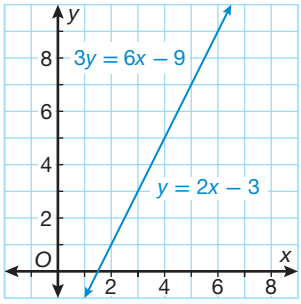
d. *(Ex 3)* Determine if the linear system at the right is consistent and independent, consistent and dependent, or inconsistent. If the system is consistent, give the solution.



e. *(Ex 3)* Determine if the linear system at the right is consistent and independent, consistent and dependent, or inconsistent. If the system is consistent, give the solution.



f. *(Ex 3)* Determine if the linear system at the right is consistent and independent, consistent and dependent, or inconsistent. If the system is consistent, give the solution.



g. (Ex 4) Superfit Gym charges nonmembers \$20 for each fitness class. Members pay \$75 per year and \$5 for each fitness class. After how many fitness classes will the total cost be the same for members and nonmembers?

Practice Distributed and Integrated

Given $h(x) = 6x + 3$ and $p(x) = 2x^2 - x$, find each of the following.

*1. (4) $p(-5)$

2. (4) $h\left(\frac{1}{2}\right)$

Identify whether the equation is a direct, inverse, or joint variation.

3. (12) $\frac{1}{3}y = x$

*4. (12) $y = \frac{1}{2}xz$

5. (12) $xy = 10$

Geometry **For each equation identify the kind of variation it represents.**

6. (12) Circumference $= 2\pi \times$ Radius

7. (12) Area of a circle $= \pi(\text{Radius})^2$

*8. (6) **Multi-Step** A jacket is on sale for 75% off the original price of \$39.99. Find the amount of the price reduction. Then find the new price, which includes a sales tax of 5%.

9. (14) **Multiple Choice** Evaluate $\begin{vmatrix} -1 & 3 \\ 2 & 3 \\ 7 & -4 \end{vmatrix}$.

A 0 **B** -9

C -29 **D** This matrix does not have a determinant.

10. (13) Given the equation $8x - 2y = -10$,

a. identify the slope and the y-intercept.

b. graph the line.

Use the slope formula to find the missing coordinate.

*11. (13) The slope of a line is $\frac{4}{9}$. The points $(3, 8)$ and $(-6, y)$ lie on the line. Find y.

12. (13) The slope of a line is $-\frac{7}{3}$. The points $(4, -4)$ and $(x, 3)$ lie on the line. Find x.

*13. (6) **Error Analysis** The price of a product decreased from \$9 to \$6. A student incorrectly computed the percent of decrease to be 50%. What was the student's error?

14. (14) Evaluate $\begin{vmatrix} -5 & 2 \\ 1 & -3 \end{vmatrix}$.

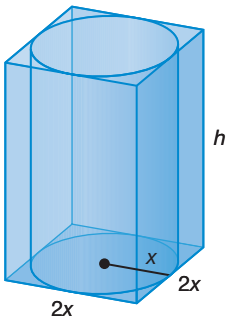
15. (14) Find x. $\begin{vmatrix} 2x & 3 \\ -x - 4 & -1 \end{vmatrix} = 10$

*16. (15) **Graphing Calculator** Graph to solve the system. Then classify the system.

$2x - y = -2$

$2y = 4x - 4$.

*17. (11) **Woodworking** A lathe operator has a piece of wood in the shape of a square prism with dimensions $2x$, $2x$, and h. He wants to make the largest possible cylinder by turning the wood on the lathe and cutting it down. The cylinder will have radius x and height h. Write a polynomial that represents the volume of wood he will cut off to make the cylinder. (The formula for the volume of a cylinder is $V = \pi r^2 h$.)



18. (12) **Physics** According to Ohm's Law, a direct current flowing in a wire is inversely proportional to the resistance of the wire. If the current in a wire is 0.01 amperes (A) and the resistance is 1200 ohms (Ω), what resistance will result in a current of 0.1 A?

19. (7) **Travel** Aaron is traveling 955 miles from Phoenix, Arizona, to Jackson, Wyoming. He leaves Phoenix, driving 410 miles before stopping for gas in Cortez, Colorado. He makes one last stop in Rock Springs, Wyoming, before traveling the final 180 miles to his destination. Write and solve an equation to find the number of miles Aaron drove between Cortez and Rock Springs.

*20. (4) **Formulate** Write a function that represents the data set $(-4, 16)$, $(1, 1)$, $(3, 9)$, $(4, 16)$, $(7, 49)$.

21. (10) **Business** A snow cone vendor wants to make at least \$2,500 the first month. If he charges \$1.25 per snow cone, how many snow cones will he have to sell? Write and solve an inequality to find how many snow cones he will have to sell.

22. (10) **Verify** Solve $-2(7x + 1) \leq -8x - 9$. Check the answer by substituting an integer that makes the inequality true.

23. (9) **Scoring** Andrew, Max, and Gracie are all taking a test where each type of question is given a certain point value. Use matrix multiplication to find out who scored the greatest number of points on the test.

Questions answered correctly

	Type 1	Type 2	Type 3
Andrew	7	5	4
Max	6	7	3
Gracie	7	4	5

Point Value	
Type 1	3
Type 2	5
Type 3	10

Use the matrices to find the products, if possible.

$$A = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 0 & 2 \end{bmatrix}$$

24. (9) AB

25. (9) BA

26. *(15)* **Multiple Choice** Which is a solution to the system below?

$$-x + 2y = 6$$
$$-3x + 4y = 4$$

A $(7, 5)$ **B** $(8, 7)$ **C** $(0, 0)$ **D** $(-16, -11)$

***27.** *(15)* **Graphing Calculator** Use a graphing calculator to determine the solution of the system.

$$-2x - y = 2$$
$$3x + 2y = 1$$

***28.** *(15)* **Write** The solution of a system is a single point. Classify the system and describe the graph.

29. *(8)* **Meaurement** In a certain recipe, the amount of sugar is directly proportional to the amount of flour. If 3 cups of sugar are used with 8 cups of flour, how many cups of sugar are used with 12 cups of flour?

30. *(3)* **Population** The U.S. population is approximately 2.99×10^8. In one year, the United States produced 4.78×10^{11} pounds of garbage. Approximately how much garbage did the average American produce that year?

LESSON 16

Using Cramer's Rule

Warm Up

1. (14) **Vocabulary** A _____ (***denominator, determinant***) is a real number associated with a square matrix.

2. (14) Find the determinant of $\begin{vmatrix} 4 & -3 \\ 5 & 2 \end{vmatrix}$.

3. (15) Solve the system. $\begin{aligned} 2x + y &= 2 \\ 4x - y &= 10 \end{aligned}$

New Concepts

Cramer's rule is a method for solving systems of linear equations using determinants. Elimination can be used to solve a general system of linear equations for x and y, as shown here.

$$\begin{aligned} ax + by &= e \;\rightarrow\; (d) \;\rightarrow\; dax + dby = de \\ cx + dy &= f \;\rightarrow\; (-b) \;\rightarrow\; -bcx - bdy = -bf \\ &\qquad\qquad (ad - bc)x = de - bf \;\rightarrow\; x = \frac{de - bf}{ad - bc} \\ ax + by &= e \;\rightarrow\; (-c) \;\rightarrow\; -cax - cby = -ce \\ cx + dy &= f \;\rightarrow\; (a) \;\rightarrow\; acx + ady = af \\ &\qquad\qquad (ad - bc)y = af - ce \;\rightarrow\; y = \frac{af - ce}{ad - bc} \end{aligned}$$

Observe that the denominators of x and y are the same and that their numerators are different. The same results are found by simplifying the following expressions.

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - fb}{ad - cb} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ce}{ad - cb}$$

Math Language

When a constant is multiplied by a variable, it is called a **coefficient.**

Note that the elements of the determinants in the denominators are the coefficients of x and y in the given equations. For this reason, this matrix is called the **coefficient matrix.** The matrices in the numerators are different. In the matrices in the numerators, the constants replace the coefficients of x when we solve for x, and the constants replace the coefficients of y when we solve for y.

Cramer's Rule

The solutions of the linear system $\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$ are

$x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D}$ and $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D}$, where D is the coefficient matrix.

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Example 1 Using Cramer's Rule to Solve Systems of Equations

Use Cramer's rule to solve $\begin{matrix} 3x + 2y = -1 \\ 4x - 3y = 10 \end{matrix}$.

SOLUTION The denominator of both x and y is the determinant of the coefficient matrix.

$$x = \frac{\begin{vmatrix} & \\ & \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} & \\ & \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}}$$

The numerator determinants are the same, except that the constants -1 and 10 replace the coefficients of x when we solve for x and replace the coefficients of y when we solve for y.

$$x = \frac{\begin{vmatrix} -1 & 2 \\ 10 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} = \frac{3 - 20}{-9 - 8} = 1 \qquad y = \frac{\begin{vmatrix} 3 & -1 \\ 4 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} = \frac{30 + 4}{-9 - 8} = -2$$

Caution

The order of the elements in the coefficient matrix is always the same.

Sometimes when using Cramer's rule, the determinant of the coefficient matrix is equal to zero. This makes the denominator of the solutions equal to zero, which makes the expression undefined.

Classifying Systems by Their Solutions		
If $D \neq 0$, the system has exactly **one unique solution.** A system that has any number of solutions is considered consistent.	If $D = 0$ but neither of the numerators is zero, the system has **no solution** and is considered inconsistent.	If $D = 0$ and at least one of the numerators is zero, the system has an **infinite number of solutions,** which means it is dependent and consistent.

Example 2 Interpreting a Denominator of Zero

a. Use Cramer's rule to solve $\begin{matrix} 3x + 2y = 5 \\ 3x + 2y = 8 \end{matrix}$.

SOLUTION Set up and simplify the equations for x and y using the coefficient matrix and the constants.

$$x = \frac{\begin{vmatrix} 5 & 2 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix}} = \frac{10 - 16}{6 - 6} = \frac{-6}{0} \qquad y = \frac{\begin{vmatrix} 3 & 5 \\ 3 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix}} = \frac{24 - 15}{6 - 6} = \frac{9}{0}$$

Division by zero is undefined, so using Cramer's rule did not provide a solution. Neither of the numerators is zero, so there is no solution. This is because the graphs of the equations are parallel lines and parallel lines never intersect.

Math Reasoning

Generalize Multiplying $3x + 2y = 5$ by 2 results in $6x + 4y = 10$, which is the same line. Give one more equation of this line.

b. Use Cramer's rule to solve the system. $\begin{aligned} 3x + 2y &= 5 \\ 6x + 4y &= 10 \end{aligned}$

SOLUTION Set up and simplify the equations for x and y.

$$x = \frac{\begin{vmatrix} 5 & 2 \\ 10 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix}} = \frac{20 - 20}{12 - 12} = \frac{0}{0} \qquad y = \frac{\begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix}} = \frac{30 - 30}{12 - 12} = \frac{0}{0}$$

In addition to the denominators being zero, both of the numerators are equal to zero, so there is an infinite number of solutions to this system. This is because the graphs of the equations are the same line.

Example 3 Application: Science

The molecular weight of H_2O is 18 atomic mass units, which means that 2 hydrogen (H) atoms and 1 oxygen (O) atom have a combined weight of 18 amu. Write a system of equations that describes the data in the table and use Cramer's rule to find the atomic weights of hydrogen and carbon (C).

Name	Formula	Molecular weight
Benzene	C_6H_6	78
Naphthalene	$C_{10}H_8$	128

Reading Math

The subscript number after each letter in a formula stands for the number of atoms of each element in that compound.

SOLUTION Using C and H as the variables, the system that represents the data is

$$\begin{aligned} 6C + 6H &= 78 \\ 10C + 8H &= 128 \end{aligned}$$

Solving this system with Cramer's rule will result in the weights of each element.

Remember that the coefficient matrix will be in the denominator for each variable. Use the constants in the numerator.

$$C = \frac{\begin{vmatrix} 78 & 6 \\ 128 & 8 \end{vmatrix}}{\begin{vmatrix} 6 & 6 \\ 10 & 8 \end{vmatrix}} = \frac{624 - 768}{48 - 60} = \frac{-144}{-12} = 12$$

$$H = \frac{\begin{vmatrix} 6 & 78 \\ 10 & 128 \end{vmatrix}}{\begin{vmatrix} 6 & 6 \\ 10 & 8 \end{vmatrix}} = \frac{768 - 780}{48 - 60} = \frac{-12}{-12} = 1$$

Check The atomic weight of carbon is 12 amu and that of hydrogen is 1 amu. Substitute the values back into each equation to make sure they are correct.

$$6(12) + 6(1) = 72 + 6 = 78$$
$$10(12) + 8(1) = 120 + 8 = 128$$

Lesson Practice

a. *(Ex 1)* Use Cramer's rule to solve the system. $\begin{aligned} 2x + 2y &= 3 \\ 3x + 8y &= 7 \end{aligned}$

b. *(Ex 2)* Use Cramer's rule to solve the system. $\begin{aligned} 2y - x &= 6 \\ 2y - x &= -2 \end{aligned}$

c. *(Ex 2)* Use Cramer's rule to solve the system. $\begin{aligned} x + 2y &= -4 \\ 3x + 6y &= -12 \end{aligned}$

d. *(Ex 3)* The molecular weight of H_2O is 18 atomic mass units, which means that 2 hydrogen (H) atoms and 1 oxygen (O) atom have a combined weight of 18 amu. Write a system of equations that describes the data in the table and use Cramer's rule to find the atomic weights of nitrogen (N) and oxygen (O).

Name	Formula	Molecular weight
Nitrogen oxide	N_2O_4	92
Nitrous oxide	N_2O	44

Practice Distributed and Integrated

1. *(4)* **Multi-Step** One T-shirt costs \$15.50. This can be expressed as the ordered pair (1, 15.5). The store is having a buy-one-get-one-free sale on T-shirts. Find the total cost if 2, 3, 4, 5, and 6 T-shirts are purchased. Express the answers as ordered pairs in set notation. Identify the domain and range. Determine if the set of ordered pairs represents a function. Explain.

***2.** *(15)* **Error Analysis** After examining the two function tables below, a student concluded that the two equations were inconsistent.

$y = -\frac{7}{3}x + 3$

x	-2	-1	0	1
y	$-\frac{5}{2}$	-1	2	5

$y = \frac{1}{3}x - 1$

x	-2	-1	0	1
y	$-\frac{5}{3}$	$-\frac{4}{3}$	-1	$-\frac{2}{3}$

a. Describe the student's error.

b. How can this error be avoided?

c. Solve the system.

3. *(4)* **Model** One day the value of one U.S. dollar was equivalent to 0.77 euro. On the same day, one U.S. dollar was equivalent to 1.24 Canadian dollars. Write a function to represent the value of Canadian dollars in euros.

***4.** *(5)* **Graphing Calculator** Solve the system: $\begin{aligned} x - 2y &= 6 \\ -6x + 3y &= 1 \end{aligned}$. Round to the nearest tenth.

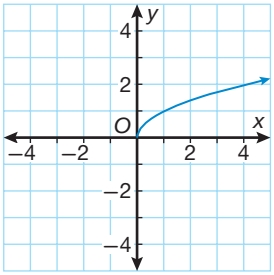
5. (6) The number of students attending a certain sports event decreased from 2080 to 1575 from one school year to the next.

a. Estimate Estimate the percent of change.

b. Find the actual percent of change. Round to the nearest percent.

Which of the following depict functions? Identify the domain and the range.

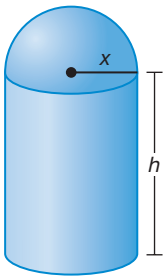
6. (4)



7. (4)

x	y
−2	−1
−2	3
−2	7

8. Measurement (11) This figure consists of a cylinder with a hemisphere on top. Write a polynomial in x that represents the total surface area of the figure. For reference, formulas for surface area are given below.



$$\text{Cylinder: } SA = 2\pi r^2 + 2\pi rh$$

$$\text{Sphere: } SA = 4\pi r^2$$

(Hint: The figure has only one circular base.)

Given matrices A, B, and C, multiply, if possible.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$$

9. (9) CA **10.** (9) AB **11.** (9) BC

Solve.

12. (7) $\frac{4}{7} + \frac{x+2}{3} = \frac{5}{3}$ **13.** (7) $\frac{5}{3} - \frac{x-4}{2} = \frac{1}{2}$

14. (7) $x - 7 + \frac{x}{4} = -\frac{1}{3}$

***15.** (15) **Demography** The population of Sacramento, California, in 2005 was 451,743 and increased by 2038 people by 2006. The population of Miami, Florida, in 2005 was 386,619 and increased by 17,429 people by 2006. If the population of these cities continued to grow at these rates, then the following equations would represent the population of the cities, where x is the number of years after 2005 and y is the population.

$y = 2038x + 451{,}743$ Sacramento's Population

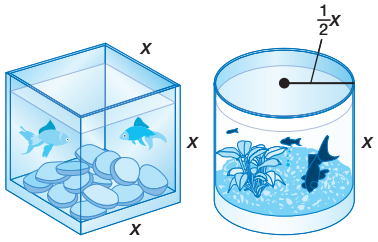
$y = 17{,}429x + 386{,}619$ Miami's Population.

a. In what year would the populations of Sacramento and Miami be equal?

b. About how many people are in each city when the populations are equal?

16. (14) Find x. $\begin{vmatrix} x & 2 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 5$

17. (11) **Aquarium Design** The aquariums shown each require the same-sized shelf, but they hold different amounts of water. Write a polynomial that represents how much more water the cube-shaped aquarium holds than the cylindrical aquarium. (Hint: The formula for the volume of a cylinder is $V = \pi r^2 h$.)



18. (10) **Explain** Why is there no solution for the compound inequality $-10 > n > -1$?

Solve the inequalities.

19. (10) $3x + 1 > -2$ or $6 < 2x - 4$

20. (10) $\frac{1}{3}x + 5 \leq 6$ or $-8 \leq \frac{1}{2}x - 7$

***21.** (16) **Construction** A contractor is replacing the tile in two bathrooms. For one bathroom, she purchased 416 large tiles and 256 small tiles at a cost of $233.60. For the other bathroom, she purchased 400 large tiles and 512 small tiles at a cost of $251.20. How much does each size tile cost?

Use Cramer's rule to solve each system.

***22.** (16) $\begin{aligned} -3x + 4y &= 8 \\ x + 4y &= 4 \end{aligned}$

***23.** (16) $\begin{aligned} -3x + 5y &= 8 \\ -3x + 5y &= 4 \end{aligned}$

***24.** (16) $\begin{aligned} 2x + 4y &= 10 \\ -5x - 10y &= -25 \end{aligned}$

***25.** (16) **Multiple Choice** Match the solution with the system of equations.

$$x = \frac{\begin{vmatrix} 8 & 2 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 5 & 2 \\ 3 & -1 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 5 & 8 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & 2 \\ 3 & -1 \end{vmatrix}}$$

A $\begin{aligned} 8x + 2y &= 5 \\ x - y &= 3 \end{aligned}$

B $\begin{aligned} 5x + 8y &= 2 \\ 3x + y &= -1 \end{aligned}$

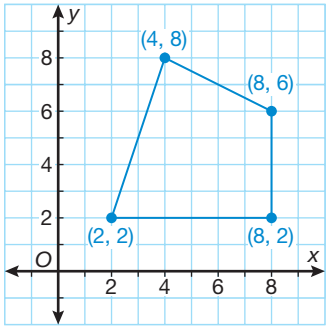
C $\begin{aligned} 5x + 2y &= 8 \\ 3x - y &= 1 \end{aligned}$

D $\begin{aligned} 8x + y &= 5 \\ 2x - y &= 2 \end{aligned}$

***26.** (16) **Write** When using Cramer's rule to solve systems of equations, what types of systems could exist if the determinant of the coefficient matrix equals zero? Describe the graphs of the lines.

***27.** (14) **Geometry** A quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) has an area equal to the absolute value of

$$\frac{1}{4}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{vmatrix}.$$ Find the area of the quadrilateral.



28. (13) Find the x- and y-intercepts of the line with equation $3x - \frac{1}{2}y = 6$.

29. (5) **Meteorology** The average temperatures were calculated, in degrees Fahrenheit, for the city of Kodiak, Alaska, and for the city of Valdez, Alaska, from the year 1971 to 2000.

City	Jan	Feb	Mar	Apr	May	Jun
Kodiak	29.7	29.9	32.6	37.3	43.5	49.2
Valdez	21.9	24.8	29.8	37.7	45.8	52.2

a. Model Record this information in a matrix and name it Matrix A.

b. Let $B = \begin{bmatrix} -32 & -32 & -32 & -32 & -32 & -32 \\ -32 & -32 & -32 & -32 & -32 & -32 \end{bmatrix}$. Convert the temperatures from Fahrenheit to Celsius by applying the operations on Matrix A. Round each element to the nearest tenth.

$$C = \frac{5}{9}([A] + [B])$$

30. (13) **Geography** Christine is hiking a trail to the summit of a mountain. The mountain has a vertical change of 15 feet upward for every 25 feet of horizontal distance.

a. What is the slope of the mountain?

b. If the trail has changed vertically by 450 feet, how much has it changed horizontally?

LAB

4

Changing the Line and Window of a Graph

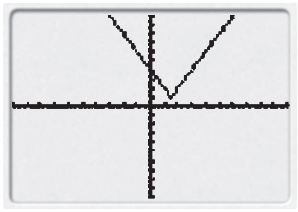
Graphing Calculator Lab (Use with Lessons 17 and 39)

Graphing Calculator Tip

For help with graphing equations, see the graphing calculator keystrokes on page 19.

Adjusting the Viewing Window

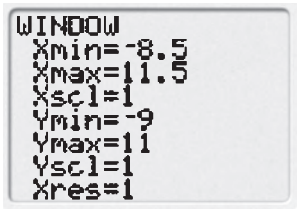
The viewing window of the graphing calculator can be adjusted to best display a graph.

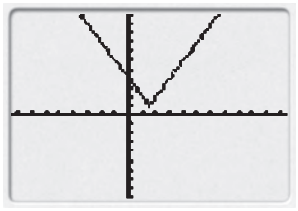


1. Graph the function $y = |2x - 3| + 1$.

2. Center the window on the vertex.

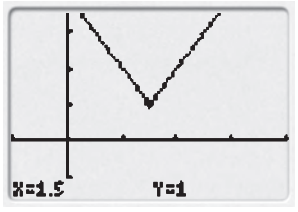
   The vertex of this function is located at (1.5, 1). To center the graph on this vertex, the x- and y-values need to be translated: $(x, y) \rightarrow (x + 1.5, y + 1)$. Press WINDOW, use the arrow keys to highlight Xmin, enter the value −8.5, and press ENTER. Repeat for Xmax, Ymin, and Ymax. Press GRAPH to display the graph centered on the vertex.





3. Use the zoom features to better see the graph.

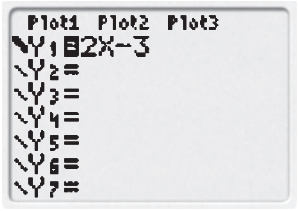
   To zoom in on a certain area of a graph, press ZOOM and press 2 to select **2: Zoom In**. The graph will then be displayed with a cursor. Use the arrow keys to move the cursor to the area you would like to zoom in on, and then press ENTER. Follow the same method but select **3: Zoom Out** to zoom the graph out.



   To return a graph to the standard settings where $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$, press ZOOM, use the arrow keys to select **6: ZStandard** and press ENTER.

Using Different Drawing Tools

A graphing calculator can be used to change the style of a line.



1. Graph the equation $y = 2x - 3$.

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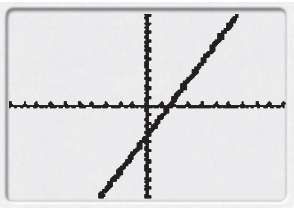
2. Change the graph to thick-line style.

Press Y= and ◀ ◀ to highlight the line symbol to the left of the equation $y = 2x - 3$.

Press ENTER to change to the thick-line symbol.

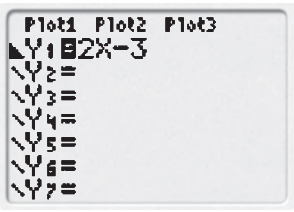
Press GRAPH to see the thick-line graph displayed.



3. Change the graph to a less-than-line style.

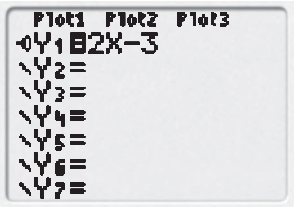
Press Y= and highlight the line symbol again.

Press ENTER twice this time until you see the less-than-line symbol. Press GRAPH to see the less-than-line graph of $y \leq 2x - 3$ displayed. (The same method can be used to graph $y \geq 2x - 3$ by selecting the greater-than-line tool.)

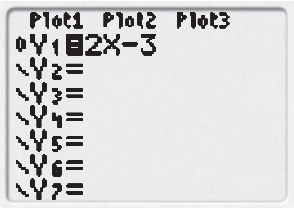


4. Investigate the other drawing tools.

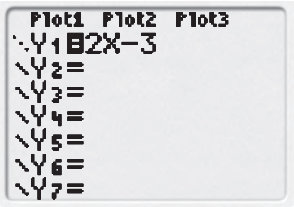
After the less-than-line symbol is the path symbol. This tool traces the graph and draws the path. This is especially helpful in drawing parametric equations.



After the path symbol is the animate symbol. This tool traces the graph without drawing the path. This tool is primarily used in physics applications.



The last tool is the dot tool, which will represent each plotted point as a dot. This tool is helpful when drawing step functions or other noncontinuous functions.



Graphing Calculator Practice

a. Use the graphing calculator to graph $y \geq 5 - x$.

b. Use the graphing calculator to graph $y = x^2 - 4x + 9$ and center the graph on (2, 5).

c. Use the graphing calculator to graph $y = |6x + 3| - 5$ and zoom in on the vertex.

LESSON 17

Solving Equations and Inequalities with Absolute Value

Warm Up

1. (10) **Vocabulary** A _____ (***conjunction, disjunction***) is a compound inequality containing the word *and*.

2. (10) Solve $x \geq -1$ and $x \leq 4$.

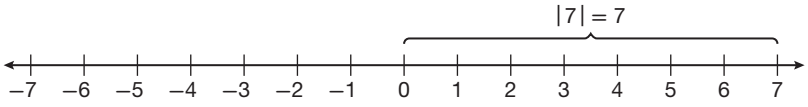
3. (10) Solve $x \geq -1$ or $x \leq 4$.

New Concepts

The **absolute value** of a number is the distance along the x-axis from the origin to the graph of the number. It is important to note that the absolute value of every number except zero is greater than zero.

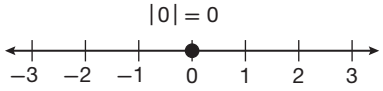
$$\text{If } x > 0, |x| = x.$$

$$|7| = 7$$



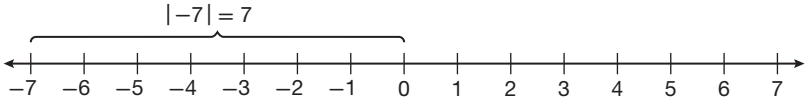
$$\text{If } x = 0, |x| = x.$$

$$|0| = 0$$



$$\text{If } x < 0, |x| = -x.$$

$$|-7| = -(-7) = 7$$



Reading Math

The equation $|x - a| = k$ can be read as "the distance from x to a is k units."

To solve absolute value equations, use inverse operations to isolate the absolute value expression on one side of the equation. Then consider the two cases below.

If $|x - a| = k$

$$x - a = -k \quad \text{or} \quad x - a = k$$

$$\text{then } x = a - k \quad \text{or} \quad x = a + k$$

The two equations above are derived, or obtained, from the original absolute value equation. Derived equations may result in **extraneous solutions.** These solutions do not satisfy the original absolute value equation. Therefore, you need to check all possible solutions.

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Example 1 Solving Absolute Value Equations

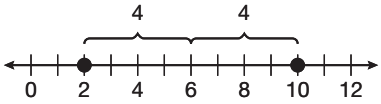
a. Solve $|x - 6| = 4$. Graph the solution.

SOLUTION

$x - 6 = -4$ or $x - 6 = 4$

$x = -4 + 6$ $x = 4 + 6$

$x = 2$ $x = 10$



Check $|2 - 6| = |-4| = 4$ $|10 - 6| = |4| = 4$

The solutions are 2 and 10.

b. Solve $|2x + 4| - 5 = 1$.

SOLUTION

$|2x + 4| - 5 = 1$

$|2x + 4| = 6$

$2x + 4 = 6$ or $2x + 4 = -6$

$2x = 2$ $2x = -10$

$x = 1$ $x = -5$

Check $|2(1) + 4| - 5 \stackrel{?}{=} 1$ $|2(-5) + 4| - 5 \stackrel{?}{=} 1$

$|2 + 4| \stackrel{?}{=} 6$ $|-10 + 4| \stackrel{?}{=} 6$

$6 = 6$ ✓ $|-6| \stackrel{?}{=} 6$

$6 = 6$ ✓

The solutions are 1 and -5.

Hint

An extraneous solution may occur anytime there is a variable outside the absolute value symbol.

c. Solve $|3x + 5| = -10$.

SOLUTION No solution. The absolute value of a number is never negative, as it represents distance.

d. Solve $|3x + 18| = 6x$. Check for extraneous solutions.

SOLUTION $3x + 18 = 6x$ or $3x + 18 = -6x$

$18 = 3x$ $18 = -9x$

$6 = x$ $-2 = x$

Check Evaluate $|3x + 18| = 6x$ for $x = 6$ and $x = -2$.

$|3(6) + 18| \stackrel{?}{=} 6(6)$ $|3(-2) + 18| \stackrel{?}{=} 6(-2)$

$|18 + 18| \stackrel{?}{=} 36$ $|-6 + 18| \stackrel{?}{=} -12$

$|36| \stackrel{?}{=} 36$ $|12| \stackrel{?}{=} -12$

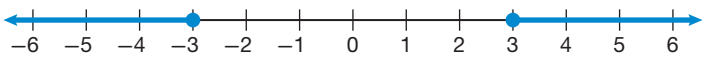
$36 = 36$ ✓ $12 \neq -12$

The solution is 6. -2 is an extraneous solution.

Absolute value inequalities can be solved by rewriting and solving them as compound inequalities. Absolute value inequalities are either conjunctions or disjunctions.

$x \leq 3$ and $x \geq -3$

$|x| \leq 3$



$x \geq 3$ or $x \leq -3$

$|x| \geq 3$

Conjunctions can be replaced with absolute value statements of less than. Disjunctions can be replaced with absolute value statements of greater than.

If a is a positive number, then

$\|x\| < a$	is the same as the conjunction	$x < a$ and $x > -a$ or $-a < x < a$
$\|x\| > a$	is the same as the disjunction	$x > a$ or $x < -a$

Note that statements like $|x| < -4$ and $|x| > -4$ are special cases. The first inequality has no solution, while the second is all real numbers.

Reading Math

"And" means to include in the solution all numbers common to both inequalities.

Example 2 Solving Absolute Value Inequalities with Conjunctions

a. Solve $|x - 5| \leq 2$. Graph the solution.

SOLUTION Rewrite the inequality as a conjunction, then solve for x.

$x - 5 \leq 2$ and $x - 5 \geq -2$

$x \leq 7$ and $x \geq 3$

$3 \leq x \leq 7$

b. Solve $-2|3x| + 4 \geq 6$.

SOLUTION Always isolate the absolute value term before rewriting the inequality as a conjunction or disjunction.

$-2|3x| + 4 \geq 6 \rightarrow -2|3x| \geq 2 \rightarrow |3x| \leq -1$

Since the absolute value of a number is always greater than or equal to zero, there is no solution.

Example 3 Solving Absolute Value Inequalities with Disjunctions

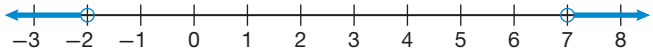
Reading Math

"Or" means to include in the solution all numbers described by both inequalities.

a. Solve $|2x - 5| > 9$. Graph the solution.

SOLUTION Rewrite the inequality as a disjunction and then solve for x.

$$2x - 5 > 9 \quad \text{or} \quad 2x - 5 < -9$$
$$2x > 14 \qquad\qquad 2x < -4$$
$$x > 7 \quad \text{or} \quad x < -2$$



$x < -2$ or $x > 7$

b. Solve $-|4x - 2| < 1$.

SOLUTION Isolate the absolute value term.

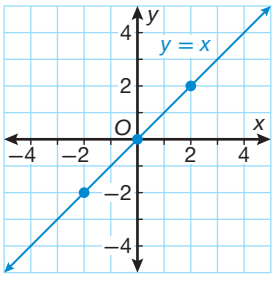
$$-|4x - 2| < 1 \quad \rightarrow \quad |4x - 2| > -1$$

Since the absolute value of a number is always greater than or equal to zero, the inequality will be true for any number substituted for x.

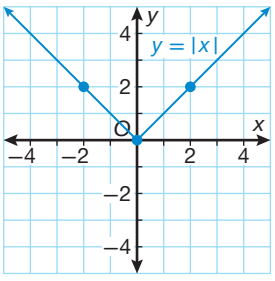
The solution is all real numbers.

The graphs in the previous examples are one-dimensional because the only variable is x. An **absolute value function** is a function that contains an absolute value expression. The graph of the equation $y = |x|$, or $f(x) = |x|$, will be two-dimensional because there are two variables, x and y.

x	y
2	2
0	0
−2	−2



x	y
2	2
0	0
−2	2



The graph on the left is the graph of $y = x$. The graph on the right is the graph of $f(x) = |x|$, the parent function of absolute value functions. It does not go below the x-axis because y equals the absolute value of x and the absolute value of a number is never negative.

Math Reasoning

Generalize Give the parent function for a linear function.

A **parent function** is the simplest function of a particular type. New functions can be graphed by **transformations,** or changes, to the graph of the parent function. These changes may involve a change in the size, shape, orientation, or position of the parent function.

Hint

Since $f(x) = |x + 4| = |x - (-4)|$, the h value is negative, not positive.

Exploration Transforming $f(x) = |x|$

Use a table of values to graph each of the following equations.

a. $f(x) = -|x|$ **b.** $f(x) = \frac{1}{2}|x|$ **c.** $f(x) = |x - 4|$

d. $f(x) = |x + 4|$ **e.** $f(x) = |x| - 4$ **f.** $f(x) = |x| + 4$

The graph of the absolute value function $y = a|x - h| + k$ can be used to model the transformations of the graph of the parent function $f(x) = |x|$.

g. Use the graphs of the functions to describe the effect of h and k on the graph of the parent function.

h. How is the shape and orientation of the graph determined?

i. Can the equation $y = a|x - h| + k$ be used to determine the **vertex,** or turning point, of the graph?

Hint

The number inside the absolute value symbol will be the "starting point"; the number outside the absolute value symbol will be the variance.

Example 4 Application: Chemistry

For hydrogen to be a liquid, it must be within 2° of −257°C. Write an absolute value equation to determine the least and greatest temperatures at which hydrogen will remain a liquid. Solve the equation and graph the solution.

SOLUTION Let t = temperature.

$$|t - (-257)| = 2$$

$$|t + 257| = 2$$

$$t + 257 = 2 \quad \text{and} \quad t + 257 = -2$$

$$t = -255 \qquad t = -259$$

The least temperature is −259°C and the greatest temperature is −255°C.

Lesson Practice

(Ex 1) **a.** Solve $|x + 3| = 7$. Graph the solution.

(Ex 1) **b.** Solve $|3x - 3| + 5 = 8$.

(Ex 1) **c.** Solve $|7x - 2| = -8$.

(Ex 1) **d.** Solve $|4x - 1| = 2x$. Check for extraneous solutions.

(Ex 2) **e.** Solve $|x + 2| < 8$. Graph the solution.

(Ex 2) **f.** Solve $3|5x| - 6 \leq 6$.

(Ex 3) **g.** Solve $|-2x + 9| \geq 7$. Graph the solution.

(Ex 3) **h.** Solve $-|2x + 3| \leq 5$.

(Ex 4) **i.** For nitrogen to be a liquid, it must be within 13° of −333°F. Write an absolute value equation to determine the least and greatest temperatures at which nitrogen will remain a liquid. Solve the equation and graph the solution.

Practice Distributed and Integrated

*1. (17) **Construction Engineering** A company's safety regulations require the wire rope used to suspend a 100-ton load to have a thickness that is within $\frac{3}{8}$ inch of $1\frac{3}{4}$ inches. Write and solve an absolute value equation to find the minimum safe thickness of rope.

2. (13) **Explain** Two points, $(3, -7)$ and $(3, 5)$, lie on a line. Using the slope formula, explain why the slope does not exist. What kind of line do these points lie on?

*3. (17) **Generalize** Will the inequality $3|x - 7| > -12$ have a solution?

Use Cramer's rule to solve each system.

*4. (16) $\begin{aligned} 4x - 2y &= 6 \\ 10x - 5y &= 15 \end{aligned}$

5. (16) $\begin{aligned} -5x + 2y &= 7 \\ 10x - 4y &= 3 \end{aligned}$

6. (16) $\begin{aligned} -3x + 2y &= 4 \\ x + 2y &= 10 \end{aligned}$

*7. (15) **Multiple Choice** Which is *not* a solution to the system below?

$$\begin{aligned} 2x - 5y &= -3 \\ -10x + 25y &= 15 \end{aligned}$$

A $(1, 1)$ **B** $(0, 0)$

C $(6, 3)$ **D** All of these are solutions.

Find the product.

8. (9) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \\ 2 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & -3 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$

9. (9) $\begin{bmatrix} 2 & -3 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -3 & 2 \\ 1 & 0 \\ 4 & 6 \end{bmatrix}$

10. (8) **Temperature** Juanita is conducting an experiment where the temperature of a liquid varies directly with time. Her observations are recorded in the table below.

Temperature	80 degrees	120 degrees	? degrees
Time	120 seconds	180 seconds	240 seconds

a. Using the values in the table, identify the constant of variation.

b. Find the missing temperature in the table.

c. Determine an accurate prediction for the temperature of the liquid after 360 seconds.

11. (13) **Multiple Choice** Identify the slope of the line with the equation $2x + 9y = 18$.

A 2 **B** $\frac{2}{9}$ **C** $-\frac{9}{2}$ **D** $-\frac{2}{9}$

12. (8) **Baseball** When a baseball pitcher throws a baseball, the ball is slowed down by wind resistance. The relationship between the force (F) of wind resistance and velocity (v) of the baseball is a direct variation of the form $F = kv^2$. The data tables below represent sample pitches from two ball games.

Pitcher A

Ball Speed (mph)	Force
90	1620
91	1656
92	1693
93	1730
94	1767
95	1805

Pitcher B

Ball Speed (mph)	Force
90	2025
91	2070
92	2116
93	2162
94	2209
95	2256

a. Find the constant of variation for each pitcher.

b. Based on your results, which pitcher was throwing the ball harder?

c. What could account for the difference in the force of wind resistance?

13. (16) **Multi-Step** Use Cramer's Rule to solve the system. Check your solution by graphing.

$$-2x + 3y = 0$$
$$x + 3y = 6$$

14. (11) **Error Analysis** A student says that the polynomial $ab^4 + a^2b^2 - 3$ is of higher degree than the polynomial $a^2b^3 + ab + 2$. What is the error? Explain.

Solve the equation or inequality.

15. (7) $\frac{y-2}{4} = 7y - 2 - 6(y+1)$

16. (10) $2 + \frac{1}{3}x \le 1\frac{2}{3}$ and $-2x - 5 \le 7$

17. (10) $-\frac{1}{3} - \frac{2x}{3} > \frac{1}{3}$ and $-7x - \frac{1}{2} < 6\frac{1}{2}$

***18.** (17) $|-2x + 7| - 3 = 10$.

***19.** (17) **Verify** Solve $|8x - 3| = 16x$. Check for extraneous solutions by substituting the solution into the equation.

***20.** (17) Solve $|3x + 2| \le 14$. Graph the solution.

21. (15) **Geometry**

a. Graph the system below. Describe the shape that the lines form.

$$3x - 2y = 2$$
$$-3x - 2y = -34$$
$$x - 2y = 6$$

b. Find the vertices of the shape.

22. (14) A triangle has vertices (1, 2), (1, −4), and (4, −4). Find the area of the triangle by writing and solving a determinant.

Use a table to graph the equation. (Hint: Choose x-values that are multiples of the slope's denominator.)

23. *(13)* $y = -\frac{5}{3}x - 2$

24. *(13)* $y = -\frac{1}{2}x + 5$

***25.** *(14)* **Graphing Calculator** Use a graphing calculator to evaluate

$$2\begin{vmatrix} -2 & 1 & 4 \\ -5 & 7 & 0 \\ 8 & -3 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 & 4 \\ 1 & 7 & 0 \\ -6 & -3 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & -2 & 4 \\ 1 & -5 & 0 \\ -6 & 8 & 1 \end{vmatrix} - 5\begin{vmatrix} 0 & -2 & 1 \\ 1 & -5 & 7 \\ -6 & 8 & -3 \end{vmatrix}.$$

26. *(3)* **Solar System** The average distance between the earth and sun is approximately 1.495×10^{11} m. Earth's orbit around the sun is nearly circular. For estimation purposes, assume that Earth's orbit is circular.

a. Estimate the distance in kilometers that the earth travels around the sun in one year. Write your answer in scientific notation. Round to the nearest hundredth.

b. Estimate the average speed at which the earth travels around the sun in km/s. Use 365.25 days per year.

***27.** *(13)* Find the slope of the line between $(-2, -7)$ and $(2, 1)$. Determine if the line rises, falls, or is horizontal or vertical.

***28.** *(8)* **Landscaping** When mixing concentrated fertilizer for a garden, the amount of fertilizer varies directly with the amount of water. If 2 cups of fertilizer are used in 16 gallons of water, how much fertilizer is used in 48 gallons of water?

Determine the kind of variation, if any, for each equation.

29. *(12)* $y = ax^2 + bx + c$, for $a, b, c \neq 0$

30. *(12)* $y = ax^2 + bx + c$, for any values of a, b, and c.

LESSON 18

Calculating with Units of Measure

Warm Up

1. **Vocabulary** 1 foot and 12 inches are _____. (*equivalent, proportional*)

2. *(3)* Simplify $\frac{x^4 \cdot x^2}{x^3}$.

3. *(SB)* Simplify $\frac{6}{7} \times \frac{5}{6} \times \frac{3}{5} \times \frac{2}{3}$.

New Concepts

You can use unit **conversion factors** to change one unit of measure to another. Dimensional analysis involves choosing the appropriate conversion factor to produce the appropriate unit.

Example 1 Choosing the Appropriate Conversion Factor

a. Change 600 inches to feet.

SOLUTION Put 600 in. over 1.

$$\frac{600 \text{ in.}}{1}$$

There are 12 inches in a foot, so the conversion factor is either $\frac{1 \text{ ft}}{12 \text{ in.}}$ or $\frac{12 \text{ in.}}{1 \text{ ft}}$. If you choose $\frac{1 \text{ ft}}{12 \text{ in.}}$ as the conversion factor, the inches will cancel.

$$\frac{600 \text{ in.}}{1} \times \frac{1 \text{ ft}}{12 \text{ in.}} = 50 \text{ ft}$$

b. Change 8 hours to minutes.

SOLUTION Put 8 hours over 1.

$$\frac{8 \text{ hr}}{1}$$

There are 60 minutes in an hour, so the conversion factor is either $\frac{1 \text{ hr}}{60 \text{ min}}$ or $\frac{60 \text{ min}}{1 \text{ hr}}$. You want the hours to cancel, so choose $\frac{60 \text{ min}}{1 \text{ hr}}$.

$$\frac{8 \text{ hr}}{1} \times \frac{60 \text{ min}}{1 \text{ hr}} = 480 \text{ min}$$

Reading Math

The slash (/) is used to symbolize the canceling of factors or, as in this lesson, units.

Sometimes you may need to use more than one conversion factor.

Example 2 Using Multiple Conversion Factors

a. Convert 42 yd^2 to in^2.

SOLUTION Put 42 yd^2 over 1.

$$\frac{42 \text{ yd}^2}{1}$$

There are 3 feet in a yard and 12 inches in a foot, so

$$\frac{42 \text{ yd}^2}{1} \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{12 \text{ in.}}{1 \text{ ft}} \times \frac{12 \text{ in.}}{1 \text{ ft}} = 54{,}432 \text{ in}^2$$

b. Convert 3000 seconds to hours.

SOLUTION Put 3000 s over 1.

$$\frac{3000 \text{ s}}{1}$$

There are 60 seconds in a minute and 60 minutes in an hour, so

$$\frac{3000 \text{ s}}{1} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{1 \text{ hr}}{60 \text{ min}} = \frac{5}{6} \text{ hr}$$

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Math Language

Per means "for each" and implies division.

Example 3 Converting Rates

Convert 50 feet per second to miles per hour.

SOLUTION Write 50 feet per second as a fraction. $\frac{50 \text{ ft}}{1 \text{ s}}$

Convert feet to miles and seconds to hours. There are 5280 feet in a mile, so the conversion factor is $\frac{1 \text{ mi}}{5280 \text{ ft}}$. There are 60 seconds in a minute and 60 minutes in an hour, so the other conversion factor is $\frac{3600 \text{ s}}{1 \text{ hr}}$.

$$\frac{50 \cancel{\text{ft}}}{1 \cancel{\text{s}}} \times \frac{1 \text{ mi}}{5280 \cancel{\text{ft}}} \times \frac{3600 \cancel{\text{s}}}{1 \text{ hr}} = \frac{(50)(3600)}{5280} \frac{\text{mi}}{\text{hr}}$$

Using a calculator, this simplifies to about 34 mph.

The **accuracy** of a measurement refers to how closely it corresponds to the actual value being measured. The **precision** of a measurement refers to the number of **significant digits** it contains. The following table illustrates how to find the number of significant digits in a measured value.

Significant Digits

Rule	Example	Number of Significant Digits
Nonzero digits are significant.	325.6	4
Zeros between significant digits are significant.	5003.09	6
Leading zeros are not significant.	0.00047	2
Zeros after the last nonzero digit that are also to the right of a decimal point are significant.	0.0001800	4

The following rules are used to determine the number of significant digits that should be in the answer to a calculation with measurements:

Rules for Significant Digits in Measurements
When adding or subtracting, the answer should have the same number of decimal places as the measurement with the fewest decimal places. When multiplying or dividing, the answer should have the same number of significant digits as the measurement with the fewest significant digits.

Example 4 Using Significant Digits

Simplify each expression.

a. 5.38 mi + 6.495 mi + 0.5 mi

SOLUTION Add the three distances.

5.38 mi + 6.495 mi + 0.5 mi = 12.375 mi

Since 0.5 mi has only one decimal place, the answer must be rounded to one decimal place, resulting in 12.4 mi.

b. 2.09 ft × 0.050 ft × 303 ft

SOLUTION Multiply the three lengths.

2.09 ft × 0.050 ft × 303 ft = 31.6635 ft^3

Since 0.050 ft has only two significant digits, the answer can only have two significant digits, resulting in 32 ft^3.

Example 5 Application: Sports

Natcar is an annual robotic car race. In 2007, the winning car completed the course with an average speed of 7.57 feet per second. What is this speed in miles per hour?

SOLUTION Write the given speed as a fraction.

$$\frac{7.57 \text{ ft}}{1 \text{ s}}$$

One of the conversion factors needs to have *ft* in its denominator and the other needs *s* in its numerator.

$$\frac{7.57 \cancel{\text{ft}}}{1 \cancel{\text{s}}} \times \frac{1 \text{ mi}}{5280 \cancel{\text{ft}}} \times \frac{3600 \cancel{\text{s}}}{1 \text{ hr}} = \frac{(7.57)(3600)}{5280} \frac{\text{mi}}{\text{hr}}$$

This answer simplifies to about 5.16 mph.

Math Reasoning

Predict If the builder of a car wanted it to reach 10 mph, about how fast would it need to go in feet per second?

Lesson Practice

a. (Ex 1) Change 720 inches to feet.

b. (Ex 1) Change 6 hours to minutes.

c. (Ex 2) Convert 52 yd^2 to in^2.

d. (Ex 2) Convert 2700 seconds to hours.

e. (Ex 3) Convert 80 feet per second to miles per hour.

f. (Ex 4) Simplify the expression 6.21 mi + 3.672 mi + 0.8 mi.

g. (Ex 4) Simplify the expression 8.08 ft × 0.020 ft × 407 ft.

h. (Ex 5) RaceUSA is an annual boxcar race. In 2007, the winning car completed the course with an average speed of 24 feet per second. What is this speed in miles per hour?

Practice Distributed and Integrated

Use Cramer's rule to solve each system.

***1.** (16) $x = y + 1$
$3x + 2y = 8$

2. (16) $3x - y = 22$
$2x + 3y = -11$

3. (16) $x + y = 20$
$5x + 10y = 200$

Solve.

***4.** (17) $-5|2x - 7| - 4 = -34$

5. (17) $\left|\frac{1}{2}x + 1\right| \leq -\frac{1}{2}$

6. (17) $|8 - 3x| > 9$

Identify whether the equation is a direct, inverse, or joint variation. Find the constant of variation.

7. (12) $\frac{y}{x} = \sqrt{2}$

8. (12) $y = \frac{15}{x}$

***9.** (15) **Chemical Mixture** A chemist has one solution that is 10% iodine and another that is 50% iodine. How many milliliters of each should the chemist use to make 100 mL of a mixture that is 20% iodine?

a. Formulate Set up a system of equations that models the problem.

b. Graphing Calculator Use a graphing calculator to solve the system.

10. (12) **Multiple Choice** In the data set at right, what is the relationship between x and y?

x	1	2	3	4
y	4	2	$1\frac{1}{3}$	1

A joint variation

B direct variation

C inverse variation

D none of the above

11. (15) **Sports** Suppose you and a friend are practicing basketball. To make practice more interesting, you invent a game. Every time you make a basket, you will gain 2 points. Every time your friend makes a basket, you will lose 1 point. Likewise, every time your friend makes a basket, he will gain 2 points, and every time you make a basket, he will lose 1 point. Both of you start with 0 points. At the end of the game, you have 16 points and your friend has 10.

a. Formulate Write a system of equations to represent this situation.

b. How many baskets did each of you make?

12. (12) **Geometry** As the diameter (d) of a circle increases in size, the circumference (C) increases. Likewise, as the diameter decreases in size, so does the circumference. The constant of variation between C and d is π. Describe the kind of variation between circumference and diameter. Write the equation.

13. (14) **Error Analysis** Explain the error in the work below. Then find the correct determinant.

$$\begin{vmatrix} -2 & 6 \\ 3 & -1 \end{vmatrix} = (6)(3) - (-2)(-1) = 18 - 2 = 16$$

Convert the units and unit rates.

***14.** (18) Change 0.075 hours to seconds.

***15.** (18) Convert 1224 in^2 to yd^2.

***16.** (18) Convert 0.26 miles per hour to feet per minute.

***17.** (18) **Space Travel** The Endeavor, a space shuttle, has speeds of about 27,404 feet per second. Convert the speed into kilometers per hour and miles per hour.

18. (17) **Verify** Show that $|2x + 3| = 16x$ has two solutions, one of which is extraneous.

***19.** (17) **Analyze** Write the four resulting inequalities that are used to solve $|-3x + 5| \geq |x|$.

20. (17) **Chemical Elements** For oxygen to be a liquid, it must be within 35.845° of −332.925°F.

a. Write an absolute value equation to determine the least and greatest temperatures at which oxygen will remain a liquid.

b. Solve the equation.

21. (13) **Explain** Give a written explanation on how to graph an equation using intercepts.

22. (11) **Analyze** Give an example of two trinomials whose sum has 6 terms.

Solve for x.

***23.** (7, 14) $\begin{vmatrix} \frac{x}{2} & 4 \\ 3 & -10 \end{vmatrix} = -(-x + 7)$

24. (7, 14) $\begin{vmatrix} -3 & 2 \\ 4x - 1 & 2x - \frac{1}{3} \end{vmatrix} = -16x + 7$

Multi-Step Given the slope and a point of the line, graph the line. Determine the y-intercept of the line.

25. (13) $m = \frac{5}{3}\ (-3, 1)$

26. (13) $m = -\frac{4}{5}\ (5, -5)$

***27.** (3, 18) **Solar System** The earth revolves about the sun at about 29.8 kilometers per second. Convert the rate into meters per hour. Write the answer in scientific notation rounded to the thousandths.

$A = \begin{bmatrix} 6 & -3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -2 & 6 \end{bmatrix}$. Find each product.

28. (9) Find AB.

29. (9) Find BA.

30. (9) Find A^2.

LESSON
19

Multiplying Polynomials

Warm Up

1. **Vocabulary** The 5 in the term $5x$ is called the _____. (11)
2. Simplify using the Distributive Property: $5(3 + 4)$. (1)
3. Combine like terms: $3x^2 + 5x - 8x^2 - x$. (2)
4. Simplify $2a^3 \cdot 3a^3$. (3)

New Concepts

To multiply two polynomials, each term of the first polynomial is multiplied by each term of the other polynomial. After each of these products is found, the like terms are combined. To multiply two binomials a method called FOIL is often used. The letters in FOIL stand for where the terms are found in the binomials.

The FOIL Method

Multiply $(x + 3)(x - 5)$.

The **F**irst terms are x and x. Their product is x^2.

The **O**utside terms are x and -5. Their product is $-5x$.

The **I**nside terms are 3 and x. Their product is $3x$.

The **L**ast terms are 3 and -5. Their product is -15.

Combine the like terms in $x^2 - 5x + 3x - 15$.

$$(x + 3)(x - 5) = x^2 - 2x - 15$$

Example 1 Multiplying Using the FOIL Method

Multiply using the FOIL method.

a. $(a + 8)(a - 5)$

SOLUTION

F: $a \cdot a = a^2$ **O:** $a \cdot (-5) = -5a$ **I:** $8 \cdot a = 8a$ **L:** $8 \cdot (-5) = -40$

$= a^2 - 5a + 8a - 40$ Multiply.

$= a^2 + 3a - 40$ Combine like terms.

b. $(2x - 3)(3x + 2)$

SOLUTION

F: $2x \cdot 3x = 6x^2$ **O:** $2x \cdot 2 = 4x$ **I:** $-3 \cdot 3x = -9x$ **L:** $-3 \cdot 2 = -6$

$= 6x^2 + 4x - 9x - 6$ Multiply.

$= 6x^2 - 5x - 6$ Combine like terms.

Math Reasoning

Write Explain in writing why $(-5 + a)(8 + a)$ would produce the same result as $(a + 8)(a - 5)$.

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Example 2 Multiplying a Binomial by a Trinomial

Multiply $(a + 5)(3a^2 - 2a + 7)$.

SOLUTION Multiply each term of the binomial by each term of the trinomial.

$$(a + 5)(3a^2 - 2a + 7)$$

$$= a(3a^2) + a(-2a) + a(7) + 5(3a^2) + 5(-2a) + 5(7)$$

$$= 3a^3 - 2a^2 + 7a + 15a^2 - 10a + 35 \quad \text{Multiply.}$$

$$= 3a^3 + 13a^2 - 3a + 35 \quad \text{Combine like terms.}$$

More than two polynomials can be multiplied together. The Associative Property of Multiplication holds for polynomials. Multiply two of the polynomials and then multiply that product by the remaining polynomial.

Example 3 Multiplying Three Binomials

Multiply $(r + 3)(r - 2)(r + 5)$ in two different ways using the Associative Property of Multiplication.

SOLUTION

Multiply $(r + 3)(r - 2)$.

$$(r + 3)(r - 2)$$

$$= r(r) + r(-2) + 3(r) + 3(-2) \quad \text{Use FOIL.}$$

$$= r^2 - 2r + 3r - 6 \quad \text{Multiply.}$$

$$= r^2 + r - 6 \quad \text{Combine like terms.}$$

$$(r + 5)(r^2 + r - 6) \quad \text{Multiply by } (r + 5).$$

$$= r(r^2) + r(r) + r(-6) + 5(r^2) + 5(r) + 5(-6) \quad \text{Distribute.}$$

$$= r^3 + r^2 - 6r + 5r^2 + 5r - 30 \quad \text{Multiply.}$$

$$= r^3 + 6r^2 - r - 30 \quad \text{Combine like terms.}$$

SOLUTION

Multiply $(r - 2)(r + 5)$.

$$(r - 2)(r + 5)$$

$$= r(r) + r(5) + (-2)r + (-2)5 \quad \text{Use FOIL.}$$

$$= r^2 + 5r - 2r - 10 \quad \text{Multiply.}$$

$$= r^2 + 3r - 10 \quad \text{Combine like terms.}$$

$$(r + 3)(r^2 + 3r - 10) \quad \text{Multiply by } (r + 3).$$

$$= r(r^2) + r(3r) + r(-10) + 3(r^2) + 3(3r) + 3(-10) \quad \text{Multiply.}$$

$$= r^3 + 3r^2 - 10r + 3r^2 + 9r - 30 \quad \text{Multiply.}$$

$$= r^3 + 6r^2 - r - 30 \quad \text{Combine like terms.}$$

Math Reasoning

Verify Does multiplying $(r + 3)(r + 5)$ and then multiplying that result by $(r - 2)$ lead to the same result as the two methods shown in Example 3?

Some binomial products appear so frequently that memorizing the patterns they form results in convenience and efficiency while solving problems.

Caution

$(a + b)^2 \neq a^2 + b^2$

$(a - b)^2 \neq a^2 - b^2$

Special Product Patterns	
Sum and difference	$(a + b)(a - b) = a^2 - b^2$
Square of a sum	$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$
Square of a difference	$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$

Example 4 Multiplying Using Special Product Patterns

Multiply.

a. $(2y - 5)^2$

SOLUTION

$(2y - 5)^2$

$= (2y - 5)(2y - 5)$

$= (2y)^2 - 2(2y)(5) + (5)^2$

$= 4y^2 - 20y + 25$

b. $(5x + 2y)(5x - 2y)$

SOLUTION

$(5x + 2y)(5x - 2y)$

$= (5x)^2 - (2y)^2$

$= 25x^2 - 4y^2$

c. $(3q + 2r)^2$

SOLUTION

$(3q + 2r)^2$

$= (3q + 2r)(3q + 2r)$

$= (3q)^2 + 2(3q)(2r) + (2r)^2$

$= 9q^2 + 12qr + 4r^2$

Many problems involving unknown quantities in geometric figures and designs can be solved using polynomials to represent dimensions.

Example 5 Application: Geography

The National Mall in Washington, D.C., forms a rectangle and includes many important sites. The length of the mall from the steps of the U.S. Capitol to the Lincoln Memorial can be expressed as $(x - 0.1)$ miles. The width of the mall is measured across the Washington Monument and can be expressed as $(x - 0.9)$ miles. Write a polynomial to express the area of the National Mall.

SOLUTION

The area of a rectangle is equal to the length times the width.

Area of the mall $= (x - 0.1)(x - 0.9)$

$= x(x) + x(-0.9) + (-0.1)x + (-0.1)(-0.9)$ Use FOIL.

$= x^2 - 0.9x - 0.1x + 0.09$ Multiply.

$= x^2 - x + 0.09$ Combine like terms.

The area of the National Mall is $x^2 - x + 0.09$ square miles.

Lesson Practice

a. (Ex 1) Multiply using the FOIL method. $(a + 7)(a - 4)$

b. (Ex 1) Multiply using the FOIL method. $(4x + 5)(3x - 2)$

c. (Ex 2) Multiply $(a + 9)(2a^2 - 6a + 5)$.

d. (Ex 3) Multiply $(r + 3)(r - 6)(r + 2)$ in two different ways using the Associative Property of Equality.

e. (Ex 4) Multiply $(4y + 3)^2$.

f. (Ex 4) Multiply $(7a + 3b)(7a - 3b)$.

g. (Ex 4) Multiply $(5q + 4r)^2$.

h. (Ex 5) The food court at the Galleria Mall forms a rectangle with eateries on all sides. The length of the food court can be expressed as $(x - 15)$ yards. The width of the food court can be expressed as $(x - 12)$ yards. Write a polynomial to express the area of the food court at the Galleria Mall.

Practice Distributed and Integrated

Simplify each expression.

1. (18) $2.25 \text{ yd} + 7.6 \text{ yd} + 0.58 \text{ yd}$

2. (18) $2.08 \text{ ft} \times 0.033 \text{ ft} \times 15.5 \text{ ft}$

3. (8, 12) **Geometry** Which type of variation is represented by the equation Area = length × width, if the Area = 25 ft^2? Find the constant of variation.

4. (8, 12) **Analyze** Which type of variation is represented by the equation $d = r \times t$ if d is held constant? r is held constant? t is held constant?

5. (18) Kellie's last three walks were 2.25 mi, 3.2 mi, and 2.31 mi. What is the sum of the distances Kellie walked?

6. (17) **Multiple Choice** Solve $|2x + 1| < 4$.

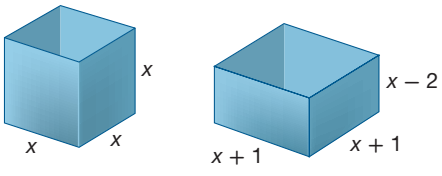
A $-\frac{5}{2} < x < \frac{3}{2}$

B $\frac{3}{2} < x < -\frac{5}{2}$

C $x < -\frac{5}{2}$ or $x > \frac{3}{2}$

D $x < \frac{3}{2}$

***7.** (11, 19) **Packaging Technology** An engineer has a box in the shape of a cube. She wants to know how the volume is affected if she adds 1 unit to each base edge and subtracts 2 units from the height. Does the volume increase, decrease, or remain the same? Explain your answer.



8. (13) Graph to determine the solution of $\begin{aligned} x + 2y &= 2 \\ y &= -\frac{1}{2}x - 3 \end{aligned}$.

9. (18) **Explain** How can feet per second be converted to miles per hour?

*10. (15) **Chemical Mixture** Ocean water is about 3.5% salt. A chemist has a solution that is 25% salt and 75% water. If the chemist needs to make 1600 mL of a solution that is 15% salt, about how much ocean water should the chemist use? Round the answer to the nearest whole number.

a. Set up a system of equations that models the problem.

b. Graphing Calculator Use a graphing calculator to solve the system.

11. (12) **Data Analysis** Each of the data sets shown is either a direct or an inverse variation. For each data set, find the constant of variation.

a.

x	y
1	1.5
2	3
3	4.5
4	6
5	7.5

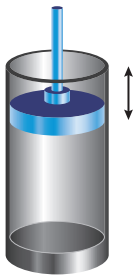
b.

x	y
1	4
2	2
3	$\frac{4}{3}$
4	1
5	$\frac{4}{5}$

c.

x	y
2	$\frac{4}{3}$
3	2
4	$\frac{8}{3}$
5	$\frac{10}{3}$

*12. (12) **Generalize** In the diagram, there is a volume (V) of air in a cylinder with a movable pressure device that applies pressure (P) on the column of air. When downward pressure is applied, the volume of air decreases. When pressure is released by moving the pressure device up, the volume of air increases. What would need to be true for the relationship between P and V to be an inverse variation? What would be the equation to model this situation?



Multiply.

*13. (19) $(8x + 6)(7x - 9)$

*14. (19) $(y - 3)(y^2 + 2y - 5)$

15. (18) **Error Analysis** Two students converted 200 feet per second to miles per hour. Which student is correct? Explain the error in the other student's work.

Student A

$$\frac{200 \text{ ft}}{1 \text{ s}} \times \frac{1 \text{ mi}}{5280 \text{ ft}} \times \frac{360 \text{ s}}{1 \text{ hr}} = \frac{(200)(360)}{5280} \frac{\text{mi}}{\text{hr}}$$

Using a calculator, this simplifies to about 13.6 miles per hour.

Student B

$$\frac{200 \text{ ft}}{1 \text{ s}} \times \frac{1 \text{ mi}}{5280 \text{ ft}} \times \frac{3600 \text{ s}}{1 \text{ hr}} = \frac{(200)(3600)}{5280} \frac{\text{mi}}{\text{hr}}$$

Using a calculator, this simplifies to about 136 miles per hour.

Convert each of the following measurements.

16. (18) $8\ \text{yd}^2$ to in^2

17. (18) 72 hours to seconds

18. (15) **Multiple Representations** The ordered pairs $(-1, -\frac{5}{2})$, $(0, -1)$, $(2, 2)$, and $(4, 5)$ are values in one function of a system that is consistent and dependent. Complete the table to represent ordered pairs of the other function in the same system.

x	0	2	4	6
y				

19. (18) **Multi-Step** While bicycling one weekend, Jess recorded the total distance and time he traveled each time that he stopped. The table shows his records.

Time (hours)	Distance Traveled (miles)
0.5	6
1	12
1.5	18
3	36

a. What was Jess's average speed, in miles per hour, for the total trip?

b. Convert Jess's speed in miles per hour to feet per second.

c. How long, in seconds, would it take Jess to travel 500 feet?

Simplify.

20. (3) $\frac{5x^2y}{7y^5} \cdot \frac{28z}{115x^{-2}y^{-3}}$

21. (3) $\frac{3x^2yz^{-4}}{18xy^{-2}} \div \frac{9y^4}{54x^6z^8}$

***22.** (17) **Chemical Elements** For helium to be a liquid, it must be within 3.0935° of −454.9065°F.

a. Formulate Write an absolute value equation to determine the least and greatest temperatures at which helium will remain a liquid.

b. Solve the equation.

***23.** (17) **Analyze** Write the distance from each point to the origin as an absolute value expression. Then simplify the expression.

a. (5, 0) **b.** (−7, 0) **c.** (0, 9) **d.** (0, −8)

Simplify the expressions.

***24.** (19) $(-3y + 7)^2$

***25.** (19) $(4a + 6b)(4a - 6b)$

26. *(11)* **Generalize** Consider the following polynomials: $P_1 = 2x^5 + x^3$, $P_2 = -2x^5 - x^3 + x$, $P_3 = x^2$, and $P_4 = x^5$. Find the number of terms in each of these sums: $P_1 + P_2$, $P_1 + P_3$, and $P_1 + P_4$. Use the results to make a generalization about the number of terms in the sum of two polynomials compared to the number of terms in the polynomials that make up that sum.

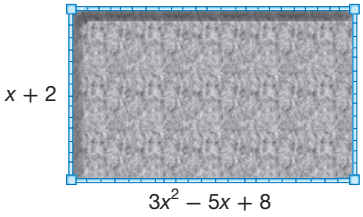
27. *(3)* **Solar System** The mass of the moon is 7.3477×10^{22} kg. If the mass of Earth is 5.9736×10^{24} kg, how many times larger is Earth than the moon?

***28.** *(19)* **Exterior Design** The diagram at the right shows the area in Lucy's backyard that she is preparing for planting.

a. Write an expression for the area of the garden.

b. Simplify the expression using multiplication.

c. Find the area of the garden if $x = 3$ feet.



Calculate the slope of the lines that contain the following pairs of points. Tell whether the line rises, falls, is horizontal, or is vertical.

29. *(13)* $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{3}{4}\right)$

30. *(13)* $\left(-1, \frac{1}{2}\right)$ and $\left(3, \frac{1}{2}\right)$

LESSON
20

Performing Operations with Functions

Warm Up

1. (7) **Vocabulary** A _____ is a mapping between two sets that associates with each element in the domain a unique element in the range.

2. (4) Evaluate $f(x) = 3x + 5$, for $x = -2$.

3. (4) True/False: The points (0, 4), (1, 7), and (1, –2) represent a function.

New Concepts

You can add, subtract, multiply, and divide functions in several ways.

Notation for Operations with Functions

Operation	Notation
Addition	$(f + g)(x) = f(x) + g(x)$
Subtraction	$(f - g)(x) = f(x) - g(x)$
Multiplication	$(fg)x = f(x) \cdot g(x)$
Division	$(f/g)(x) = \frac{f(x)}{g(x)}$, when $g(x) \neq 0$

The domain of the sum of two functions is all the numbers that are common to both of the original domains.

Example 1 Finding the Sum and Difference of Functions Numerically and Algebraically

Given $g(x) = x + 4$; $D = \{\text{Integers}\}$, $h(x) = x - 2$; $D = \{\text{Reals}\}$

a. Find $(h + g)(2)$ numerically.

SOLUTION Note that $D = \{\text{Integers}\}$ is common to both functions. Find $h(2)$ and $g(2)$. Then add.

$$h(2) = 2 - 2 \qquad g(2) = (2) + 4$$
$$h(2) = 0 \qquad g(2) = 6$$
$$(h + g)(2) = (0) + (6) = 6$$

The numerical sum of $(h + g)(2) = 6$.

b. Find the algebraic sum $(h + g)(x)$.

$$h(x) = x - 2$$
$$\underline{g(x) = x + 4}$$
$$h(x) + g(x) = 2x + 2$$

The algebraic sum of $(h + g)(x) = 2x + 2$.

Math Reasoning

Verify Check that $(h + g)(2) = 6$ using the algebraic sum. Show your work.

c. Find $(g - h)(-2)$ numerically.

SOLUTION Find $g(-2)$ and $h(-2)$.

$$g(-2) = (-2) + 4 \qquad h(-2) = (-2) - 2$$
$$g(-2) = 2 \qquad h(-2) = -4$$
$$(g - h)(-2) = (2) - (-4) = 6$$

The numerical difference of $(g - h)(-2) = 6$.

d. Find the algebraic difference $(g - h)(x)$.

$$g(x) - h(x) = (x + 4) - (x - 2) = x + 4 - x + 2 = 6$$

The algebraic difference of $(g - h)(x) = 6$.

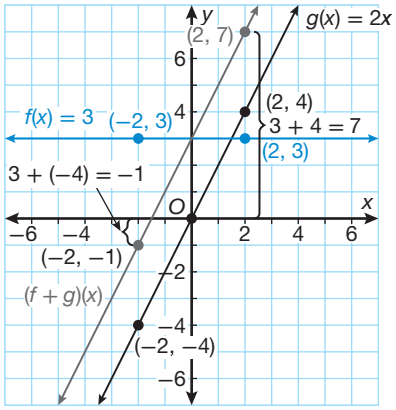
You can also find the sum and difference of functions using a coordinate plane.

Example 2 Finding the Sum and Difference of Functions Geometrically

Given $f(x) = 3$; $D = \{\text{Reals}\}$, $g(x) = 2x$; $D = \{\text{Reals}\}$

a. Find $(f + g)(x)$ geometrically.

SOLUTION



Step 1: Write $(f + g)(x)$ as $f(x) + g(x)$ and replace both notations $f(x)$ and $g(x)$ with y.

Step 2: Graph the equations on the coordinate plane.

Step 3: Given D for both f and g is $\{\text{Reals}\}$, let $x = 2$ and $x = -2$.

Step 4: Find the vertical distance from the x-axis to the y-coordinates $f(2)$ and $g(2)$. Find $(2, 0)$ on the x-axis and count the units vertically up or down to the y-coordinate of each function, f and g. Add the vertical distances from the x-axis to $f(2)$ and $g(2)$: $3 + 4 = 7$. Plot the point $(2, 7)$.

Step 5: Find the y-coordinate of each function at $x = -2$. Add the vertical distances from the x-axis to $f(-2)$ and $g(-2)$: $3 + (-4) = -1$. Plot the point $(-2, -1)$. Draw a line through $(2, 7)$ and $(-2, -1)$.

Math Reasoning

Verify How can the algebraic sum $(f + g)(x)$ be used to verify that the line graphed as the sum of f and g is correct?

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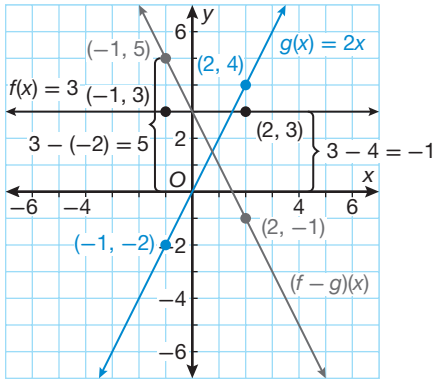
b. Find $(f - g)(x)$ geometrically.



SOLUTION

Step 1: Write $(f - g)(x)$ as $f(x) - g(x)$ and replace both notations $f(x)$ and $g(x)$ with y.

Step 2: Graph the equations.

Step 3: Given D for both f and g is {Reals}, let $x = 2$ and $x = -1$.

Step 4: Find the vertical distance from the x-axis to the y-coordinates $f(2)$ and $g(2)$. Find $(2, 0)$ on the x-axis and count the units vertically up or down to the y-coordinate of each function, f and g. Subtract the vertical distances from the x-axis to $f(2)$ and $g(2)$: $3 - 4 = -1$. Plot the point $(2, -1)$.

Step 5: Repeat the process in **Step 4** for $x = -1$. Plot the point $(-1, 5)$. Draw a line through $(2, -1)$ and $(-1, 5)$.

Example 3 Multiplying Functions Numerically and Algebraically

a. Find $(hg)(-4)$ where $h(x) = x + 3$; $D = \{\text{Reals}\}$, and $g(x) = x - 6$; $D = \{\text{Negative integers}\}$.

SOLUTION

Method 1: Note that the common domain is the negative integers. Find $h(-4)$ and $g(-4)$, then multiply these answers.

$$h(-4) = -4 + 3 \qquad g(-4) = (-4) - 6$$
$$h(-4) = -1 \qquad g(-4) = -10$$
$$(hg)(-4) = (-1)(-10) = 10$$

Method 2: Find $hg(x)$ algebraically. Then find $hg(-4)$.

$$(hg)(x) = (x + 3)(x - 6)$$
$$(hg)(x) = x^2 - 3x - 18$$
$$(hg)(-4) = (-4)^2 - 3(-4) - 18$$
$$(hg)(-4) = 16 + 12 - 18$$
$$(hg)(-4) = 10$$

b. Find $(fg)(-4)$ where $f(x) = x + 3$; $D = \{\text{Reals}\}$, and $g(x) = x - 5$; $D = \{\text{Positive integers}\}$.

SOLUTION

The common domain is the positive integers. Since –4 is a negative integer, $(fg)(-4)$ cannot be found. The problem has no answer.

Math Reasoning

Generalize Find $(f/g)(0)$ for $f(x) = 2x + 1$ and $g(x) = x$. Explain your answer.

Example 4 Dividing Functions Algebraically and Numerically

Find $(f/g)(6)$ if $f(x) = x + 2$; $D = \{\text{Integers}\}$, and $g(x) = x + 3$; $D = \{\text{Positive whole numbers}\}$.

SOLUTION Note that $D = \{\text{Positive whole numbers}\}$ is the common domain.

Method 1: Find $f(6)$ and $g(6)$ and then divide these answers.

$$f(6) = 6 + 2 \qquad g(6) = 6 + 3$$
$$f(6) = 8 \qquad g(6) = 9$$
$$(f/g)(6) = \frac{8}{9}$$

Method 2: Find $(f/g)(x)$ algebraically. Then find $(f/g)(6)$.

$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{x + 2}{x + 3}$$

The algebraic solution for $(f/g)(x)$ is $\frac{x+2}{x+3}$. Now find $(f/g)(6)$.

$$\frac{6 + 2}{6 + 3} = \frac{8}{9}$$

Example 5 Application: Business

Betsy makes memory wire bracelets to earn extra cash. The cost of making a bracelet is given by the function $f(x) = \$2 + \$5.75x$. Her income from the sale of bracelets is given by the function $g(x) = \$10x$. The number of bracelets is represented by x. Find the function representing her profit, $p(x)$. Then find the profit resulting from the sale of 5 bracelets.

SOLUTION Her profit is represented by the function $p(x) = g(x) - f(x)$ since the income from each bracelet, $g(x)$, minus the cost of making each bracelet, $f(x)$, gives the profit, $p(x)$, for each bracelet, x.

$$p(x) = g(x) - f(x) = 10(x) - (2 + 5.75(x))$$
$$10x - 2 - 5.75(x) = 4.25x - 2$$

So $p(x) = 4.25x - 2$. The profit resulting from the sale of 5 bracelets is found by evaluating $p(5)$.

$$p(5) = 4.25(5) - 2 = 21.25 - 2 = 19.25$$

Betsy's profit on 5 bracelets is \$19.25.

Lesson Practice

(Ex 1) **a.** Given $f(x) = 3x - 5$; $D = \{\text{Reals}\}$, $g(x) = x - 3$; $D = \{\text{Integers}\}$ find $(f + g)(6)$ numerically.

(Ex 1) **b.** Given $f(x) = 3x - 5$; $D = \{\text{Reals}\}$, $g(x) = x - 3$; $D = \{\text{Integers}\}$ find the algebraic sum $(f + g)(x)$.

(Ex 1) **c.** Given $f(x) = 7x - 8$; $D = \{\text{Reals}\}$, $g(x) = x + 5$; $D = \{\text{Integers}\}$ find $(f - g)(3)$ numerically.

d. (Ex 1) Given $f(x) = 7x - 8$; $D = \{\text{Reals}\}$, $g(x) = x + 5$; $D = \{\text{Integers}\}$ find the algebraic difference $(f - g)(x)$.

e. (Ex 2) Given $f(x) = 8$; $D = \{\text{Reals}\}$, $g(x) = 3x$; $D = \{\text{Reals}\}$ find $(f + g)(x)$ geometrically.

f. (Ex 2) Given $f(x) = 8$; $D = \{\text{Reals}\}$, $g(x) = 3x$; $D = \{\text{Reals}\}$ find $(f - g)(x)$ geometrically.

g. (Ex 3) Find $(hg)(-4)$ where $h(x) = x + 6$; $D = \{\text{Reals}\}$, and $g(x) = x - 5$; $D = \{\text{Negative integers}\}$.

h. (Ex 3) Find $(fg)(-4)$ where $f(x) = x + 9$; $D = \{\text{Reals}\}$, and $g(x) = x - 4$; $D = \{\text{Positive integers}\}$.

i. (Ex 4) Find $(f/g)(7)$ where $f(x) = x + 3$; $D = \{\text{Integers}\}$, and $g(x) = x + 5$; $D = \{\text{Whole numbers}\}$.

j. (Ex 5) Patty sells pastries to bakeries to earn extra cash. The cost of making one pastry is given by the function $f(x) = \$3 + \$6.25x$. Her income from the sale of pastries is given by the function $g(x) = \$12x$. The number of pastries is represented by x. Find the function representing Patty's profit $p(x)$. Then find her profit from the sale of 6 pastries.

Practice Distributed and Integrated

1. (18) **Write** Explain how to simplify this problem: 12.65 in. + 30.1 in. + 15.843 in.

2. (16) **Multi-Step** The solution to the system of equations $\begin{matrix} x + y = 5 \\ px + qy = -4 \end{matrix}$ is (3, 2).

a. Use Cramer's rule to set up equations for x and y.

b. Set $x = 3$ and $y = 2$. Simplify to write an equation with variables p and q.

c. Graph the line from part **b.**

d. Choose a point (p, q) along the line in part **c.** Substitute those values for p and q into the original system to write a system of equations with variables x and y.

e. Graph your system from part **d** to check your answer. How can you tell if your answer is correct?

Solve the equations.

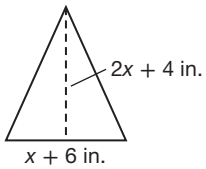
3. (7) $3(5r - 10) = 8(2r - 4) + 5r$

4. (7) $\frac{3}{4}\left(\frac{1}{2}x - 4\right) = (x - 7)$

***5.** (16) **Graphing Calculator** Solve the system using Cramer's Rule. Verify the solution by using a graphing calculator.

$$\begin{cases} 2x + 3y = 54 \\ -x + 6y = 48 \end{cases}$$

6. (19) **Geometry** Write an expression for the area of the triangle and simplify.



*7. (18) **Transportation** The rate of one knot equals one nautical mile per hour. One nautical mile is 1852 meters. What is the speed in meters per second of a ship traveling at 20 knots?

*8. (18) **Bird Flight** The peregrine falcon is perhaps the fastest animal on earth. In a stoop, or dive, the peregrine has been clocked at speeds of over 180 miles per hour. Convert this to feet per seconds.

9. (15) A scientist has one solution that is 10% iodine and another that is 40% iodine. How much of each should the scientist use to produce 100 mL of a mixture that is 25% iodine?

Multiply the polynomials.

10. (19) $(x + 7)(x - 9)$

11. (19) $(x + 3)^2 (x - 2)$

12. (19) **Write** Explain how to multiply $(2x + 8)(3x - 6)$ using the FOIL method.

13. (12) **Justify** Is the equation $xyz = 0$ a joint variation? Why or why not?

*14. (16) **Graphing Calculator** The population of Tulsa, Oklahoma, in 2005 was 381,479 and increased by 1393 people by 2006. The population of Tampa, Florida, in 2005 was 325,800 and increased by 7088 people by 2006. If the population of these cities continued to grow at these rates, then the following equations would represent the population of the cities, where x is the number of years after 2005 and y is the population.

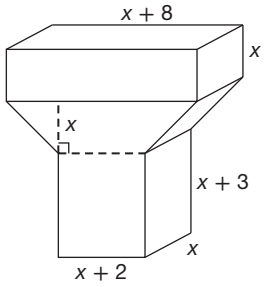
$y = 1393x + 381{,}479$ Tulsa's Population

$y = 7088x + 325{,}800$ Tampa's Population

a. In what year would the populations of Tulsa and Tampa be equal?

b. About how many people would be in each city when the populations are equal?

*15. (11) **Architecture** A concrete support for a highway bridge consists of (from the bottom up) a rectangular prism, a trapezoidal prism, and another rectangular prism, with the dimensions shown. Write a polynomial in standard form to represent the total volume. (Hint**:** The formula for the area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$.)



Find the percent of change in each situation. Round to the nearest tenth.

16. (6) The population changes from 135,000 people to 150,000 people.

17. (6) A price changes from \$2.49 for a cup of tea to \$2.59.

18. (6) The energy usage changes from 25,500 trillion Btu to 22,500 trillion Btu.

Determine the kind of variation, if any, for each equation.

19. (12) $y = \frac{1}{x}$, for $x \neq 0$

20. (12) $z = \frac{1}{xy + a}$, for $x, y \neq 0$ and any value of a

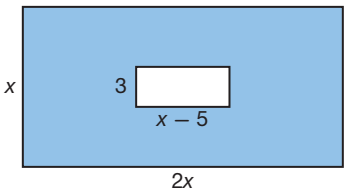
21. *(14)* Write the determinant as an equation, and then solve the equation.

$$\begin{bmatrix} -x & 2 & -2 \\ 0 & -3 & 1 \\ 2x & 8 & -3 \end{bmatrix} = -4x + 10$$

***22.** *(20)* **Error Analysis** Two students were evaluating $(f - g)(3)$, given $f(x) = 5x - 2$ and $g(x) = 9x + 20$. Who is correct? Explain the error.

Student A	Student B
$(f-g)(3) = 5(3) - 2 - (9(3) + 20)$ $= 15 - 2 - 27 - 20$ $= -34$	$(f-g)(3) = 5(3) - 2 - (9(3) + 20)$ $= 15 - 2 - 27 + 20$ $= 6$

23. *(11, 19)* **Geometry** Write a polynomial in standard form that represents the shaded area formed by the two rectangles.



24. *(20)* **Spending and Saving** The amount of money Hannah has saved is given by the function $f(x) = 20x + 100$. Jess, Hannah's sister, has started spending her money and this is given by the function $g(x) = -10x + 400$. The number of weeks the girls are spending and saving is represented by x.

a. Formulate Find the function representing the amount of money that the girls have combined, $h(x)$.

b. Find the amount of money that the girls would have together in 3 weeks.

c. Who will have more money in 15 weeks?

***25.** *(20)* Given $f(x) = 6x + 13$ and $g(x) = 7x - 15$, find $(f + g)(x)$.

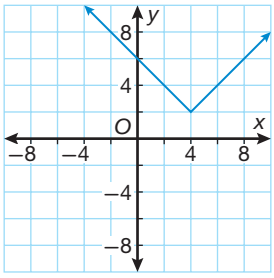
***26.** *(20)* Explain how to find $(f \cdot g)(x)$, given $f(x) = 5x - 9$ and $g(x) = 9x + 8$.

***27.** *(20)* **Multiple Choice** Given $f(x) = -2x + 11$ and $g(x) = 6x - 1$, find $(f + g)(8)$.

A 42 **B** 52 **C** 74 **D** 76

28. *(17)* Solve $|-3x - 4| \geq 2$, then graph the solution. Is $x = 4$ a solution to the inequality?

29. *(17)* **Multiple Representations** Write the absolute value function shown in the graph.



***30.** *(17)* **Chemical Elements** For neon to be a liquid, it must be within 1.2715° of −247.1755°C.

a. Formulate Write an absolute value inequality to determine the least and greatest temperatures at which neon will remain a liquid.

b. Solve the equation.

INVESTIGATION

2

Solving Parametric Equations

House Painting A house painter charges $30 per hour. On average, she paints 200 square feet of a wall per hour.

In t hours, the painter paints x square feet and earns y dollars. The following equations describe the area painted and amount earned in terms of t.

$$x = 200t$$
$$y = 30t$$

1. Suppose the painter works for 7 hours.
   - **a.** How many square feet does she paint?
   - **b.** How much does she earn?
2. Complete the table.

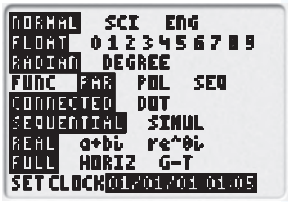
t (hours)	0	1	2	3	4	5
x (square feet)	0	200				
y (dollars)	0	30				

3. **Model** Use the table to write six (x, y) coordinate pairs. (Do not use the t-values.) Graph and connect the points.
4. **Write** Describe the graph. What is the real-world meaning of the point (200, 30)?

Hint

Time is often the parameter when a real-world situation is modeled by parametric equations.

When two variables are expressed in terms of a third variable, the equations used are called **parametric equations.** In the activity above, the parametric equations are $x = 200t$ and $y = 30t$. The variable in both equations, t, is the **parameter.**



5. **Graphing Calculator** A graphing calculator can graph parametric equations.
   - **a.** To put the calculator in parametric mode, press MODE. Use the arrow keys to highlight "PAR," and then press ENTER.
   - **b.** Press the Y= button to enter the parametric equations. In $X1_T$, enter 200, and then press the X,T,θ,n key to insert the variable t. In $Y1_T$, enter 30, and then press X,T,θ,n.
   - **c.** Press the WINDOW key. Enter 0 for Tmin, 10 for Tmax, and 1 for Tstep. Enter 0 for Xmin, 2000 for Xmax, and 500 for Xscl. Enter 0 for Ymin, 300 for Ymax , and 100 for Yscl.
   - **d.** Press GRAPH to graph the equation. Compare the graph to the one from question 3 above.

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A projectile is an object that falls to earth due to the force of gravity. Parametric equations can describe the path of a projectile.

Golf A golfer hits his first shot on a hole. When the golf ball leaves the club, it is traveling 120 feet per second horizontally and 96 feet per second vertically.

The parametric equations that describe the golf ball's motion are

$$x = 120t$$
$$y = -16t^2 + 96t$$

where x is the horizontal distance the golf ball has traveled and y is the height of the ball.

Math Reasoning

Formulate Using the parametric equations describing the path of the golf ball, how can an equation be written for y in terms of x?

Notice where the initial speeds appear in the equations. The initial horizontal speed, 120 ft/s, is the coefficient of t in the equation for x. The initial vertical speed, 96 ft/s, is the coefficient of t in the equation for y.

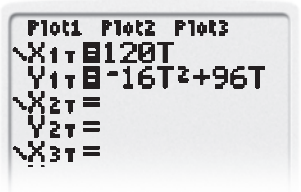
6. Complete the table.

t (seconds)	0	1	2	3	4	5	6
x (feet)	0						
y (feet)	0						

7. Write Use the table to write (x, y) coordinate pairs. Graph and connect the points. Describe the graph.

8. Graphing Calculator Use a graphing calculator to graph the parametric equations.

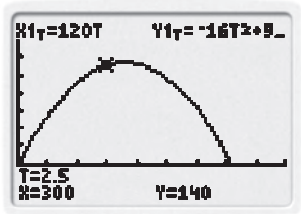
a. Press the Y= button. Enter the parametric equations, using the x^2 button to enter the exponent.



b. Press the WINDOW key. Enter 0 for Tmin, 6 for Tmax, and 0.5 for Tstep. Enter 0 for Xmin, 900 for Xmax, and 100 for Xscl. Enter –50 for Ymin, 200 for Ymax, and 25 for Yscl. Then press GRAPH to see the graph.

c. Press the TRACE key. Notice the blinking cursor at the origin, and the x-, y-, and t-values for that point at the bottom of the screen.

d. Use the right and left arrows to move along the graph. Watch the values of the variables change. When does the golf ball reach its highest point, and what is its height at that time? How long does it take the golf ball to hit the ground, and how many feet does it travel horizontally?



Math Reasoning

Generalize Gravity on the moon is about one-sixth as strong as it is on the earth. Would a golf ball hit in the same way on the moon go higher, farther, or both, than the one in this Activity did on the earth?

Investigation Practice

Suppose t represents a Great Dane puppy's age in months, x is its shoulder height in inches, and y is its weight in pounds. Then these parametric equations approximate the typical growth of a puppy from age 2 months to age 12 months.

$$x = 1.8t + 11.4$$
$$y = 10t$$

a. Complete the table.

t (months)	2	4	6	9	12
x (inches)	15	18.6			
y (pounds)	20	40			

b. Graph the (x, y) coordinate pairs. Describe the graph.

c. **Predict** According to the graph, how heavy will a 24-inch-tall puppy be?

A bus traveling at 40 miles per hour uses 5 gallons of fuel per hour. The fuel tank has a capacity of 60 gallons.

d. Let t be the time in hours, x be the distance traveled, and y be the number of gallons remaining in the fuel tank. Write parametric equations for x and y in terms of t.

e. Make a table of values and graph the (x, y) coordinate pairs. Describe the graph.

f. Use the graph to determine how far the bus can travel on one tank of fuel. How many hours can the bus be driven before it needs refueling?

Graphing Calculator **Suppose the golfer in the activity on page 144 hits a second shot. This time, the horizontal speed of the ball is only 60 feet per second, but the vertical speed is 128 feet per second.**

g. Write a pair of parametric equations for the horizontal distance traveled by the golf ball (x) and the height of the golf ball (y).

h. Graph the equations on a graphing calculator. Adjust Tmax, Xmax, and/or Ymax to see the complete flight of the ball.

i. Use TRACE to move along the graph. What do the coordinates of each point mean?

j. For this shot, when does the golf ball reach its highest point, and what is its height at that time? How long does it take the golf ball to hit the ground, and how many feet does it travel?

k. **Write** Compare the maximum heights and the distances traveled for the two shots. Explain why the differences make sense.

LESSON 21

Solving Systems of Equations Using the Substitution Method

Warm Up

1. (15) **Vocabulary** A _____ of a system is an ordered pair or a set of ordered pairs that satisfy all the equations in the system.
   A check **B** evaluation **C** solution

2. (13) Solve for y: $-6x + 3y = -9$.

3. (2) Distribute $7xy(5x - 3 + 4y)$.

New Concepts

One way a system of equations can be solved is through graphing. Sometimes a solution to a system may be difficult to determine on a graph, especially if the solution is an intersection point with values that are fractions or decimals.

However, there is an algebraic way to find the exact solution of a system of equations. The **substitution** method is a method used to solve systems of equations by solving an equation for one variable and substituting the resulting expression into the other equation.

Example 1 Solving a System of Equations by Substitution

Solve the system of equations by substitution. $\begin{aligned} 2y - x &= -6 \\ y - 2x &= 1 \end{aligned}$

SOLUTION

Step 1: Choose an equation and solve for one of the variables.

$$y - 2x = 1$$

$$y = \boxed{2x + 1}$$ ← This expression will be used in the next step.

Step 2: Substitute the expression for y into the other equation. Solve for x.

$$2\boxed{y} - x = -6$$

$$2(2x + 1) - x = -6 \quad \text{Substitute for } y.$$

$$x = -\frac{8}{3} \quad \text{Solve for } x.$$

Step 3: Substitute the value of x into either equation to solve for y.

$$2y - \boxed{x} = -6$$

$$2y - \left(-\frac{8}{3}\right) = -6$$

$$y = -\frac{13}{3}$$

The solution is $\left(-\frac{8}{3}, -\frac{13}{3}\right)$.

Hint

Check the answer by substituting the x-value and the y-value into both equations. The answer is correct if the values make both equations true.

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If a system of equations forms intersecting lines, then the equations have one common solution. When solving independent systems algebraically, the result is a coordinate. Recall the three types of possible systems.

Classifying Systems			
Type of Lines	**Graph of Lines**	**Type of Solution**	**Type of System**
Intersecting Lines	The lines have one point in common.	Consistent	Independent
Coinciding Lines	The lines overlap and share all points in common.	Consistent	Dependent
Parallel Lines	The lines have no points in common.	Inconsistent	No solution

If a system of equations forms coinciding lines, then the equations are describing the same line. Thus, all of the points on the line are solutions to the system. When solving dependent systems algebraically, the result is a true numerical statement (one without variables) such as $0 = 0$ or $5 = 5$.

Example 2 Recognizing a System of Coinciding Lines

Solve the system of equations by substitution.

$$y = x - 2$$
$$2y = 2x - 4$$

SOLUTION

Step 1: Choose an equation and solve for one of the variables. Use the first equation, since it has been solved for y.

$$y = \boxed{x - 2}$$

Step 2: Substitute the expression for y into the other equation. Solve for x.

$$\begin{aligned} 2\boxed{y} &= 2x - 4 \\ 2(x - 2) &= 2x - 4 \qquad \text{Substitute for } y. \\ 2x - 4 &= 2x - 4 \\ 0 &= 0 \qquad \text{True.} \end{aligned}$$

The substitution produced an equation that is always true. Therefore, these two equations form coinciding lines. Since all of the coordinates on the line are solutions, write the solution as $\{(x, y) \mid y = x - 2\}$.

Hint

If the two equations in the system are rewritten into the slope-intercept form, then it is easier to identify coinciding lines.

If a system of equations forms parallel lines, then the lines have no points in common. When attempting to solve systems with no solution, the result is a false numerical statement such as $0 = 6$ or $-4 = 4$.

Example 3 Recognizing a System of Parallel Lines

Solve the system of equations by substitution. $\begin{aligned} y &= 2x + 2 \\ -2x + y &= -2 \end{aligned}$

SOLUTION

Step 1: Choose an equation and solve for one of the variables. Use the first equation. Since y has been solved for, move to the next step.

$$y = \boxed{2x + 2}$$

Step 2: Substitute the expression for y into the other equation. Solve for x.

$$\begin{aligned} -2x + \boxed{y} &= -2 \\ -2x + (2x + 2) &= -2 && \text{Substitute for } y. \\ +2 &= -2 && \text{False.} \end{aligned}$$

The substitution produced an equation that is always false. Therefore, these two equations form parallel lines. Since parallel lines do not intersect, this system has no solution.

Example 4 Substituting with Fractions

Solve the system of equations by substitution. $\begin{aligned} 3x + 2y &= -3 \\ 4x - 3y &= 13 \end{aligned}$

SOLUTION Choose an equation and solve for one of the variables.

$$\begin{aligned} 3x + 2y &= -3 \\ 2y &= -3x - 3 \\ y &= \frac{-3x - 3}{2} \end{aligned}$$

$$4x - 3\boxed{y} = 13$$

$$4x - 3\left(\frac{-3x - 3}{2}\right) = 13 \qquad \text{Substitute the expression for } y \text{ into the other equation.}$$

$$4x(2) - 3\left(\frac{-3x - 3}{2}\right)(2) = 13(2) \qquad \text{Eliminate the denominator by multiplying every term on both sides by 2.}$$

$$8x - 3(-3x - 3) = 26 \qquad \text{Simplify.}$$

$$x = 1 \qquad \text{Solve for } x.$$

$$3x + 2y = -3 \qquad \text{Choose an equation for substituting the value of } x.$$

$$3(1) + 2y = -3 \qquad \text{Substitute the value for } x \text{ into the equation.}$$

$$y = -3 \qquad \text{Solve for } y.$$

The solution is $(1, -3)$.

Math Reasoning

Analyze How can the expression $\frac{-3x - 3}{2}$ be written so the denominator is eliminated?

Example 5 Application: Demography

Predict This table shows the increase of the population of Maine and of Hawaii from 2004 to 2005. If the populations continue to grow at these constant rates, in what year will these two states have the same population?

State	2004	2005	Increase
Maine	1,317,253	1,321,505	4252
Hawaii	1,262,840	1,275,194	12,354

SOLUTION

1. **Understand** The population of the states and the rate at which the population is increasing are given. Find the year that Maine and Hawaii will have the same population.

2. **Plan** Set up a system of equations and find the point of intersection. Use the substitution method to ensure a better estimate.

3. **Solve** Formulate and solve the system of equations. Let $x = 0$ represent the year 2004.

$$y = 4252x + 1{,}317{,}253$$
$$y = 12{,}354x + 1{,}262{,}840$$

Step 1: Choose an equation and solve for one of the variables.

$$y = \boxed{4252x + 1{,}317{,}253}$$

Step 2: Substitute.

$$\boxed{4252x + 1{,}317{,}253} = 12{,}354x + 1{,}262{,}840$$
$$54{,}413 = 8102x$$
$$x \approx 6.716 \text{ years later}$$

The projected year is sometime during the year of 2010.

4. **Check** Verify your solution by substituting your value for x into each equation.

$$y = 4252\,(6.716) + 1{,}317{,}253 \longrightarrow y \approx 1{,}345{,}809$$
$$y = 12{,}354\,(6.716) + 1{,}262{,}840 \longrightarrow y \approx 1{,}345{,}809$$

About 6.716 years after 2004, Maine and Hawaii should have the same population.

Graphing Calculator

Use the intersection feature described in Lab 3 on p. 85 to verify the solution to this system of equations.

Lesson Practice

a. *(Ex 1)* Solve the system of equations by substitution.
$$y + 3x = -4$$
$$3y - 2x = -2$$

b. *(Ex 2)* Solve the system of equations by substitution.
$$3y = 12 - 6x$$
$$y = -2x + 4$$

c. (Ex 3) Solve the system of equations by substitution. $\begin{aligned} y &= 6x - 5 \\ 2y - 10 &= 12x \end{aligned}$

d. (Ex 4) Solve the system of equations by substitution. $\begin{aligned} 3y - 7x &= 5 \\ 4x - 9y &= 2 \end{aligned}$

e. (Ex 5) The table shows the increase in the number of rushing yards for two football players from 2005 to 2006. If each player's yardage continues to grow at these constant rates, in what year will these two players have the same total rushing yards?

Player	2005	2006	Increase
Corey	733	812	79
Marion	538	654	116

Practice Distributed and Integrated

Use substitution to solve each system of linear equations.

***1.** (21) $\begin{aligned} 3x + y &= 10 \\ -2x - 5y &= -11 \end{aligned}$

***2.** (21) $\begin{aligned} -x + 3y &= 10 \\ 3x + 4y &= -4 \end{aligned}$

3. (17) Graph $|-2x - 1| \geq 3$.

***4.** (20) Find $(f + g)(3)$ if $f(x) = -2x + 4$ and $g(x) = -(x^2)$.

5. (9) Find the product. $\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & -1 \\ 1 & 5 & 3 \end{bmatrix}$

***6.** (21) **Verify** Use the system of equations: $\begin{aligned} 2y - 4x &= 6 \\ y + 5x &= -10 \end{aligned}$.

a. Solve the system of equations by solving the first equation for y, then substituting.

b. Verify your answer by solving the second equation for y, then substituting. Show your work.

Solve each equation.

7. (17) $|x + 3| = 16$

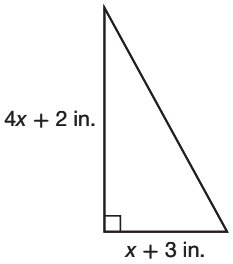
8. (17) $|2x + 3| = 2 - x$

9. (17) $2|x + 6| = 8x$

***10.** (20) **Error Analysis** Two students evaluated $(f - g)(2)$, given $f(x) = 5x - 2$ and $g(x) = 8x + 10$. Who is correct? Explain the error.

Student A	Student B
$\begin{aligned}(f - g)(2) &= 5(2) - 2 - (8(2) + 10) \\ &= 10 - 2 - 16 - 10 \\ &= -18\end{aligned}$	$\begin{aligned}(f - g)(2) &= 5(2) - 2 - (8(2) + 10) \\ &= 10 - 2 - 16 + 10 \\ &= 2\end{aligned}$

*11. (19) **Geometry** Write an expression for the area of the triangle.



12. (14) Evaluate the determinant: $\begin{vmatrix} -2 & 6 \\ 1 & 5 \end{vmatrix}$

*13. (16) **Multiple Choice** Phyllis collects silver dollars and half-dollars. She has 5 times as many half-dollars as silver dollar coins. She has 192 coins. Which solution gives the number of silver dollars Phyllis has?

A $\dfrac{\begin{vmatrix} 192 & 1 \\ 0 & -5 \end{vmatrix}}{6}$ **B** $\dfrac{\begin{vmatrix} 1 & 192 \\ -5 & 0 \end{vmatrix}}{-6}$ **C** $\dfrac{\begin{vmatrix} 1 & 192 \\ -1 & 0 \end{vmatrix}}{6}$ **D** $\dfrac{\begin{vmatrix} 192 & 1 \\ 0 & 1 \end{vmatrix}}{-6}$

14. (10) **Multiple Choice** Tickets to the community theatre cost $20. It is also possible to purchase a season pass for $170. Use an inequality to determine how many productions you can attend at the $20 price before spending more than it costs for the season pass.

A 7 **B** 8 **C** 9 **D** 10

15. (20) Given $f(x) = 4x - 1$ and $g(x) = 2x - 1$, find the sum $(f + g)(x)$.

16. (6) **Estimate** Estimate the new amount when 14.082 is increased by 9.68%. Explain your reasoning.

*17. (21) **Formulate** Bill leaves his house for Makayla's house riding his bicycle at 8 miles per hour. At the same time, Makayla leaves her house heading toward Bill's house walking at 3 miles per hour. Write a system of equations that could be used to determine how long it takes for them to be the same distance d from Makayla's house in h hours. They live 8.25 miles apart.

18. (14) **Computer Graphics** A computer programmer wants a shape to move and expand on the screen. The programmer decides to multiply the coordinates of each vertex of the shape by the matrix $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$. The area of the shape is multiplied by a factor that equals the determinant of this matrix. By what factor is the area multiplied? Explain.

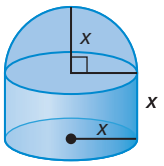
19. (11) Write the polynomial $3x + 6x^5 - 3x^2 + 43 - 5x^3$ in standard form.

20. (11) **Measurement** This figure consists of a cylinder with a hemisphere on top. Write a polynomial in terms of x that represents the total volume of the figure. Formulas for volume are given below.

Cylinder: $V = \pi r^2 h$

Sphere: $V = \frac{4}{3}\pi r^3$

(Hint: The figure has only one circular base.)



21. (12) **Justify** Is the equation $abc = 0$ a joint variation? Why or why not?

22. (9) **Basketball** The table shows the types of shots made in a game by the three top scoring L.A. Lakers and the point value of each type of shot. Use matrix multiplication to find out which player scored the most points in the game.

Shots Made

Player	3 Point	2 Point	1 Point
K. Bryant	1	5	6
M. Evans	2	5	5
L. Walton	1	7	1

23. (13) **Pyramids** The Great Pyramid of Giza has a height of 138 m and a horizontal distance from the side to the center of about 115 m. What is the slope of the pyramid to the nearest tenth?

24. (19) Simplify the product: $\frac{a^{-3}x}{y^{-3}}\left(\frac{xxx^{-2}}{y^{-2}yy} - 3\right)$

25. (8) **Planets** The weight of an object on Mercury varies directly with the weight of that object on Earth. An object that weighs 150 lb on Earth weighs 57 lb on Mercury. Find the weight of an object on Mercury that weighs 25 lb on Earth.

26. (7) Margaret wants to determine her average speed on her way to work. She knows she drives for about 10 minutes through her neighborhood, then $\frac{1}{2}$ hour on the highway, and finally 10 minutes to her office. Her odometer tells her the total distance from her home to her work is 40 miles. Write and solve an equation to find her average speed in miles per hour.

27. (1) Your dinner total at your favorite restaurant is \$24. Use the distributive property to mentally calculate a 15% tip.

***28.** (9) **Graphing Calculator** Find the product. Check using the graphing calculator.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} -2 & 3 & -4 \\ 0 & -1 & 1 \end{bmatrix}$$

***29.** (19) **Multi-Step** The length of a regulation basketball court is represented by the expression $(x + 12)$ feet. The width of a regulation basketball court is represented by the expression $(x - 32)$ feet.

a. Write an expression that represents the area of a regulation NBA basketball court.

b. Evaluate the area of the court if $x = 82$.

30. (10) Solve and graph the inequality: $-2(x + 3) \leq 4$.

LAB
5

Storing and Plotting a List of Data

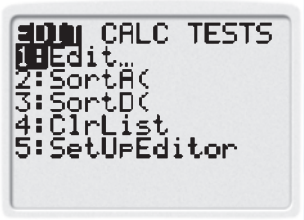
Graphing Calculator Lab ***(Use with Lesson 22)***

A graphing calculator can be used to store and plot data. Use the graphing calculator to analyze the following two sets of data.

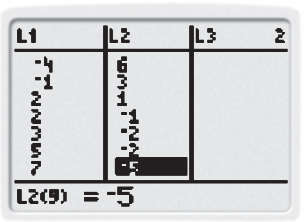
Data Set 1	Data Set 2
–5	7
–4	5
–4	6
–1	3
2	1
2	–1
3	–2
5	–2
7	–5

Storing a List of Data

1. Enter the Data Set 1 into the calculator.
   - **a.** Press STAT to open the statistics menu. **[1: Edit…]** will already be highlighted. Press ENTER.
   - **b.** If there is data already displayed in the lists, then clear the list. To clear L1 use the arrow keys ◄ ▲ ► ▼ to highlight the L1 column heading and press CLEAR ENTER. Repeat the step to clear L2 if necessary.



   - **c.** Once the columns are empty, use the arrow keys ◄ ▲ ► ▼ to move to the first entry in the L1 column. Press (–) 5 ENTER to enter the first data point. Continue inputting the data set, pressing ENTER after each number.



2. Enter the Data Set 2 into the calculator.
   - **a.** Once L1 is inputted, press ► to move to the L2 column and press 7 ENTER to enter the first data point. Continue inputting data set 2, pressing ENTER after each number.
   - **b.** The data sets are now inputted as L1 and L2 and can be analyzed and plotted.

Plotting Data

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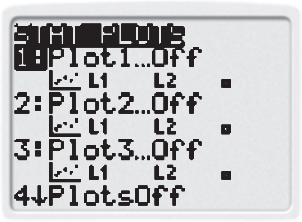
Before plotting the data sets, make sure that the graph is clear. Press Y= and use the arrow keys ◄ ▲ ► ▼ to select any equations in the Y= equation editor. Press CLEAR ENTER to delete them.

1. Turn on the plot feature.



   a. Press 2nd MODE to return to the home screen. Press 2nd Y= to open the [**STAT PLOT**] menu. Press ENTER to open the first Stat plot.

   b. Press ENTER to turn Plot1 on. This will display the plot on the graph.

2. Customize the graph as a scatter plot.

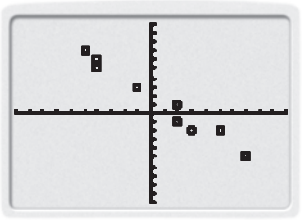
   a. Press ▼ to move to **[Type:]**. The first graph should be highlighted. This will display the data as a scatter plot. (If not, use the arrow keys ◀ ▲ ▶ ▼ to highlight the scatter plot and press ENTER.)

   b. Press ▼ to move to **[Xlist:]**. L1 should be displayed, meaning it will use the data entered from data set 1 for the x-values. (If L1 is not displayed, press 2nd 2 ENTER.)

   c. Press ▼ to move to **[Ylist:]**. L2 should be displayed, meaning it will use the data entered from data set 2 for the y-values. (If L2 is not displayed, press 2nd 2 ENTER.)

   d. Press ▼ to move to **[Mark:]**. Make sure the first symbol, the square, is highlighted. This means that the data points in the scatter plot will be displayed as squares.

3. Plot the data on the graph.



   a. Press GRAPH. The scatter plot of data sets 1 and 2 is now displayed.

Graphing Calculator Practice

1. Use the graphing calculator to enter the following data sets as L1 and L2.

2. Use the graphing calculator to make a scatter plot, with L2 as x-values and L1 as y-values.

Data Set 1	Data Set 2
5	–8
4	–6
4	–5
5	–2
5	1
4	3
4	6

LESSON
22

Analyzing Continuous, Discontinuous, and Discrete Functions

Warm Up

1. (4) **Vocabulary** The vertical line test states that a relation is a function if any vertical line intersects the graph of the relation no more than _____.

2. (4) Identify the domain and range of $\{(5, 2), (7, 0), (-1, 9)\}$.

3. (4) Identify the range of $f(x) = 2x + 7$ given the domain of $\{0, 3, 4\}$.

New Concepts

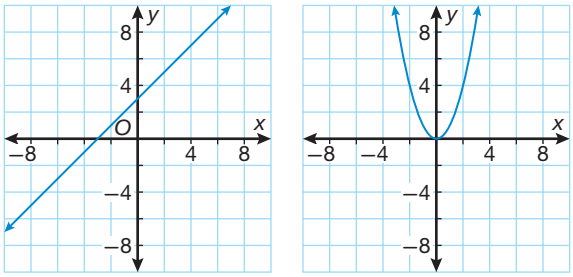
A relation pairs input values x from the domain with output values y from the range. A function is a specific type of relation where every value in the domain must be matched to only one value in the range.

Hint

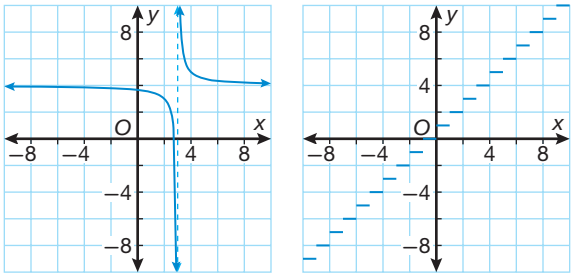
A graph of a continuous function can be drawn without lifting the pencil.

A **continuous function** is a function with a graph that has no gaps, jumps, or asymptotes. Any function that has gaps, jumps or asymptotes is considered a **discontinuous function.** A **discrete function** is a type of function made up of separate, disconnected points.

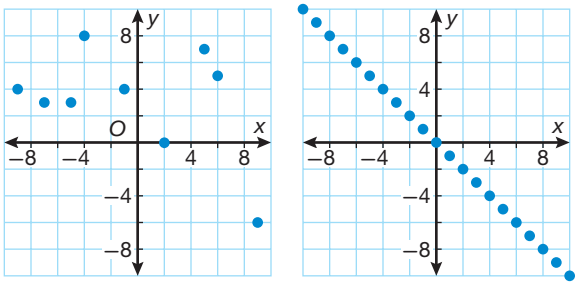
Continuous Functions



Discontinuous Functions



Discrete Functions



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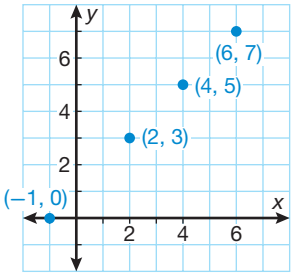
Example 1 Graphing Discrete and Continuous Functions

Graph the function and state the domain and range of:

a. the discrete function

$f = \{(2, 3), (4, 5), (6, 7), (-1, 0)\}$.

SOLUTION

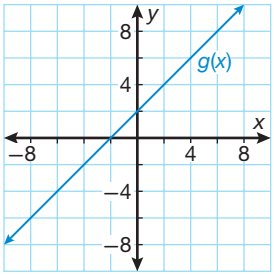


Domain: $\{2, 4, 6, -1\}$
Range: $\{3, 5, 7, 0\}$

b. the continuous function

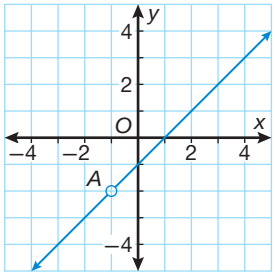
$g(x) = x + 2$.

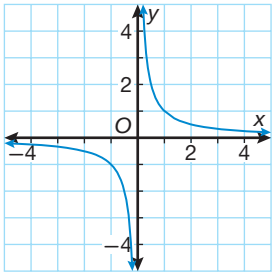
SOLUTION



Domain and Range: all real numbers

Discontinuous functions have specific places where they are not continuous.



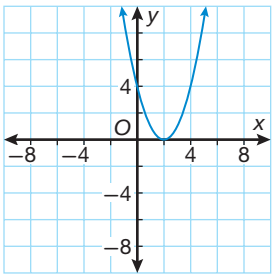


The point A above is called a **point of discontinuity** because it is the point at which the graph of the relation or function is not connected. The second graph above is discontinuous at the line $x = 0$. Therefore, the second graph is discontinuous but has no point of discontinuity.

Example 2 Identifying Types of Functions

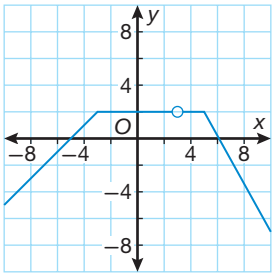
Determine if each graph is continuous, discontinuous, or discrete. If the function is discontinuous, then name the x-value(s) where the discontinuity occurs.

a.



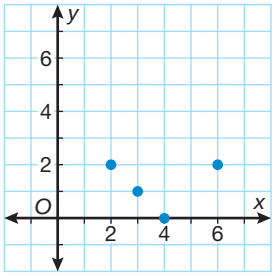
SOLUTION Continuous function

b.

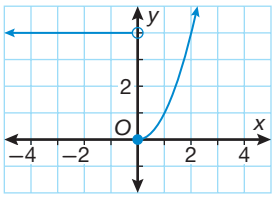


SOLUTION Discontinuous function; point of discontinuity at about (3, 2).

c.



SOLUTION Discrete function



The vertical line test can be used on continuous, discontinuous, and discrete relations to determine if the relations are functions. The discontinuous relation shown at right is a function because it passes the vertical line test even at the point of discontinuity. Recall that to pass the vertical line test, any vertical line can only intersect the graph at one point. The domain of this function is all real numbers and the range is $y \geq 0$.

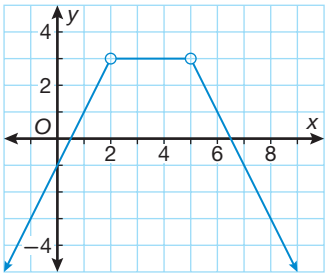
Example 3 Identifying Functions and Domain and Range

Determine if each graph is continuous, discontinuous, or discrete. Determine if it is a function or a relation. Determine the domain and range of each.

Hint

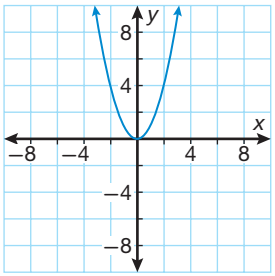
To find the domain, imagine the graph pressed to the *x*-axis. How much of the *x*-axis is covered?

a.



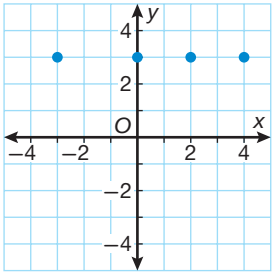
SOLUTION Discontinuous function; Domain: All real numbers except $x = 2, 5$; Range: $y \leq 3$

b.



SOLUTION Continuous function; Domain: All real numbers; Range: $y \geq 0$

c.



SOLUTION Discrete function; Domain: $x = -3, 0, 2, 4$; Range: $y = 3$

Math Reasoning

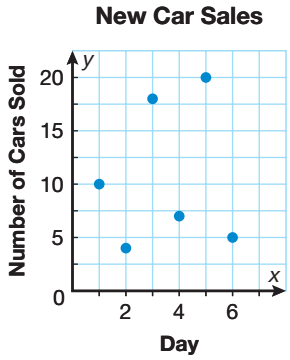
Verify Use a graphing calculator to store and plot the points. Then compare the display to the graph in the solution.

Example 4 Application: Consumer Math

A company records its sales of new cars for a six day period. The results are listed in the table below. Graph the points. Identify the domain and range. Show that the graph is a function by using the vertical line test and analyzing the domain and range.

Day	Number of Cars Sold
1	10
2	4
3	18
4	7
5	20
6	5

SOLUTION Domain: 1, 2, 3, 4, 5, 6; Range: 10, 4, 18, 7, 20, 5; A vertical line may be placed on the graph to show that the relation passes the vertical line test and therefore is a function. Examining the domain and range indicates that for every x there is exactly one y.

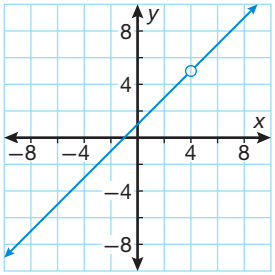


Lesson Practice

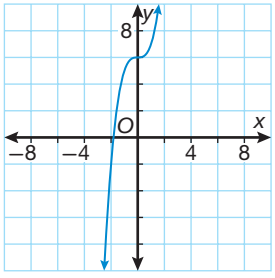
(Ex 1) **a.** Graph and state the domain and range of the discrete function $G = \{(0, 0), (2, 3), (-4, 6), (9, 10)\}$.

(Ex 1) **b.** Graph and state the domain and range of the continuous function $h(x) = 5$.

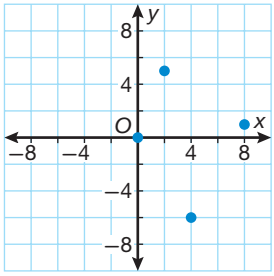
(Ex 2) **c.** Determine if the graph is continuous, discontinuous, or discrete. If the function is discontinuous, name the x-value(s) where the discontinuity occurs.



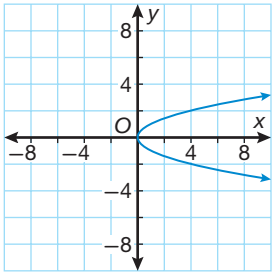
(Ex 2) **d.** Determine if the graph is continuous, discontinuous, or discrete. If the function is discontinuous, name the x-value(s) where the discontinuity occurs.



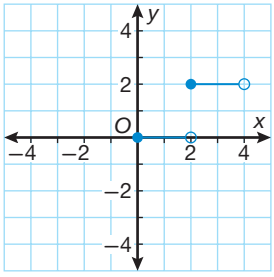
e. *(Ex 2)* Determine if the graph is continuous, discontinuous, or discrete. If the function is discontinuous, then name the x-value(s) where the discontinuity occurs.



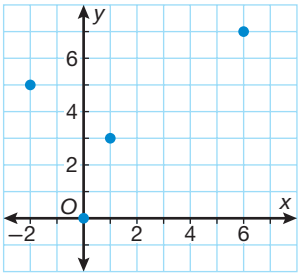
f. *(Ex 3)* Determine if the graph is continuous, discontinuous, or discrete. Determine if it is a function or a relation. Determine the domain and range.



g. *(Ex 3)* Determine if the graph is continuous, discontinuous, or discrete. Determine if it is a function or a relation. Determine the domain and range.



h. *(Ex 3)* Determine if the graph is continuous, discontinuous, or discrete. Determine if it is a function or a relation. Determine the domain and range.



i. *(Ex 4)* A restaurant's carryout menu list meals by numbers. Beside each meal is the total number of side orders that can be added to a particular meal. The results are listed in the table below. Graph the points. Identify the domain and range. Show that the graph is a function by using the vertical line test and analyzing the domain and range.

Meal	Number of Sides
1	2
2	0
3	3
4	3
5	1

Practice Distributed and Integrated

1. (7) **Population** According to the 2000 census the three largest states by population in the U.S. are California, Texas, and New York. The total population of the three states is 73,699,925. The population of California is 13,019,828 more than the population of Texas and the population of New York is 1,875,363 less than the population of Texas. Find the population of Texas.

Graph and state the domain and range of each function. State whether the function is discrete or continuous.

***2.** (22) $\{(1, -2), (3, 0), (-6, 5), (6, 3)\}$

***3.** (22) $f(x) = 3x$

***4.** (20) Find $(f)(g)(-2)$ if $f(x) = 3x + 1$ and $g(x) = x + 6$.

5. (16) Use Cramer's Rule to solve the system of equations. $\begin{aligned} 3x + 7y &= -18 \\ -4x + 11y &= -37 \end{aligned}$

6. (Inv 2) **Rocket** A physics class is about to test their first rocket out on the football field. The teacher fires the rocket at a 75° angle. At ignition, the rocket travels 38 feet per second horizontally and 110 feet per second vertically.

In t seconds, the rocket will travel x and y feet. The parametric equations that the rocket will travel are

$$\begin{aligned} x &= 38t \\ y &= -16t^2 + 110t \end{aligned}$$

a. How many seconds does it take the rocket to hit the ground? Round your answer to the nearest thousandth of a second.

b. How far does the rocket travel?

c. Approximately how far has the rocket traveled when it has reached its highest point in flight? Round your answer to the nearest foot.

State the degree of each polynomial.

7. (11) $x^4 + 2x - x^7$

8. (11) $5x^2 + 2x^5 - 3$

***9.** (19) The Temple of Kukulcan, located in the ancient Mayan city of Chichen Itza, was built on a square base. If the expression $(4x + 12)$ represents one side of the base, write an expression that represents the area of the base.

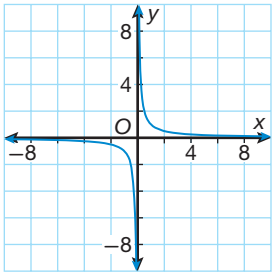
10. (13) **Mountains** Mt. McKinley in Alaska is the highest mountain in North America with an elevation of 20,320 ft. From its center the mountain runs horizontally approximately 119,727 ft. Find the approximate slope of Mt. McKinley. Round your answer to the thousandths place.

Simplify the expressions. Use negative exponents when necessary.

11. (3) $\dfrac{(4x)^{-2}y^0(y^{-2})^2y}{32^{-1}x^2(yx^0)^{-3}}$

12. (3) $\dfrac{9x^{-4}(y^3x^2)^{-2}}{2^{-3}x^2y^3x(x^{-4}y^2)}$

*13. (22) James examines the graph below and states that the domain of the function is all real numbers. Identify the error in his decision.



14. (17) **Error Analysis** Two students solve the inequality $32 + \frac{|x-7|}{13} < 7$ and their answers are below.

Which student is correct? Explain how you know.

Ben: No solution.

Ralph: $x = 332$ or $x = 318$.

Calculate the percent of change.

15. (6) from 5 to 8.2

16. (6) from 5 to 9.4

17. (Inv 2) **Rugby** Jonny Wilkinson, the fly half of the England rugby team, is at the 41 meter line on the rugby pitch. He runs up to kick the ball for a field goal. The ball moves at a horizontal speed of 11 meters per second and a vertical speed of 23 meters per second.

In t seconds, the rugby ball will travel x and y meters. The parametric equations that the ball will travel are

$$x = 11t$$
$$y = -4.9t^2 + 23t$$

a. How many seconds does it take for the ball to get to the field goal post? Round to the nearest hundredth of a second.

b. The field goal posts are 1.2 meters tall. Assume Jonny kicks the ball directly center of the field goal. Will the ball travel through the goal posts? Explain why or why not.

*18. (21) **Multi-Step** Kate and Riley are reading the same book. Kate reads $\frac{1}{3}$ page per minute, and Riley reads $\frac{3}{4}$ page per minute. Kate has already read 70 pages, while Riley has read 30 pages. If they both resume reading together, eventually Riley will catch up to Kate.

a. On what page will that occur?

b. How many minutes have they read when Riley catches up?

19. (8) **Engines** Two-stroke engines require a pre-mix of gas and oil. The amount of gas used varies directly with the amount of oil used. In most two-stroke engines 3 gal of gas requires 19.2 oz of oil.

a. Write the proportion that can be used to find how many ounces of oil to use with 5 gal of gas.

b. Solve the proportion in part **a.**

c. Check the answer from part **b.** with the proportion from part **a.**

20. (8) Convert 420 inches to feet.

21. (14) **Geometry** If one corner of a parallelepiped is at the origin, then all the vertices of the shape can be determined by knowing that the three other vertices with edges to the origin are at (1, 3, 4), (−3, 2, 1), and (4, −2, 3). The volume of the parallelepiped is the determinant of the matrix with these coordinates as rows. Find the volume of the parallelepiped.

22. (9) **Multiple Choice** The following matrix is the product of which two matrices? [19]

A $\begin{bmatrix}6 & -1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}$ **B** $\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}6 & -1\end{bmatrix}$ **C** $\begin{bmatrix}1 & 0 & 3\end{bmatrix}\begin{bmatrix}4\\-2\\5\end{bmatrix}$ **D** $\begin{bmatrix}4\\-2\\5\end{bmatrix}\begin{bmatrix}1 & 0 & 3\end{bmatrix}$

23. (12) **Justify** Is the equation $xy = 0$ an inverse variation? Why or why not?

Use substitution to solve the system of equations.

***24.** (21) $\begin{aligned}-x + y &= -2\\ x + y &= 8\end{aligned}$

***25.** (21) $\begin{aligned}3x + y &= 4\\ -3x + y &= 8\end{aligned}$

26. (10) **Write** Write a compound inequality using *and* and a compound inequality using *or*. Explain the difference when you connect two inequalities with *and* and when you connect two inequalities with *or*.

27. (8) Find the constant of variation if x varies directly with y and $x = 28$ when $y = 7$.

***28.** (6) **Graphing Calculator** The US minimum wage increased from \$0.75 in 1950 to \$5.15 in 2000. Use your calculator to find the percent increase to the nearest percent.

29. (1) **Justify** Simplify: $3 \cdot 4 \cdot 7$. Justify your answer by identifying which property you used for each step.

30. (13) **Multi-Step** Use Point 1 (−8, −3) and Point 2 (1, −8).

a. Find the slope of the line that goes through Point 1 and Point 2.

b. Based on the slope, is the line rising or falling?

c. Sketch a graph of the line that goes through Point 1 and Point 2.

d. Based on the graph, is the line rising or falling?

LESSON
23

Factoring Polynomials

Warm Up

1. (SB) **Vocabulary** Numbers or expressions that are multiplied to form a product are called _____.
2. (SB) Write all the positive integer factor pairs for the product 15.
3. (SB) Write all the integer factor pairs for the product −35.
4. (SB) Write all the integer factor pairs for the product 16. Use both positive and negative factors.

New Concepts

To completely **factor** a number means to express it as a product of prime factors. For example, $18 = 2 \cdot 3 \cdot 3$, or $2 \cdot 3^2$. Similarly, to completely factor a polynomial of two or more terms means to express it as a product of prime polynomial factors. A **prime polynomial** is a polynomial that cannot be factored. Use what you know about multiplying polynomials to factor them. As the first step in factoring any polynomial, factor out the **greatest common monomial factor** if it is other than 1.

Math Language

Unless otherwise specified, *factor a polynomial* means *completely factor the polynomial.*

Example 1 Factoring Out the Greatest Common Monomial Factor

Factor each polynomial.

a. $3x^2 + 6x - 3$

SOLUTION

$3x^2 + 6x - 3 = 3 \cdot x \cdot x + 2 \cdot 3 \cdot x - 3 \cdot 1$ 3 is the GCF.

$= 3(x^2 + 2x - 1)$

$x^2 + 2x - 1$ cannot be factored, so the factoring is complete. Check by using the distributive property to multiply.

Check $3(x^2 + 2x - 1) = 3x^2 + 6x - 3$

b. $-4x^3 - 10x$

SOLUTION When the leading coefficient is negative, include the negative in the monomial that you factor out. This will yield a positive leading coefficient in the polynomial inside the parentheses.

$-4x^3 - 10x = -2 \cdot 2 \cdot x \cdot x \cdot x - 2 \cdot 5 \cdot x$ $2x$ is the GCF.

$= -2x(2x^2 + 5)$ Factor out $-2x$. Then $2x^2 + 5$ has a positive leading coefficient.

Online Connection
www.SaxonMathResources.com

The following table shows some useful factoring strategies to try after factoring out the greatest common monomial factor (if it is other than 1).

Factoring Polynomials where b, c, u and v are nonzero integers.		
Type of Polynomial	**Factoring Pattern**	**Examples**
Basic Quadratic Trinomial	$x^2 + bx + c = (x + u)(x + v)$ $u + v = b$ and $uv = c$	$x^2 + 9x + 20 = (x + 4)(x + 5)$ $4 + 5 = 9$ and $4 \cdot 5 = 20$
Perfect Square Trinomial	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$	$x^2 + 8x + 16 = (x + 4)^2$ $x^2 - 6x + 9 = (x - 3)^2$
Difference of Two Squares	$a^2 - b^2 = (a + b)(a - b)$	$x^2 - 9 = (x + 3)(x - 3)$

Example 2 Factoring a Perfect Square Trinomial

a. Factor $x^2 - 10x + 25$.

SOLUTION

In a perfect square trinomial, the square of half the coefficient of the linear term is the constant term. So, $x^2 - 10x + 25$ is a perfect square trinomial because half of -10 is -5, and the square of -5 is 25. Use the factoring pattern $a^2 - 2ab + b^2 = (a - b)^2$.

$$a^2 - 2ab + b^2 = (a - b)^2$$
$$x^2 - 10x + 25 = (x - 5)^2$$
$$x^2 - 10x + 25 = (x - 5)^2$$

Check $(x - 5)^2 = (x - 5)(x - 5)$
$= x^2 - 5x - 5x + 25$
$= x^2 - 10x + 25$ ✓

b. Factor $4m^2 + 12mn + 9n^2$.

SOLUTION

Use the factoring pattern $a^2 + 2ab + b^2 = (a + b)^2$

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$4m^2 + 12mn + 9n^2 = (2m + 3n)^2$$

Check $(2m + 3n)^2 = (2m + 3n)(2m + 3n)$
$= 4m^2 + 6mn + 6mn + 9n^2$
$= 4m^2 + 12mn + 9n^2$

Hint

In the quadratic trinomial $x^2 - 10x + 25$, x^2 is the quadratic term, $-10x$ is the linear term, and 25 is the constant term.

Example 3 Factoring a Difference of Two Squares

Factor $x^2 - 49$.

SOLUTION $x^2 - 49 = x^2 - 7^2$, so it is a difference of two squares.

$$x^2 - 49 = (x + 7)(x - 7)$$

Check $(x + 7)(x - 7) = x^2 - 7x + 7x - 49$

$$= x^2 - 49 \quad \checkmark$$

The Zero Product Property says that if a product is zero, then at least one of its factors is zero. You can use this property to solve some polynomial equations.

Hint

The Zero Product Property applies to any number of factors. If $abc = 0$, then $a = 0$ or $b = 0$ or $c = 0$.

Zero Product Property
Let a and b be real numbers. If $ab = 0$, then $a = 0$ or $b = 0$.

Example 4 Solving a Basic Quadratic Trinomial Equation

Solve $x^2 + 3x - 10 = 0$.

SOLUTION First factor the quadratic polynomial $x^2 + 3x - 10$. Find two numbers whose product is -10 and whose sum is 3. The factor pairs for -10 are $(1)(-10)$, $(-1)(10)$, $(2)(-5)$, and $(-2)(5)$. The factor pair whose sum is 3 is $(-2)(5)$. So, the factored expression for $x^2 + 3x - 10$ is $(x - 2)(x + 5)$. Now use the Zero Product Property to solve the equation.

$x^2 + 3x - 10 = 0$	Original equation
$(x - 2)(x + 5) = 0$	Factor.
$(x - 2) = 0$ or $(x + 5) = 0$	Zero Product Property
$x = 2$ or $x = -5$	Solve each equation.

Check To check a solution, substitute it for the variable in the original equation. If it makes the equation true, it is a correct solution.

$$x^2 + 3x - 10 = 0 \qquad x^2 + 3x - 10 = 0$$
$$2^2 + 3(2) - 10 \stackrel{?}{=} 0 \qquad (-5)^2 + 3(-5) - 10 \stackrel{?}{=} 0$$
$$4 + 6 - 10 \stackrel{?}{=} 0 \qquad 25 - 15 - 10 \stackrel{?}{=} 0$$
$$0 = 0 \quad \checkmark \qquad 0 = 0 \quad \checkmark$$

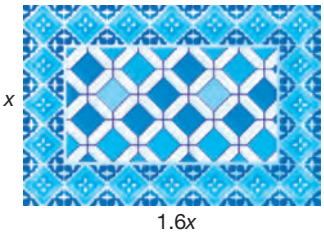
The solutions are 2 and -5 because they both make the original equation true.

Math Reasoning

Write The golden ratio is the number $\frac{1+\sqrt{5}}{2}$. Explain why it is irrational. Then approximate its value to the nearest thousandth.

Example 5 Application: The Golden Ratio in Art

The Golden Ratio is approximately 1.6 to 1. It has been used by artists for centuries to create visually pleasing art works. An artist wants to make a rectangular tapestry with an area of 90 square feet whose length and width are in the ratio 1.6 to 1. What length and width should she use?



SOLUTION

Understand The dimensions of the rectangle can be represented by x and $1.6x$.

Plan Write and solve an equation for the area of the rectangle.

Solve

$(1.6x)(x) = 90$ Length times width equals area.

$1.6x^2 = 90$ Multiply both sides by 10 to eliminate decimals.

$16x^2 = 900$

$16x^2 - 900 = 0$ Subtract 900 from both sides.

$(4x - 30)(4x + 30) = 0$ Factor the difference of two squares.

$4x - 30 = 0$ or $4x + 30 = 0$ Apply the Zero Product Property.

$x = 7.5$ or $x = -7.5$ Solve each equation. Use the positive solution 7.5.

$x = 7.5$ and $1.6x = (1.6)(7.5) = 12$.

The artist should use the dimensions 7.5 feet and 12 feet.

Check The ratio is correct: $\frac{12}{7.5} = 1.6$. The area is correct: $(12)(7.5) = 90$. ✓

Lesson Practice

(Ex 1) **a.** Factor $12x^2 + 4x + 8$.

(Ex 1) **b.** Factor $-20x^3 + 5x^2$.

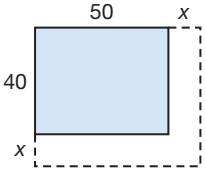
(Ex 2) **c.** Factor $x^2 + 12x + 36$.

(Ex 2) **d.** Factor $9x^2 + 30xy + 25y^2$.

(Ex 3) **e.** Factor $x^2 - 1$.

(Ex 4) **f.** Solve $x^2 - 10x + 21 = 0$.

(Ex 5) **g.** A rectangular wooden deck is 50 feet long and 40 feet wide. The owner wants to increase both dimensions by the same amount, resulting in a 50% increase in area. Write and solve an equation to find out how much he should increase each dimension.



Practice Distributed and Integrated

Evaluate each expression.

1. (2) $-a(a-b)$ if $a=-\frac{1}{2}$ and $b=\frac{1}{3}$

2. (2) $-xy(-x^2-y)$ if $x=-\frac{1}{2}$ and $y=\frac{1}{4}$

3. (4) Determine whether $f(x)=3x+x^2$ is a function.

4. (18) Convert 120 inches per second to feet per minute.

5. (19) Find the product: $\frac{-3^{-2}x}{y}\left(\frac{9y^0x}{-x}-\frac{3x}{y}\right)$.

Solve each equation for x.

6. (7) $0.003x+0.02x-0.03=0.177$

7. (7) $2\frac{1}{3}x+1\frac{3}{5}=7\frac{2}{5}$

8. (7) $\frac{1}{2}x+4=6\frac{1}{5}$

***9.** (22) Graph and state the domain and range of the discrete function: $\{(-2,-2), (4,1), (-1,6), (0,2)\}$.

Factor each polynomial.

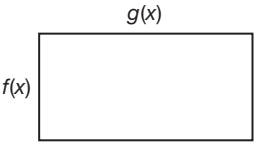
***10.** (23) $-3x^3+6x^2+3x$

***11.** (23) $-2x^2+5x+3$

***12.** (23) x^2-5x+6

***13.** (23) **Analyze** Is x^2-25 a Basic Quadratic Polynomial, a Perfect Square Trinomial, or a Difference of Two Squares? Explain how you can tell by looking at the polynomial.

14. (20) **Geometry** In the diagram of a rectangle, $f(x)=2x+5$ and $g(x)=5x+4$. Find the perimeter of the rectangle in terms of x.



15. (7) **Verify** Is $x=-1$ the solution to the equation $5(4x-1)=-7(9+4x)+8-2x$?

16. (13) **Write** Give a written explanation on how to graph a linear equation using intercepts.

17. (17) **Multi-Step** At a sugar plantation processing plant, a machine fills a bag with 5 pounds of white sugar. For quality control, another machine weighs each bag in ounces and rejects bags that differ from 5 pounds by more than y ounces.

a. Write an absolute-value function to show the minimum and maximum weight of sugar that could be in each bag.

b. Graph the solution of the function.

18. (16) **Verify** **a.** Use Cramer's Rule to solve the system. $\begin{aligned} x+2y&=0 \\ x+3y&=6 \end{aligned}$

b. Check your solution by graphing.

*19. (9) **Graphing Calculator** Find the product. Check using the graphing calculator.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 4 & -8 \\ 0 & -3 & 1 \end{bmatrix}$$

20. (11) **Multiple Choice** What is the degree of the polynomial $3x + 2x + 1$?

A 0 **B** 1 **C** 2 **D** 3

*21. (21) For the system of equations below, find values of b and c so that the system is characterized as follows.

$$\begin{aligned} x - 3y &= 12 \\ 3x + by &= c \end{aligned}$$

a. consistent and dependent

b. inconsistent

c. consistent and independent

22. (10) **Error Analysis** Describe and correct the error.

$$\begin{aligned} 4x - 3 &> x - 2 \\ 4x &> x - 5 \\ x &> -5 \end{aligned}$$

23. (9) **Justify** Show and explain each step in finding the product AB.

$$A = \begin{bmatrix} 5 & 2 & 14 \\ 9 & 11 & 18 \\ 3 & 7 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 15 & 6 \\ 7 & 13 \\ 10 & 16 \end{bmatrix}$$

24. (8) **Planets** The weight of an object on the Sun varies directly with the weight of that object on Earth. An object that weighs 12 oz on Earth weighs 336 oz on the Sun. Find the weight of an object on the Sun that weighs 7 oz on Earth.

*25. (22) **Data Analysis** Death Valley has the lowest elevation in North America. Below is a table of the temperature over the course of one day in the desert.

Time	1 a.m.	6 a.m.	10 a.m.	2 p.m.	4 p.m.	8 p.m.	11 p.m.
Temperature	83°	75°	90°	101°	105°	99°	90°

a. Graph the time and temperature data. Connect the points with line segments.

b. Explain why the temperature data creates a continuous relation.

c. Determine if the graph is a function. Explain.

26. (15) **Nutrition** Nutritionists recommend a daily diet consisting of at least 85 g of carbohydrate and at least 50 g of protein. A cup of fried rice contains 48.7 g of carbohydrate and 9.1 g of protein. A cup of sweet and sour prawn contains 28.6 g of carbohydrate and 16.5 g of protein. How many cups of each dish will give the minimum daily allowance of carbohydrate and protein? Let r represent the number of cups of fried rice and p the number of cups of prawn.

*27. (23) **Envelopes** The United States Postal Service will mail an envelope that is no greater than $2x + 1.5$ inches long and no greater than $x + 1$ inches high. The total area of an envelope with the maximum dimensions is approximately 69 in^2. Calculate the approximate maximum dimensions of an envelope that can be mailed through the USPS.

28. (13) **Population** The table below lists the population estimates in millions for Hawaii from 2000 to 2002.

Year	2000	2001	2002
Population	1.21	1.22	1.23

a. Write the data above as ordered pairs where x is the number of years after 2000 and y is the population in millions.

b. Use slope and y-intercept to find the equation of the line representing this data.

Factor each expression.

*29. (23) $4x^2 - 25$

*30. (23) $x^2 + 3x - 18$

LESSON
24

Solving Systems of Equations Using the Elimination Method

Warm Up

1. (15) **Vocabulary** A system of equations with an infinite number of solutions is a(n) _____ **(*consistent, inconsistent*)** and a(n) _____ **(*independent, dependent*)** system.

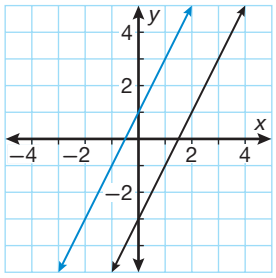
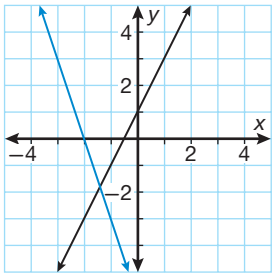
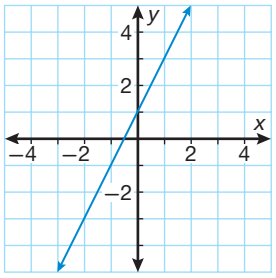
2. (15) Solve $\begin{aligned} y &= x + 1 \\ y + 3x &= 5 \end{aligned}$ by graphing.

3. (21) Solve $\begin{aligned} y &= x + 9 \\ 2x - y &= -13 \end{aligned}$ by substitution.

New Concepts

Recall that a linear system with no solutions is an inconsistent system while a system with *at least* one solution is a consistent system. If there is exactly one solution to a system, it is independent. If there are infinitely many solutions, the system is dependent.

The solution of a system is the intersection of the graphs of the equations of the system, making it possible to describe a system just by looking at its graph.

Inconsistent System	Consistent System	
	Independent System	Dependent System
		

Therefore, one method of solving a linear system is by graphing and finding the intersection point or points if they exist. Another method previously discussed is the substitution method, which involves substituting an equivalent expression for x or y in one equation to get an equation in one variable. When that equation is solved, the value of that variable is then substituted into either original equation and the second variable is solved for.

Math Language

The **elimination method** is sometimes referred to as the **addition method.**

The last method for solving a linear system is the **elimination method.** When the elimination method is used, the two equations are added. The sum of one of the variables is 0, leaving an equation in one variable, which is solved. This value is then substituted into either original equation to solve for the second variable.

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Sometimes extra steps are needed to be sure that the sum of one of the variables is 0. This could mean multiplying one or both equations by a constant.

Example 1 Solving Systems Using the Elimination Method

a. Solve $\begin{aligned} 15 &= 3x - 2y \\ x &= -11 - 2y \end{aligned}$ and classify the system.

Math Reasoning

Justify Why is the value of x substituted into one of the original equations and not one of the equations written in standard form?

SOLUTION Begin by writing each equation in standard form.

$$\begin{aligned} 15 &= 3x - 2y \\ x &= -11 - 2y \end{aligned} \quad \rightarrow \quad \begin{aligned} 3x - 2y &= 15 \\ x + 2y &= -11 \end{aligned}$$

Add vertically. The y-terms will be eliminated. Solve for x and substitute the value for x into either original equation to find y.

$$\begin{aligned} 3x - 2y &= 15 \\ +\ x + 2y &= -11 \\ \hline 4x \quad &= 4 \\ x &= 1 \end{aligned} \quad \longrightarrow \quad \begin{aligned} 15 &= 3x - 2y \\ 15 &= 3(1) - 2y \\ 15 &= 3 - 2y \\ 12 &= -2y \\ -6 &= y \end{aligned}$$

The solution is $(1, -6)$. The system is consistent and independent.

Check Substitute 1 for x and -6 for y into both original equations.

$$\begin{aligned} 15 &= 3x - 2y \\ 15 &\stackrel{?}{=} 3(1) - 2(-6) \\ 15 &= 15 \quad \checkmark \end{aligned} \qquad \begin{aligned} x &= -11 - 2y \\ 1 &\stackrel{?}{=} -11 - 2(-6) \\ 1 &= 1 \quad \checkmark \end{aligned}$$

Hint

To eliminate like terms, multiply each equation by a constant to make the terms opposites.

b. Solve $\begin{aligned} 2x + 5y &= -34 \\ -3x + 2y &= -44 \end{aligned}$ and classify the system.

SOLUTION The equations are already in standard form. Multiply the first equation by 3 and the second equation by 2.

$$\begin{aligned} 2x + 5y &= -34 \\ -3x + 2y &= -44 \end{aligned} \quad \rightarrow \quad \begin{aligned} 3(2x + 5y) &= 3(-34) \\ 2(-3x + 2y) &= 2(-44) \end{aligned} \quad \rightarrow \quad \begin{aligned} 6x + 15y &= -102 \\ -6x + 4y &= -88 \end{aligned}$$

Add the equations. Solve for x and y.

$$\begin{aligned} 6x + 15y &= -102 \\ +\ -6x + 4y &= -88 \\ \hline 19y &= -190 \\ y &= -10 \end{aligned} \quad \longrightarrow \quad \begin{aligned} 2x + 5y &= -34 \\ 2x + 5(-10) &= -34 \\ 2x - 50 &= -34 \\ 2x &= 16 \\ x &= 8 \end{aligned}$$

The solution is $(8, -10)$. The system is consistent and independent.

Check Substitute 8 for x and -10 for y into both original equations.

$$\begin{aligned} 2x + 5y &= -34 \\ 2(8) + 5(-10) &\stackrel{?}{=} -34 \\ -34 &= -34 \quad \checkmark \end{aligned} \qquad \begin{aligned} -3x + 2y &= -44 \\ -3(8) + 2(-10) &\stackrel{?}{=} -44 \\ -44 &= -44 \quad \checkmark \end{aligned}$$

Example 2 Solving Systems with Fractional Coefficients

a. Solve $\begin{aligned} \frac{3}{4}x &= 3y \\ 15 + 7y &= 3x \end{aligned}$.

SOLUTION Begin by writing each equation in standard form.

$$\begin{aligned} \frac{3}{4}x &= 3y \\ 15 + 7y &= 3x \end{aligned} \longrightarrow \begin{aligned} \frac{3}{4}x - 3y &= 0 \\ 3x - 7y &= 15 \end{aligned}$$

Math Reasoning

Analyze How is multiplying one equation by −1 and then adding the equations the same as subtracting one equation from the other?

Clear the fraction in the first equation by multiplying the equation by 4. Then multiply one of the equations by −1 to make the x-terms opposites.

$$\begin{aligned} 4\left(\frac{3}{4}x - 3y\right) &= 4(0) \\ 3x - 7y &= 15 \end{aligned} \longrightarrow \begin{aligned} 3x - 12y &= 0 \\ 3x - 7y &= 15 \end{aligned} \longrightarrow \begin{aligned} 3x - 12y &= 0 \\ -3x + 7y &= -15 \end{aligned}$$

Solve for x and y.

$$\begin{aligned} 3x - 12y &= 0 \\ +\ -3x + 7y &= -15 \\ \hline -5y &= -15 \\ y &= 3 \end{aligned} \longrightarrow \begin{aligned} 15 + 7y &= 3x \\ 15 + 7(3) &= 3x \\ 15 + 21 &= 3x \\ 36 &= 3x \\ 12 &= x \end{aligned}$$

The solution is (12, 3).

b. Solve $\begin{aligned} \frac{5}{6}x + \frac{7}{8}y &= 25 \\ \frac{x}{3} + 2y &= 76 \end{aligned}$.

SOLUTION Clear the fractions in the first equation by multiplying it by the least common denominator 24. Multiply the second equation by 3.

After the fractions are cleared, multiply the second equation by −20 + 0 eliminate the x-terms.

$$\begin{aligned} 24\left(\frac{5}{6}x + \frac{7}{8}y\right) &= 24(25) \\ 3\left(\frac{x}{3} + 2y\right) &= 3(76) \end{aligned} \longrightarrow \begin{aligned} 20x + 21y &= 600 \\ x + 6y &= 228 \end{aligned} \longrightarrow \begin{aligned} 20x + 21y &= 600 \\ -20x - 120y &= -4560 \end{aligned}$$

Solve for x and y.

$$\begin{aligned} 20x + 21y &= 600 \\ +\ -20x - 120y &= -4560 \\ \hline -99y &= -3960 \\ y &= 40 \end{aligned} \longrightarrow \begin{aligned} \frac{x}{3} + 2y &= 76 \\ \frac{x}{3} + 2(40) &= 76 \\ \frac{x}{3} &= -4 \\ x &= -12 \end{aligned}$$

The solution is (−12, 40).

Example 3 Solving Systems with Infinitely Many and No Solutions

a. Solve $\begin{aligned} 2x - y &= 3 \\ -6x + 3y &= -9 \end{aligned}$ and classify the system.

SOLUTION Multiply the first equation by 3 to make opposite x-terms.

$$\begin{aligned} 3(2x - y) &= 3(3) \\ -6x + 3y &= -9 \end{aligned} \quad \rightarrow \quad \begin{aligned} 6x - 3y &= 9 \\ -6x + 3y &= -9 \end{aligned}$$

Solve for x and y.

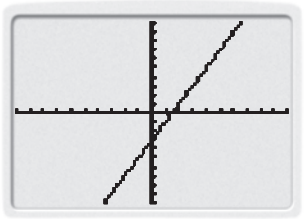
$$\begin{aligned} 6x - 3y &= 9 \\ +\ -6x + 3y &= -9 \\ \hline 0 &= 0 \end{aligned}$$

$0 = 0$ is a true statement.

Math Language

Two lines that **coincide** share the same points.

There are infinitely many solutions. The system is consistent and dependent. The lines coincide.



b. Solve $\begin{aligned} 3x - 4y &= -24 \\ 6x - 8y &= 16 \end{aligned}$ and classify the system.

SOLUTION Multiply the first equation by -2 to make opposite x-terms.

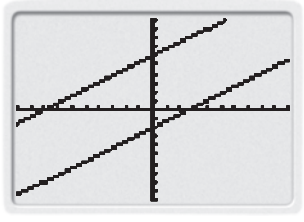
$$\begin{aligned} -2(3x - 4y) &= -2(-24) \\ 6x - 8y &= 16 \end{aligned} \quad \rightarrow \quad \begin{aligned} -6x + 8y &= 48 \\ 6x - 8y &= 16 \end{aligned}$$

Solve for x and y.

$$\begin{aligned} -6x + 8y &= 48 \\ +\ 6x - 8y &= 16 \\ \hline 0 &= 64 \end{aligned}$$

$0 = 64$ is a false statement.

There are no solutions. The system is inconsistent. The lines are parallel.



Example 4 Application: Vacationing

In 2007, the costs to take the elevator to the top of the Eiffel Tower were such that a group of 6 adults and 4 children paid 94.20 euros and a group of 5 adults and 14 children paid 145.70 euros. Find the cost for each adult and each child.

Math Reasoning

Estimation How can estimation be used to check the answer for reasonableness?

SOLUTION Write a system where a represents the cost for an adult and c the cost for a child. Multiply so that one of the variables will be eliminated.

$$\begin{matrix} 6a + 4c = 94.20 \\ 5a + 14c = 145.70 \end{matrix} \longrightarrow \begin{matrix} -5(6a + 4c) = -5(94.20) \\ 6(5a + 14c) = 6(145.70) \end{matrix} \longrightarrow \begin{matrix} -30a - 20c = -471 \\ 30a + 84c = 874.2 \end{matrix}$$

Solve for x and y.

$$\begin{aligned} -30a - 20c &= -471 \\ +\ 30a + 84c &= 874.2 \\ \hline 64c &= 403.2 \\ c &= 6.3 \end{aligned} \longrightarrow \begin{aligned} 6a + 4c &= 94.2 \\ 6a + 4(6.3) &= 94.2 \\ 6a + 25.2 &= 94.2 \\ 6a &= 69 \\ a &= 11.5 \end{aligned}$$

The cost for one adult is 11.50 euros and the cost for one child is 6.30 euros.

Lesson Practice

Solve and classify each system.

a. *(Ex 1)* $\begin{matrix} -2y = -6x + 62 \\ -x = -22 - 2y \end{matrix}$

b. *(Ex 1)* $\begin{matrix} -x - y = -3 \\ 3x = 2y - 11 \end{matrix}$

Solve.

c. *(Ex 2)* $\begin{matrix} 5x + 7y = -36 \\ \frac{2}{3}x - 8y = 22 \end{matrix}$

d. *(Ex 2)* $\begin{matrix} \frac{1}{2}x - y = 15 \\ \frac{1}{4}x + \frac{1}{3}y = 0 \end{matrix}$

Solve and classify each system.

e. *(Ex 3)* $\begin{matrix} -7 = y + 3x \\ 15x = 50 - 5y \end{matrix}$

f. *(Ex 3)* $\begin{matrix} 4x = 10 + 2y \\ 5 = 2x - y \end{matrix}$

g. *(Ex 4)* In 2007, the costs of tickets to the top of the Gateway Arch in St. Louis, Missouri, were such that a group of 10 adults and 3 children paid $109 and a group of 5 adults and 3 children paid $59. Find the cost for each adult and each child.

Practice Distributed and Integrated

Solve for x.

1. (14) $\begin{vmatrix} -2 & 1 \\ 1 & x \end{vmatrix} = 16$

2. (14) $\begin{vmatrix} -3+x & x \\ 1 & 6 \end{vmatrix} = 5$

3. (14) $\begin{vmatrix} 4 & -5 \\ 2 & x \end{vmatrix} = 8$

4. (18) Convert 120 inches per second to feet per minute.

5. (13) Calculate the slope of the line that passes through the points (2, 1) and (0, 5).

Find each product.

6. (9) $\begin{bmatrix} -5 & 2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 4 \\ 1 & 8 \end{bmatrix}$

7. (9) $[-1 \;\; 5] \times \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

8. (9) $[-3 \;\; 4] \times \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

***9.** (23) Factor the polynomial: $x^4 - 16$.

Solve each system of equations using elimination.

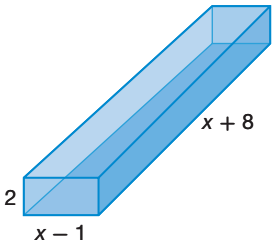
***10.** (24) $\begin{aligned} 2x - 5y &= 5 \\ -5x + y &= -1 \end{aligned}$

***11.** (24) $\begin{aligned} -4x + 7y &= -47 \\ 3x - 5y &= 34 \end{aligned}$

***12.** (24) $\begin{aligned} 5x + y &= 14 \\ -3x - 4y &= -5 \end{aligned}$

***13.** (24) $\begin{aligned} 2x - 5y &= -3 \\ -10x + 25y &= 15 \end{aligned}$

***14.** (23) **Geometry** Find the possible value(s) of x if Volume = 104.



15. (21) Solve $\begin{aligned} 2x + y &= -4 \\ x + y &= 7 \end{aligned}$ by substitution.

16. (19) **Model** Model FOIL for the multiplication of any two binomials in the form $(a + b)(c + d)$.

17. (9) **Multi-Step** Use $A = \begin{bmatrix} 5 & 3 \\ 6 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 7 & 4 \\ 8 & 7 \end{bmatrix}$.

a. Find AB.

b. Find BA.

c. Is $AB = BA$?

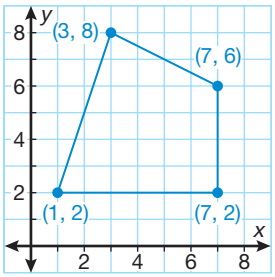
***18.** (24) **Sightseeing** The cost of tickets to the observatory on the 86th floor of the Empire State Building in 2007 were such that 5 adult tickets and 4 senior citizen tickets cost $154 and 3 adult tickets and 8 senior citizen tickets cost $182. Find the cost of each type of ticket.

19. (11) **Justify** If polynomial P_1 has a greater degree than polynomial P_2, does P_1 have a greater value than P_2? Explain your answer and justify it by giving examples.

20. (20) If $f(x) = 3x - 9$ and $g(x) = x - 3$, find $(fg)(x)$.

21. (14) **Coordinate Geometry** A quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) has an area equal to the absolute value of $\frac{1}{4}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{vmatrix}$.

Find the area of the quadrilateral.



***22.** (13) **Graphing Calculator** The table shows the total number of birdhouses Sal made, y, after x amount of days.

Number of Days	2	3	7
Birdhouses	$5\frac{1}{3}$	8	$18\frac{2}{3}$

a. Plot the data on your graphing calculator.

b. Find the line that best fits the data using linear regression and graph the line on your graphing calculator.

c. Use the line to predict the number of birdhouses Sal can make in 10 days.

***23.** (24) **Error Analysis** Find and correct the error a student made below.

$$\begin{aligned} 4x + y &= 9 \\ 4x - 3y &= 5 \end{aligned} \quad \rightarrow \quad \begin{aligned} 4x + y &= 9 \\ -4x + 3y &= 5 \end{aligned}$$

$$\begin{array}{rcl} 4x + y = 9 & & 4x + y = 9 \\ \underline{+\ -4x + 3y = 5} & \longrightarrow & 4x + 3.5 = 9 \\ 4y = 14 & & 4x = 5.5 \\ y = 3.5 & & x = 1.375 \end{array}$$

The solution is (1.375, 3.5).

24. (10) **Multi-Step** At the local bowling alley there is a tournament underway. One of the bowlers has played two games and has a total score of 530. To win the tournament, a total score of 750 for three games is needed. Use the information given to write an inequality describing the possible number of points the bowler can score and win the tournament. What is the minimum score possible in order to win the tournament?

25. (18) **Estimate** A cheetah runs at a speed of 70 miles per hour. Estimate a cheetah's speed in feet per second.

26. (17) **Grizzly Bears** The table below shows amounts of three foods that a 200-lb grizzly bear can eat in a day. Which term, *disjunction* or *conjunction*, would represent the amount of these foods required for a 200-lb grizzly bear? Explain.

Type	Deer	Salmon	Vegetation
Amount (lb)	24	30	67

27. *(8)* **Multiple Choice** The amount of water added to a dry concrete mix varies directly with the amount of dry concrete. If 80 lb of dry concrete requires 6 pt of water, how much water is needed for 200 lb of dry concrete? Circle the letter that best represents the amount of water needed.

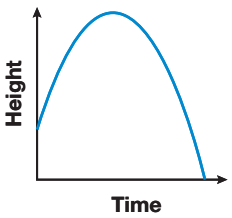
A 15 pt **B** 10 pt **C** 8 pt **D** 6.5 pt

State whether the function is discrete or continuous.

***28.** *(22)* (1, 5), (0, 9), (2, 7), (−5, 4)

***29.** *(22)* (2, 5), (4, 8), (6, 9), (−9, −6)

30. *(22)* **Sports** A baseball when hit travels in a parabolic path. If time is represented by the x-values and the height of the ball is represented by the y-values, the graph at right can be drawn. Does this example represent a continuous, discontinuous, and/or discrete function? Justify your answer.



LAB

6

Calculating 1- and 2-Variable Statistical Data

Graphing Calculator Lab ***(Use with Lesson 25, Inv. 7, 73, 80, and Inv. 8)***

A graphing calculator can be used to analyze data sets of 1 and 2 variables.

Data Set 1	Data Set 2
–5	–10
–3	–6
–2	–4
–1	–2
0	0
2	4
3	6
4	8

Calculating Statistics for 1-Variable Data

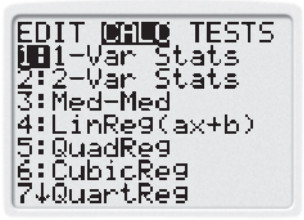
Graphing Calculator Tip

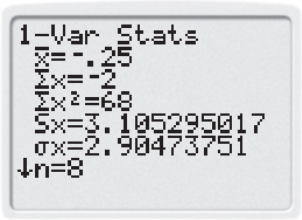
For help with entering data into lists, see graphing calculator lab 5 on pg. 153.

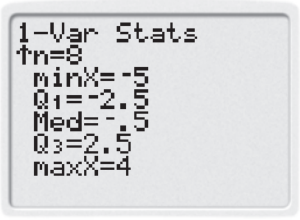
1. Use the graphing calculator to enter the data sets into L1 and L2.

2. Calculate 1-variable statistical data.

   From the home screen, press STAT, and then press ▶ to open the **[CALC]** menu. Press ENTER to select **[1: 1-Var Stats]**. Input L1 by pressing 2nd 1. Then press ENTER and the calculator will analyze the data values in L1. Press ▼ to scroll through the information.





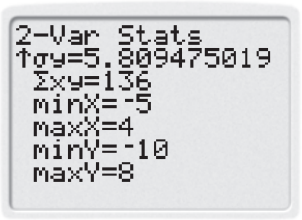


Calculating Statistics for 2-Variable Data

Use the above sets of data already entered into L1 and L2.

a. From the home screen, press STAT, and then press ▶ to open the **[CALC]** menu. Press ▼ to scroll to **[2: 2-Var Stats]**. Then press ENTER. Input L1 and L2 by pressing 2nd 1 , 2nd 2. Then press ENTER and the calculator will analyze the data values in L1 and L2.

b. The data listed for L1 is referred to as x. The data for L2 is referred to as y. There is one new statistic: Σxy is the sum of the products of the paired data values.



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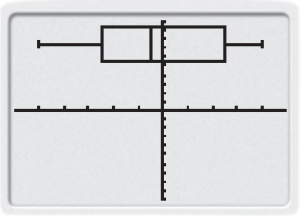
Display 1-Variable Statistical Data Through a Box-and-Whisker Plot

1. Customizing a box-and-whisker plot.
   - **a.** Press 2nd Y= to open the **[STAT PLOT]** menu. Press ENTER to open the first Stat plot.
   - **b.** Press ENTER to turn Plot1 on. This will display the plot on the graph.
   - **c.** Press ▼ to move to **[Type:]**. Press ▶ until the box-plot symbol is highlighted and press ENTER.
   - **d.** Press ▼ to move to **[Xlist:]**. L1 should be displayed, meaning it will use the data values from data set 1. (If L1 is not displayed, press 2nd 1 ENTER.)
   - **e.** Press ▼ to move to **[Freq:]**. Press 1 ENTER.

2. Graphing a box-and-whisker plot. Press ZOOM and use the arrow key ▼ to select **[9: ZoomStat]** and press ENTER. The box-and-whisker plot of L1 will now be displayed.



Graphing Calculator Practice

Use the graphing calculator to analyze the following data sets.

Data Set 1	Data Set 2
–12	5
–4	6
13	16
21	21
30	23

- **a.** Graph a box-and-whisker plot for data set 2.
- **b.** Calculate the value of Σxy for data sets 1 and 2.

LESSON 25

Finding Measures of Central Tendency and Dispersion

Warm Up

1. (22) **Vocabulary** A set of separate numbers is referred to as _____ **(*discrete, continuous*)** data.

2. (SB) Subtract 25.5 − 37.75.

3. (SB) Simplify $(-0.25)^2$.

New Concepts

Statistics is the branch of mathematics that involves the collection, analysis, and comparison of sets of numerical data.

A **measure of central tendency** is used to represent the middle of a set of data.

Measures of Central Tendency
The **mean,** or average, of n numbers is the sum of the numbers divided by n. The mean is denoted by $\overline{x}$, read "x-bar." For the data set $x_1, x_2, \ldots, x_n$, the mean is $$\overline{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}.$$
The **median** of n numbers is the middle number when the numbers are written in order. If n is even, the median is the mean of the two middle numbers.
The **mode** of n numbers is the number that appears most frequently in the list. There may be one mode, no mode, or more than one mode.

Example 1 Finding Measures of Central Tendency

Luann earned the following grades on her algebra quizzes:

18, 15, 20, 16, 17, 19, 19

Calculate the mean, median, and mode for Luann's quiz grades.

SOLUTION

Mean: $\overline{x} = \frac{18 + 15 + 20 + 16 + 17 + 19 + 19}{7} = \frac{124}{7} \approx 17.7$

Median: Arrange the data in order. 15, 16, 17, 18, 19, 19, 20

The median is the middle number, 18.

Mode: The mode is the number that appears most often, 19.

Caution

The data must be in either ascending or descending order to properly determine the median.

A **measure of dispersion** is a statistic that indicates how spread out, or dispersed, data values are. Three common measures of dispersion are the range, variance, and standard deviation.

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Graphing Calculator Tip

$\sqrt{\ }$ means "square root." Press the $\sqrt{\ }$ key first, then enter the data.

Measures of Dispersion

The **range of a data set** is the difference between the largest and smallest data values.

The **variance** of a data set is the average of the squared differences from the mean.

The **standard deviation** measures, on average, how far away from the mean the data values are. The standard deviation σ, which is the Greek letter sigma, is the square root of the variance and is calculated using the following formula for a data set with n terms, $x_1, x_2, \ldots, x_n$:

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}}$$

Example 2 Finding Measures of Dispersion

Find the range and standard deviation for the following sets of data.

a. 5, 7, 12, 14, 17

SOLUTION

The range is the difference between the largest and smallest data values.

The range is $17 - 5 = 12$.

To find the standard deviation, first calculate the mean.

$$\bar{x} = \frac{5 + 7 + 12 + 14 + 17}{5} = \frac{55}{5} = 11$$

$$\sigma = \sqrt{\frac{(5 - 11)^2 + (7 - 11)^2 + (12 - 11)^2 + (14 - 11)^2 + (17 - 11)^2}{5}}$$

$$= \sqrt{\frac{98}{5}} \approx 4.4$$

b. 10, 12, 7, 11, 20, 7, 6, 8, 9

SOLUTION

The range is $20 - 6 = 14$.

To find the standard deviation, first calculate the mean.

$$\bar{x} = \frac{10 + 12 + 7 + 11 + 20 + 7 + 6 + 8 + 9}{9} = \frac{90}{9} = 10$$

$$\sigma = \sqrt{\frac{(10 - 10)^2 + (12 - 10)^2 + \ldots + (8 - 10)^2 + (9 - 10)^2}{9}}$$

$$\sigma = \sqrt{\frac{0^2 + 2^2 + \ldots + (-2)^2 + (-1)^2}{9}} = \sqrt{\frac{144}{9}} = \sqrt{16} = 4$$

The standard deviation is 4.

An **outlier** is an item in a data set that is much larger or much smaller than the other items in the set. The presence of an outlier in a set of data can have a misleading effect on the measures of central tendency and dispersion.

Example 3 Examining the Effect of an Outlier

Identify the outlier in the data set. Then find the mean, median, mode, range, and standard deviation of the data set when the outlier is included and when it is not.

$$2, 2, 3, 3, 4, 4, 4, 6, 68$$

SOLUTION

The outlier is 68.

The range with the outlier is $68 - 2 = 66$.

The range without the outlier is $6 - 2 = 4$.

The mode with and without the outlier is 4.

The median with the outlier is 4.

The median without the outlier is 3.5, which is the mean of the numbers 3 and 4.

The mean with the outlier is:

$$\bar{x} = \frac{2 + 2 + \ldots + 68}{9} = \frac{96}{9} \approx 10.7$$

The mean without the outlier is:

$$\bar{x} = \frac{2 + 2 + \ldots + 6}{9} = \frac{28}{8} = 3.5$$

The standard deviation with the outlier is:

$$\sigma = \sqrt{\frac{(2 - 10.7)^2 + \ldots + (68 - 10.7)^2}{9}} = \sqrt{\frac{3710.01}{9}} \approx \sqrt{412.2} \approx 20.3$$

The standard deviation without the outlier is:

$$\sigma = \sqrt{\frac{(2 - 3.5)^2 + \ldots + (6 - 3.5)^2}{8}} = \sqrt{\frac{12}{8}} = \sqrt{1.5} \approx 1.2$$

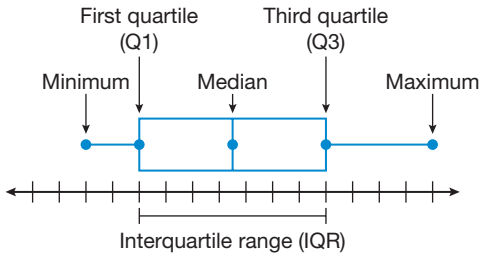
Caution

Be sure to reduce the number of data items by the number of outliers before calculating mean and standard deviation.

Math Reasoning

Verify Use a graphing calculator to check the solution in Example 3.

One convenient way to organize data is with a **box-and-whisker plot.**



The **first quartile** is the median of the lower half of the data set.
The **third quartile** is the median of the upper half of the data set.

Reading Math

σ is the lower case Greek letter sigma and represents standard deviation. $\bar{x}$ is read "x bar" and represents the mean.

Example 4 Organizing Data with a Box-and-Whisker Plot

Draw a box-and-whisker plot to display the following data:

31, 29, 28, 20, 35, 22, 32, 22, 32.

SOLUTION

Write the data in ascending order: 20, 22, 22, 28, 29, 31, 32, 32, 35.

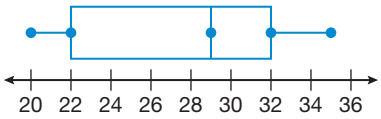
The median of the data is 29.

The lower quartile 22 is the median of the lower half of the data.

The upper quartile 32 is the median of the upper half of the data.

The minimum is 20. The maximum is 35.

Draw the box-and-whisker plot.



Example 5 Application: Sports

The National League leaders in home runs for the 2003 through 2006 seasons are found in the table below. What was the mean number of home runs hit by the league leader for those seasons and what was the standard deviation for that statistic?

Season	Leader	Home Runs
2003	Jim Thome	51
2004	Adrian Beltre	48
2005	Andruw Jones	47
2006	Ryan Howard	49

SOLUTION

The mean is $\bar{x} = \frac{51 + 48 + 47 + 49}{4} = \frac{195}{4} = 48.75.$

The standard deviation is

$$\sigma = \sqrt{\frac{(51 - 48.75)^2 + \ldots + (49 - 48.75)^2}{4}} = \sqrt{\frac{8.75}{4}} = \sqrt{2.1875} \approx 1.5.$$

Lesson Practice

(Ex 1) **a.** Jacob paid the following dollar amounts for lunch in one week: 4.5, 6.5, 7, 5, 7, 5, 7. Calculate the mean, median, and mode for Jacob's lunch costs.

(Ex 2) **b.** Find the range and standard deviation for the following set of data: 11, 8, 2, 7, 12.

(Ex 2) **c.** Find the range and standard deviation for the following set of data: 14, 9, 19, 16, 8, 17, 25, 21, 15.

(Ex 3) **d.** Identify the outlier in the data set {6, 4, 7, 47, 5, 5, 6, 5}. Then find the mean, median, mode, range, and standard deviation of the data set when the outlier is included and when it is not.

e. (Ex 4) Draw a box-and-whisker plot to display the following data: 3, 5, 1, 2, 4, 2, 6.

f. (Ex 5) Annual snowfall amounts in inches for Buffalo, New York, for 2000–2007 are listed in the table.

What was the mean amount of snowfall in Buffalo during those years and what was the standard deviation?

Season	Amount (inches)
2000–01	158.7
2001–02	132.4
2002–03	111.3
2003–04	100.9
2004–05	109.1
2005–06	78.2
2006–07	88.9

Practice Distributed and Integrated

Simplify each expression using the properties of real numbers.

1. (1) $-5[(2-1-3)-(5-8)-6]$

2. (1) $2(2-6)-(-1-1)-3$

3. (3) Simplify: $-3x^6(x^{-1}y^{-8})^{-3}$

4. (10) Solve for x and graph the solution. $3x+1>x+13$

5. (9) Find the product $\begin{bmatrix}-2 & 10\\ 3 & 2\end{bmatrix}\times\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$.

6. (19) Simplify the product: $3x^2(2x^2-3x)$.

***7.** (23) Factor the polynomial: $2x^2+9x-5$.

Solve each system of linear equations using substitution.

8. (21)
$$\begin{aligned}4x+y&=-11\\ -x+y&=9\end{aligned}$$

9. (21)
$$\begin{aligned}x+y&=4\\ x-y&=6\end{aligned}$$

Find the mean and median for each data set.

***10.** (25) {2, 2, 3, 10, 11, 13, 15}

***11.** (25) {1, 2, 3, 3}

***12.** (25) {1, 1, 13, 15, 14}

***13.** (23) **Geometry** The area of a new window is $x^2-24x+144$ in^2. What are the dimensions of the new window?

14. (20) **Multiple Choice** Given $f(x)=-3x+11$ and $g(x)=4x-10$, find $(f+g)(12)$.

A -9 **B** 13 **C** 85 **D** 105

Solve each system using elimination.

*15. (24) $\begin{cases} 5x + y = 19 \\ -2x - y = -7 \end{cases}$

*16. (24) $\begin{cases} -x + 3y = 12 \\ 6x - y = -21 \end{cases}$

*17. (24) $\begin{cases} 2x + y = 8 \\ x - y = 10 \end{cases}$

18. (11) **Write** Describe how the graph of polynomial function $P_1 + P_2$ compares to the graph of polynomial function P_1 if P_2 does not equal 0, but has degree 0. Explain.

19. (18) **Drumming** Mike "Da Man" Mangini holds the title as the world's fastest drummer. At the 2005 Summer NAMM Session, Mike drummed out 1,247 singles in 60 seconds.

a. How many singles could Mike drum in one hour?

b. Not only does Mike have fast hands, he has fast feet, pounding down a toe-stomping 13,222 singles with his feet in 15 minutes. How many singles could Mike drum with his feet in one hour?

c. How much faster does Mike drum with his hands than with his feet?

20. (23) **Quilting** Kylie is making a rectangular quilt out of 4 inch squares. So far, her quilt is 36 inches by 56 inches. She wants to add a border around her quilt that is the same number of squares on each side. A standard twin comforter covers approximately 4800 square inches.

a. How many squares should she add to each side if she wants her quilt to cover the same area?

b. Verify that her quilt will cover the same area.

21. (9) **Data Analysis** Alea, Cory, and Troy had lunch together. Alea bought 1 drink, 2 sandwiches, and 2 snacks. Cory bought 2 drinks, 2 sandwiches, and 1 snack. Troy bought 2 drinks, 1 sandwich, and 3 snacks. The cost of a drink is \$1.50, a sandwich is \$4.00, and a snack is \$1.00.

a. Organize the information into the tables below.

Lunch

Person	Drink	Sandwich	Snack

Cost	

b. Create two matrices to represent the tables in part a.

c. Who spent the most money?

*22. (14) **Graphing Calculator** **a.** Write the determinant as an equation.

$$\begin{vmatrix} -x & 2 & -2 \\ 0 & -3 & 1 \\ 8 & 2x & 0 \end{vmatrix} = 0$$

b. Solve for x.

c. Check your answer by substituting for x and finding the determinant with a graphing calculator.

23. (7) **Multi-Step** A cell phone plan costs \$29.95 a month and \$0.40 per minute used over 250 minutes. There is a one time only activation fee of \$25.00. The first month's bill is \$114.95.

a. Write an equation to solve for the minutes over 250 used.

b. Solve the equation from part a.

c. Show the work to check the answer to part b.

d. How many total minutes were used?

***24.** (24) **Formulate** Create a linear system that requires that both equations be multiplied by a constant before they are added together. Tell which constants should be used and why.

25. (19) **Write** The length of one side of a square patio is represented by the expression $(2x - 5)$. Write an expression that represents the area of the patio.

26. (17) **Exercise** For a 30-year-old to work out within her cardio range, her heart rate should be within 24 beats per minute of 147 bpm.

a. Write an absolute-value equation to determine the lowest and highest heart rate at which she will be in her cardio range.

b. Solve the equation.

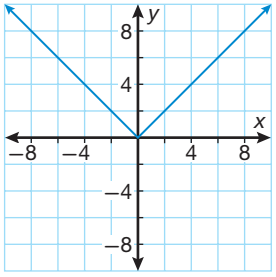
27. (13) **Error Analysis** Ani is asked find the slope of the line through the points (11, 6) and (7, 4), and then to tell whether the line rises or falls. Her solution is shown below. Identify whether there is an error in her solution. If so, give the correct solution.

Ani's solution:

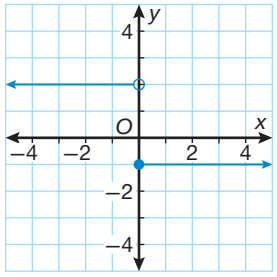
$\frac{4-6}{7-11} = \frac{-2}{-4} = \frac{1}{2}$, so $m = \frac{1}{2}$. Because the slope is a fraction, the line falls.

28. (22) Determine if each graph is continuous, discontinuous, and/or discrete. Determine if each graph is a function or a relation. Determine the domain and range of each.

a.



b.



***29.** (25) **Skateboarding** The table below shows the number of Americans who participated in skateboarding more than once during 1997–2000. Find the mean and standard deviation of the data.

Year	1997	1998	1999	2000
Participants (in millions)	8.2	7.2	7.8	11.6

30. (8) **Analyze** The table shows the amount of pasta required for each number of servings. Is this a direct variation relationship? If so, what is the constant of variation? If not, explain why.

Servings	Pasta
4	8 oz
13	26 oz
17	34 oz
29	58 oz

LESSON
26

Writing the Equation of a Line

Warm Up

1. (13) **Vocabulary** A _____ line has an undefined slope.

2. (13) Calculate the slope of the line that contains the points $(-2, 3)$ and $(4, -1)$.

3. (13) Identify the slope and y-intercept of the line $y = 2x - 5$.

4. (13) **Multiple-Choice** Which line has a slope of 0?

A A line that contains $(-2, -2)$ and $(7, -2)$

B A line that contains $(4, -2)$ and $(-1, 2)$

C A line that contains $(5, -7)$ and $(3, -2)$

D A line that contains $(0, 0)$ and $(5, 5)$

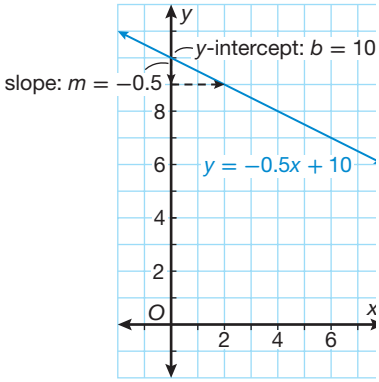
5. (13) Find the slope of the line described by $2x + 3y = 12$.

New Concepts

Linear equations are often written in slope-intercept form because that is the easiest way to sketch a graph.

Hint

The slope-intercept form of a line is $y = mx + b$, where m = slope and b = y-intercept.



However, an equation is often written in another form. For example, suppose you have \$30 dollars to spend on pens and notebooks. Notebooks are \$3 each and pens are \$1.50 per package. The following equation models all the possible combinations of pen and notebook purchases.

Let x = the number of packages of pens

Let y = the number of notebooks

$$1.5x + 3y = 30$$

This equation is written in standard form. Solving for y in the equation gives the slope-intercept form.

$$y = -0.5x + 10$$

The **standard form of a linear equation** is written below. For an equation in standard form, A, B, and C are real numbers and A and B are not both zero.

$$Ax + By = C$$

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Example 1 Writing an Equation in Standard Form

The equation for converting Fahrenheit temperatures (F) to Celsius (C) is written in this form:

$$F = \frac{9}{5}C + 32$$

Write the equation in standard form.

SOLUTION

Step 1: Move all terms with variables to the same side of the equation.

$$F - \frac{9}{5}C = 32$$

Step 2: Multiply by a factor that results in integer coefficients for F and C and an integer on the right side of the equation. Multiply both sides by 5.

$$5F - 9C = 160$$

Math Reasoning

Verify The Fahrenheit-Celsius formula is based on the fact that water boils at 212°F and 100°C, and water freezes at 32°F and 0°C. Use these coordinates to verify the accuracy of the formula.

If you know the slope m and the y-intercept b of a line, then you can write the equation of the line. The slope-intercept form of a linear equation is written as:

$$y = mx + b.$$

Example 2 Writing an Equation in Slope-Intercept Form

A line has slope 3 and crosses the y-axis at the point (0, 7). What is the equation of this line written in slope-intercept form?

SOLUTION

Recall the slope-intercept form of a linear equation: $y = mx + b$. For this equation $m = 3$ and $b = 7$. Substitute the known values in the equation.

$$y = 3x + 7$$

If you know the slope m and an ordered pair (x_1, y_1) of any point on the line, then you can use the **point-slope form** to write the equation of the line.

$$y - y_1 = m(x - x_1)$$

Example 3 Using the Point-Slope Formula

A line has slope -2 and passes through the point (10, 12). What is the equation of this line written in slope-intercept form?

SOLUTION

Substitute the known values into the point-slope form of the linear equation, then solve the equation for y to write in slope-intercept form.

$$y - 12 = -2(x - 10)$$
$$y = -2x + 20 + 12$$
$$y = -2x + 32$$

Math Reasoning

Justify Explain how the slope formula and the point-slope formula are related.

If you know the coordinates of two points on a line, you can use these to find the slope, m. To find the slope, solve the point-slope form of the equation to find a formula for m:

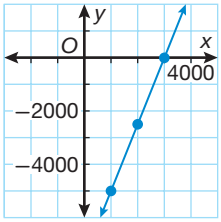
$$m = \frac{y - y_1}{x - x_1}$$

Once you have calculated m, you can substitute the value of m and one of the points in the point-slope form, then solve for y to write the equation in slope-intercept form.

Example 4 Writing the Equation of a Line Given Two Points

The chart below shows the number of units sold for a new product and the amount left to reach before the product breaks even and then becomes profitable. The points are graphed and shown to be part of the same line. Write the equation of this line in slope-intercept form.

Units Sold	Amount Before Break Even
1,000	−5,000
2,000	−2,500
3,000	0



SOLUTION

Step 1: Use the formula for slope with any two points to find the value of m.

Let $(x, y) = (1000, -5000)$ and $(x_1, y_1) = (2000, -2500)$.

$$m = \frac{y - y_1}{x - x_1}$$

$$m = \frac{-5000 + 2500}{1000 - 2000}$$

$$= \frac{-2500}{-1000}$$

$$= 2.5$$

Step 2: Use the point-slope form with one of the points given, then solve for y to find the equation in slope-intercept form.

$$y - (-5000) = 2.5(x - 1000)$$

$$y - (-5000) = 2.5(x - 1000)$$

$$y + 5000 = 2.5x - 2500$$

$$y = 2.5x - 7500$$

Math Reasoning

Analyze The equation that can be used to predict the height of a child has an infinite domain and range. How does this affect the accuracy of the equation?

Example 5 Mathematical Modeling with Linear Equations

The Centers for Disease Control has developed growth charts that pediatricians use to see if a child's height is within a certain average. The chart shows data for a boy's height in the 50th percentile. The three points lie on a line. Use the data to write the equation of the line in slope-intercept form.

Age (years)	Height (inches)
2	35
3	37.5
5	42.5

SOLUTION

Step 1: Use the formula for slope with any two points to find m.

Let $(x, y) = (3, 37.5)$ and $(x_1, y_1) = (5, 42.5)$.

$$m = \frac{y - y_1}{x - x_1}$$

$$= \frac{37.5 - 42.5}{3 - 5}$$

$$= \frac{-5}{-2}$$

$$= 2.5$$

Step 2: Use the point-slope form with one of the points given, then solve for y to find the equation in slope-intercept form.

$$y - 35 = 2.5(x - 2)$$

$$y = 2.5x - 5 + 35$$

$$y = 2.5x + 30$$

Lesson Practice

a. *(Ex 1)* The equation for converting Celsius (C) temperatures to Fahrenheit (F) is usually written in the form $C = \frac{5}{9}(F - 32)$. Write the equation in standard form.

b. *(Ex 2)* A line with a slope of -2 crosses the y-axis at the point $(0, 15)$. What is the equation of this line written in slope-intercept form?

c. *(Ex 3)* A line with a slope of 1.5 passes through the point $(-24, 0)$. What is the equation of this line in slope-intercept form?

d. *(Ex 4)* A line passes through points $(10, 0)$ and $(16, 12)$. What is the equation of the line written in slope-intercept form?

e. *(Ex 5)* A coastal city is tracking the movements of a hurricane that is expected to make landfall in the region. The table shows the distance from landfall for successive days. These points are on the same line when graphed. Write the equation of this line in slope-intercept form.

Day	Distance from landfall (miles)
1	500
2	375
3	250

Practice Distributed and Integrated

Classify each polynomial by degree and number of terms.

1. (11) $3x^4$

2. (11) $-x^3 + 5x$

3. (11) $2x^2 - 5x + 1$

4. (5) Find the sum: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 6\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$

5. (18) Convert 450 feet to yards.

Factor each polynomial.

***6.** (23) $-15x^2 + 5x$

***7.** (23) $4x^2 - 9$

***8.** (23) $6x^2 + 19x - 7$

***9.** (25) Find the mode of the data set: {100, 120, 128, 128, 129, 130, 130}.

Find the equation of the line that passes through the two points given.

***10.** (26) (0, 3) and (4, −4)

***11.** (26) (1, 5) and (4, 2)

***12.** (26) (−3, 2), (5, 0)

13. (1) **Analyze** Identify which property of real numbers is being demonstrated. Explain why this property might be helpful in solving this problem without a calculator.

$$(4 \cdot 6) \cdot 5 = 4 \cdot (6 \cdot 5) = 4 \cdot 30 = 120$$

14. (22) **Multi-Step** **a.** Graph $y = x^2 + 2$.

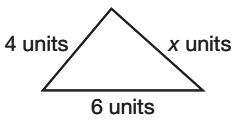
b. Is this relation a function?

c. Determine if the graph is continuous, discontinuous, and/or discrete.

15. (7) **Multiple Choice** Choose the letter that best represents the solution to the equation: $\frac{2}{5}c + 3 = -5c + \frac{3}{4}$.

A $-2\frac{2}{5}$ **B** $\frac{12}{13}$ **C** $\frac{1}{11}$ **D** $-\frac{5}{12}$

16. (10) **Geometry** Write a compound inequality using the fact that the sum of the lengths of any two sides of a triangle is greater than the length of the third side (Triangle Inequality) using the diagram. (*Hint*: Remember that length cannot be negative.)



***17.** (23) **Multi-Step** A homeowner painted her bathroom. She has a mirror that is 30 inches high and 20 inches wide. She decides to decrease both mirror dimensions by the same amount, and she will now have to paint the newly exposed wall. She only has enough paint to cover 225 square inches.

a. Calculate the new area of the mirror.

b. Write and solve an equation to find out how much she should decrease each dimension.

18. (Inv 2) **Soccer** Mia Hamm is running with the ball and is about to make a chip shot aiming to kick the ball above the goalie's head. Mia is 31 feet from the goal and kicks the ball at a horizontal speed of 14 feet per second and a vertical speed of 37 feet per second.

In t seconds, the soccer ball will travel x and y feet. The parametric equations that the ball will travel are

$$x = 14t$$
$$y = -16t^2 + 37t$$

a. How many seconds does it take the ball to travel to the goal? Round your answer to the nearest hundredth of a second.

b. The goalie is 5.5 feet tall and the goal is 8 feet tall. If the ball lands below the goalie's head, it is assumed that the goal has been blocked. Will Mia make the goal? Explain why or why not.

19. (20) **Multi-Step** Given $f(x) = -4x + 3$ and $g(x) = -x + 2$.

a. Graph $f(x)$ and $g(x)$ on the same grid.

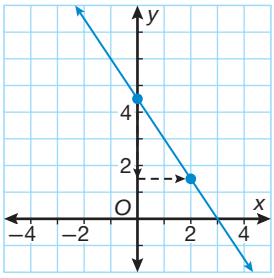
b. Graph the difference $(f - g)(x)$.

c. Use algebra to verify the difference $(f - g)(x)$.

20. (13) **Error Analysis** The work below uses the slope-intercept form of a linear equation to graph the equation $3x - 2y = 9$. Inspect the work. Is it correct? If not, explain the error and find the correct solution. If it is correct, explain in words each step of the work.

$$3x - 2y = 9$$
$$2y = -3x + 9$$
$$y = -\frac{3}{2}x + 4\frac{1}{2}$$

slope $= -\frac{3}{2}$; y-intercept $= \left(0, 4\frac{1}{2}\right)$



21. (17) **Fitness** An adult is considered to be in a healthy weight range if his or her BMI is within 3.2 of 21.7.

a. Write an absolute value inequality to determine the minimum and maximum BMI range that is considered healthy.

b. Solve the inequality.

22. (18) **Running** The record for the fastest mile ever run by a human belongs to Hicham El Guerrouj of Morocco. He ran one mile in 3.719 minutes. Determine El Guerrouj's speed in miles per hour.

23. (21) **Coffee** A coffee blend contains Sumatra beans, which cost \$12/lb, and Kona Hawaiian beans, which cost \$45/lb. If the blend costs \$30/lb, how much of each type of coffee is in 50 lb of the blend?

***24.** (24) **Write** Explain how using the elimination methods to solve systems with infinitely many solutions is similar to and different from using the elimination method to solve systems with no solutions.

25. *(14)* **Verify** Find the determinant using expansion by minors. Verify the solution using a graphing calculator.

$$\begin{vmatrix} -1 & 2 & -2 \\ 2 & 4 & 1 \\ -1 & 1 & -3 \end{vmatrix}$$

***26.** *(26)* **Justify** Is the equation $2\pi x + 3y = 20$ in standard form? Explain your reasoning.

27. *(9)* **Data Analysis** Amara, Kate, and Ava are all taking a test where each type of question is given a certain point value. Use matrix multiplication to find out who scored the greatest number of points on the test.

Questions Answered Correctly

	Type 1	Type 2	Type 3
Amara	2	3	7
Kate	5	5	4
Ava	9	6	3

Point Value

Type 1	4
Type 2	6
Type 3	8

28. *(16)* **Medicine** A 7% saline (salt) solution can help people with cystic fibrosis, but it must be mixed by hand from commercially available solutions of 0.9% and 10% solutions. How much of each commercial solution is needed to make 6 mL of the 7% solution?

a. Set up equations relating the number of mL of each solution to the saline concentrations. Let a represent the number of mL of the 0.9% solution and b the number of mL of the other solution.

b. Use Cramer's Rule to solve the system of equations.

***29.** *(26)* **Graphing Calculator** When you step on the brakes to stop a car, the car still continues forward. The distance traveled is called the "stopping distance." The faster a car is moving, the longer the stopping distance. The table shows the stopping distance at three different speeds for a passenger car traveling on dry pavement. Use the linear regression capabilities of your graphing calculator to write an equation in slope-intercept form for the stopping distance.

Initial Traveling Speed (mile/hour)	Stopping Distance (feet)
35	46.430
49	91.097
62	142.364

30. *(19)* **Write** Explain how to multiply $(x + 2)(2x - 1)$ using the FOIL method.

LESSON
27 Connecting the Parabola with the Quadratic Function

Warm Up

1. (7) **Vocabulary** An equation of the form $3x - 6 = 9$ is a _____ equation.
2. (13) Find the x- and y-intercepts of $6x - 4y = -12$.
3. (17) Identify the domain and range of the equation $y = |x|$.
4. (19) Multiply: $(x + 8)(x + 8)$.

New Concepts

A **quadratic equation** is one that can be written in the form $ax^2 + bx + c = 0$, where $x \neq 0$. Examples of quadratic equations include $x^2 = 25$, $3x^2 - 8 = 1$, and $-2x^2 - 8x = 0$.

A **quadratic function** is a function that can be written in the form $f(x) = ax^2 + bx + c$, which is called **standard form of a quadratic function,** where $a \neq 0$ and a, b, and c are real numbers. Because $f(x) = y$, a quadratic function can also be written as $y = ax^2 + bx + c$.

Caution

Remember to follow the order of operations when simplifying $f(x) = 2(x - 4)^2 + 9$. Square $(x - 4)$ before distributing the 2.

Example 1 Converting to Standard Form

a. Write $y - 6 = 2x - x^2$ in standard form.

SOLUTION

Isolate y and list the terms in decreasing order.

$$y - 6 = 2x - x^2$$
$$y = 2x - x^2 + 6$$
$$y = -x^2 + 2x + 6$$

b. Convert $f(x) = 2(x - 4)^2 + 9$ into standard form.

SOLUTION

Expand and simplify.

$$f(x) = 2(x - 4)^2 + 9$$
$$= 2(x^2 - 8x + 16) + 9$$
$$= 2x^2 - 16x + 32 + 9$$
$$= 2x^2 - 16x + 41$$

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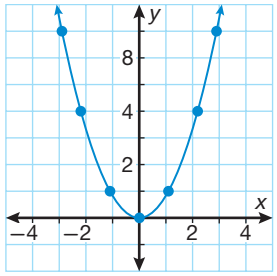
The parent function of a quadratic function is $f(x) = x^2$. So, in standard form, $a = 1$, $b = 0$, and $c = 0$.

The graph of the parent function, $f(x) = x^2$, is shown below.

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9



This curve is called a **parabola.** The graph of every quadratic function is a parabola. All parabolas have the same symmetric U shape. For the graph of $f(x) = x^2$, the point (0, 0) is the **vertex of the parabola.** The vertex indicates where the curve changes direction. It is the lowest (or highest) point on a parabola.

The **zeros** of a quadratic function are the values of x for which the function equals 0. Therefore on the graph of a quadratic function, the zeros are the x-intercepts, or where the graph intersects the x-axis. The y-intercept is the point where the graph intersects the y-axis and can be calculated by substituting x with 0.

Math Reasoning

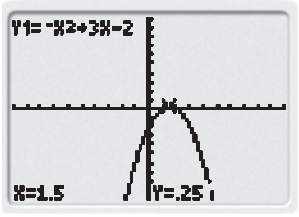
Analyze In standard form, the value of c is always the y-intercept. Why?

Exploration Finding the Zeros and Vertex of a Parabola

Find the zeros and vertex of $y = -x^2 + 3x - 2$.

1. Graph $y = -x^2 + 3x - 2$.

2. Press 2nd TRACE to access the **CALC** menu and select **2: zero.** Choose a point to the left and to the right of one of the x-intercepts and press enter; the coordinates of the first root (x-intercept) will be displayed. Repeat for the other x-intercept. What are the x-intercepts?

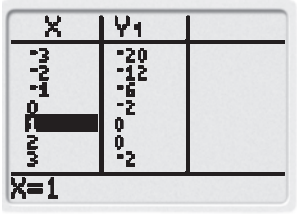


3. To find the y-intercept, press TRACE and enter 0, after **X=.** What is the y-intercept?

4. Since the vertex is halfway between the x-intercepts, the x-coordinate of the vertex is $\frac{1+2}{2}$, or 1.5. With TRACE still chosen, enter 1.5 for x.

What is the y-value? Where is the vertex located?

The **TABLE** menu, which can be accessed from 2nd GRAPH, can also be helpful when analyzing the graph of a quadratic function. The x-intercepts can be found by locating zeros in the second column.



When working with quadratic equations in standard form, the x-coordinate of the vertex of the parabola can be found using $x = -\frac{b}{2a}$. The y-coordinate can then be found using substitution. The vertex is on the axis of symmetry of the parabola.

The **axis of symmetry** is a line that divides a figure into two congruent mirror images. Therefore, the **reflection** of each point on the left side of a parabola is located on the right side of the parabola. The equation of the axis of symmetry is $x = -\frac{b}{2a}$.

Example 2 Graphing Quadratic Functions

a. Graph $f(x) = x^2 - x - 6$.

SOLUTION Use $x = -\frac{b}{2a}$ to find the x-coordinate of the vertex, then substitute to find the y-coordinate.

$$x = -\frac{b}{2a} = -\frac{(-1)}{2(1)} = \frac{1}{2}$$

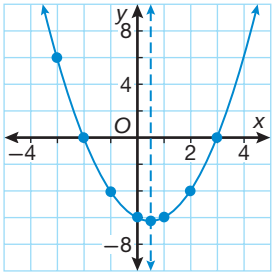
$$y = x^2 - x - 6 = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 6 = -6\frac{1}{4}$$

Vertex: $\left(\frac{1}{2}, -6\frac{1}{4}\right)$; Axis of symmetry: $x = -6\frac{1}{4}$.

Plot the axis of symmetry and the vertex.

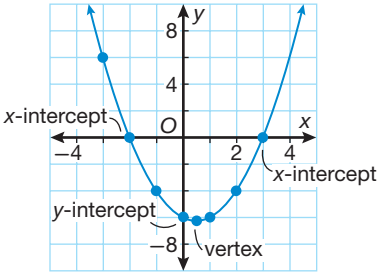
Next, make a table of ordered pairs. Plot the points and connect with a smooth line.

x	$x^2 - x - 6$	y
-3	$(-3)^2 - (-3) - 6$	6
-2	$(-2)^2 - (-2) - 6$	0
-1	$(-1)^2 - (-1) - 6$	-4
0	$(0)^2 - (0) - 6$	-6
1	$(1)^2 - (1) - 6$	-6
2	$(2)^2 - (2) - 6$	-4
3	$(3)^2 - (3) - 6$	0



b. Identify the x- and y-intercepts of $f(x) = x^2 - x - 6$ on its graph. Then use a calculator to determine the exact values.

SOLUTION The x-intercepts are the x-coordinates of the points where the graph intersects the x-axis at $(-2, 0)$ and $(3, 0)$. The y-intercept is the y-coordinate of the point where the graph intersects the y-axis at $(0, -6)$.



c. Identify the domain and range of $f(x) = x^2 - x - 6$.

SOLUTION The domain consists of the x-values. Any real number can be squared, so the domain is the set of all real numbers. The range consists of all the possible outputs. The parabola has a vertex at $(0.5, -6.25)$. There are no outputs below the vertex. Therefore, the least possible value for $f(x)$ is -6.25. The range is all real numbers greater than or equal to -6.25.

The height of an object that is dropped can be modeled by the quadratic function $h(t) = -16t^2 + b$, where t is the time in seconds, and b is the original height the object is dropped from. To find the time it takes to reach the ground, find the t-intercept where $h(t) = 0$.

Hint

Even if the graph of a quadratic function has two zeros, one of them may not make sense in a real-world application and can therefore be ignored.

Example 3 Application: Engineering

A bridge built over a stream can be modeled on the coordinate plane by the function $f(x) = -\frac{1}{4}x^2 + 16$, where the x-axis represents ground level and the units on the graph represent feet. Find the horizontal distance the bridge spans.

SOLUTION

Understand The question asks for the distance between where the bridge touches the ground on either side of the stream.

Plan Find the x-intercepts and use their values to find the distance.

Solve Use a graphing calculator. The intercepts are -8 and 8.

Because the axis of symmetry is the y-axis, add the absolute values of the x-intercepts: $8 + 8 = 16$. The bridge spans a distance of 16 feet.

Check The **TABLE** feature confirms that $y = 0$ when $x = -8$ and 8.

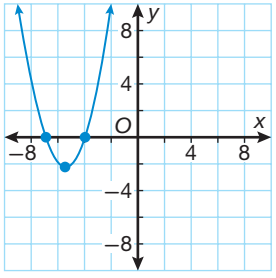
Lesson Practice

a. *(Ex 1)* Write $11 - x^2 = y + 7x$ in standard form.

b. *(Ex 1)* Convert $f(x) = -(x + 1)^2 - 5$ into standard form.

c. *(Ex 2)* Graph $f(x) = x^2 - 5x + 4$.

d. *(Ex 2)* Estimate the x- and y-intercepts and the coordinates of the vertex of $f(x) = x^2 + 11x + 28$ from its graph. Then use a calculator to determine the exact values.



e. *(Ex 2)* Identify the domain and range of $f(x) = -x^2 - 4.5$.

f. *(Ex 3)* A coin is thrown off the roof of a building that is 35 feet above the sidewalk. The height of the coin, in feet, at any second, x, during the fall can be found using the function $f(x) = -16x^2 + 35$. How long does it take for the coin to hit the sidewalk? Round your answer to the nearest hundredth of a second.

Practice Distributed and Integrated

Convert each function to standard form.

***1.** (27) $f(x) = (2x + 3)^2$

***2.** (27) $f(x) = 2(x + 5) + x^2$

***3.** (27) $f(x) = (x - 5)^2$

Graph each function.

***4.** (27) $f(x) = (-x + 1)^2$

***5.** (27) $f(x) = x^2 - 3$

Factor each polynomial.

6. (23) $2x^2 - 4x - 30$

7. (23) $-4x^3 + 20x^2 + 12x$

8. (18) Convert 3 liters per minute to milliliters per second.

9. (13) Identify the y-intercept given the linear equation $y - 4 = 2(x - 2)$.

Find the constant of variation for the given values of x and y given that x varies inversely with y.

10. (12) $x = 3, y = 12$

11. (12) $x = 10, y = 5$

12. (12) $x = 4, y = 9$

***13.** (24) **Multi-Step** Shirley has 47 coins in a jar. The jar contains only dimes and nickels and the value of the coins is \$3.20.

a. Using d for dimes and n for nickels, write an equation representing the number of coins.

b. Write an equation to show that the value of the dimes plus the value of the nickels is \$3.20 in all.

c. Write and solve a linear system to find the number of dimes and number of nickels in Shirley's jar.

14. (7) Solve the equation for r: $3(5r - 10) = 8(2r - 4) + 5r$.

A $r = \frac{1}{4}$ **B** $r = \frac{1}{3}$ **C** $r = 2$ **D** $r = 3$

***15.** (27) **Justify** Tell why the range of a quadratic function will never be the set of all real numbers.

16. (21) **Write** Without solving the system, can you tell how many solutions this system of equations has? Explain.

$$y + 1 = 3x$$
$$2y - 6x = -2$$

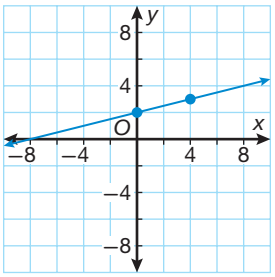
17. (12) **Geometry** Identify the kind of variation each equation represents.

a. Volume of a rectangular prism = length × width × height

b. Volume of a cube = (length)^3

18. *(13)* **Multiple Choice** Choose the letter that best represents the graph of $y = -4x + 2$.

A



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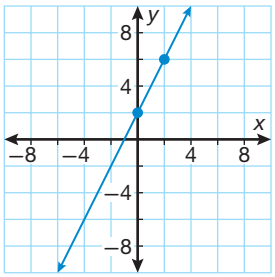

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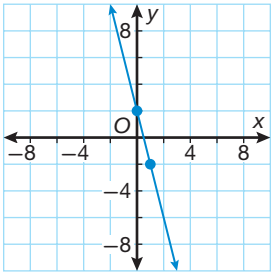

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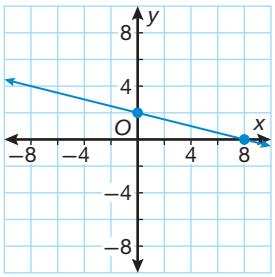

***19.** *(23)* **Multi-Step** A rectangular wooden deck is 40 feet long and 30 feet wide. The owner wants to increase the width and length to double the area of the deck. He also wants the increase in length to be twice as much as the increase in width.

a. Calculate the new area of the deck.

b. Write and solve an equation to find out how much the owner should increase each dimension.

***20.** *(21)* **Error Analysis** Two students were asked which of these lines coincide. Their answers are below. Which student is correct? What is the relationship between the lines of the incorrect answer?

Equation 1: $y + 3x = 7$ Equation 2: $y = -3x + 5$ Equation 3: $10 - 2y = 6x$

Becky: Equations 1 and 3 coincide.

Randy: Equations 2 and 3 coincide.

21. *(14)* **Justify** Justify the claim that the determinant of the product of matrices $\begin{bmatrix} 3.1 & -2.2 \\ 0.6 & 4.5 \end{bmatrix}$ and $\begin{bmatrix} -1.9 & 4.8 \\ -2.2 & 3.0 \end{bmatrix}$ equals the product of the determinants of these matrices.

22. *(18)* **Fuel Economy** A dragster can consume 6 gallons of gasoline in $\frac{1}{4}$ mile. What is the fuel economy in miles per gallon? How far would a car with a 20-gallon tank get before needing a refill?

23. *(7)* **Imports** The total imports of goods and services, in millions of dollars, for the U.S. in 2004, 2005, and 2006 was 5,971,007. The imports in 2005 were 228,100 more than in 2004 and 206,784 less than in 2006. What were the imports in millions of dollars in 2005?

24. (19) While buying a bird feeder, Karl discovers one in the shape of a rectangular prism. Karl is trying to determine the volume of bird food that the feeder will hold. He knows that the dimensions are as follows:

Length: x inches Width: $(x - 3.75)$ inches Height: $(x - 0.25)$ inches

Write an expression for the volume of the bird feeder that Karl can use in his calculations.

25. (11) **Error Analysis** A student says that the polynomial $ab^4 + a^2b^2 - 3$ is of higher degree than the polynomial $a^2b^3 + ab + 2$. Is this correct? Explain.

26. (20) **Statistics** The proceeds for three major fundraisers are represented by $f(x) = 35x - 100$, $g(x) = 52x - 60$, and $h(x) = 100x - 250$. Write a function $m(x)$ that would represent the mean of the proceeds in terms of x.

***27.** (25) The table shows the average salaries of various engineering jobs in 2006. Find the range and median of this data.

Job	Aerospace	Agricultural	Civil	Computer	Environmental	Mechanical	Petroleum
Average Salary	$50,993	$46,172	$43,679	$52,464	$47,384	$50,236	$61,156

28. (17) **Verify** Show that $|1 - 3x| = 5x$ has two solutions, one of which is extraneous.

***29.** (26) **Medicine** The Centers for Disease Control has developed growth charts that pediatricians use to see if a child's height is within a certain average. The chart shows data for a girl's height in the 50th percentile. The three points are on a line. Use the data to write the equation of the line in slope-intercept form.

Age (years)	Height (inches)
3	37
4	40
5	43

***30.** (9) **Graphing Calculator** $A = \begin{bmatrix} -5 & 3 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$ Use a graphing calculator to find AB and BA. Is $AB = BA$?

LESSON
28

Simplifying Rational Expressions

Warm Up

1. (11) **Vocabulary** A _____ is a monomial or the sum or difference of monomials.

2. (19) Multiply $-4x^2(6x^2y^3 + 2xy^2 - 1)$.

3. (23) Factor $-12x^6 + 15x^5y$.

4. (23) Which is a difference of squares?

A $(16x^2 + y^2)$ **B** $(8 - 4y^2)$ **C** $(9 - 50y^6)$ **D** $(x^{10} - 100)$

New Concepts

A **rational expression** is the quotient of two polynomials, such as $\frac{2x^2 + 4x - 4}{x + 3}$, with a denominator of a degree that is greater than or equal to one. As with any fraction, the denominator cannot equal zero. Therefore, $\frac{2x^2 + 4x - 4}{x + 3}$ is not defined when $x = -3$ because $-3 + 3 = 0$.

In any function, an excluded value is a domain value that makes the function undefined. For the function above, -3 is an excluded value. Not all rational expressions have excluded values and some have more than one excluded value.

No excluded values	More than one excluded value
$\frac{-3x - 6}{4}$, $\frac{4x^2 + 16}{x^2 + 1}$	$\frac{5}{x^2 + 3x + 2}$, excluded values: -1 and -2

Math Reasoning

Analyze Why will the expression $x^2 + 1$ never be equal to 0 for any value of x?

A rational expression is simplified when there are no common factors, other than 1, in the numerator and denominator.

Simplified	Not simplified
$\frac{x^3}{2}$, $\frac{x + 3}{x + 4}$	$\frac{x^6y^2}{x^4}$, common factor: x^4 $\frac{(x + 3)(x - 7)}{(x - 7)}$, common factor: $(x - 7)$

Rational expressions with monomial numerators and denominators are best simplified by using the Quotient of Powers Property, which states that to divide two powers with the same base, subtract the exponents.

Quotient of Powers Property
For $a \neq 0$, and integers m and n, $\frac{a^m}{a^n} = a^{m-n}$.

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Example 1 Simplifying by Using the Quotient of Powers Property

Identify any excluded values. Then simplify the expression.

a. $\frac{2m^3}{m^2}$

SOLUTION The excluded value is 0 because $0^2 = 0$.

$$\frac{2m^3}{m^2} = 2m^{3-2} = 2m$$

b. $\frac{x^4}{x^7}$

SOLUTION The excluded value is 0 because $0^7 = 0$.

$$\frac{x^4}{x^7} = x^{4-7} = x^{-3} = \frac{1}{x^3}$$

All rational expressions can be simplified by dividing out the common factors that appear in both the numerator and denominator.

Example 2 Simplifying by Dividing Out Common Factors

Identify any excluded values. Then simplify the expression.

a. $\frac{4x + 8}{8x + 16}$

SOLUTION To find what value of x makes the denominator 0, set the denominator equal to 0 and solve for x.

$$8x + 16 = 0$$
$$8x = -16$$
$$x = -2$$ The excluded value is -2.

Factor out the GCF from the numerator and denominator.

$$\frac{4x + 8}{8x + 16} = \frac{4(x + 2)}{8(x + 2)}$$
$$= \frac{\cancel{4}^{1}\cancel{(x+2)}}{\cancel{8}_{2}\cancel{(x+2)}} = \frac{1}{2}$$ Divide out 4 and $(x + 2)$.

b. $\frac{3b^2 - 27}{b + 3}$

SOLUTION The excluded value is -3.

$$\frac{3b^2 - 27}{b + 3} = \frac{3(b^2 - 9)}{b + 3}$$ Factor out the GCF of 3 in the numerator.
$$= \frac{3(b + 3)(b - 3)}{b + 3}$$ Factor the difference of squares.
$$= \frac{3(\cancel{b+3})(b - 3)}{\cancel{b+3}}$$ Divide out the common factor, $b + 3$.
$$= 3(b - 3)$$

When -1 is factored out of $b - a$, the result is $-1(-b + a)$, which can be written as $-1(a - b)$. This result allows opposite binomials to be factored out of a rational expression.

Caution

Be sure to determine any excluded values before simplifying the expression.

Math Language

Opposite binomials are binomials of the form $a - b$ and $b - a$.

Example 3 Simplifying by Factoring Out −1

Identify any excluded values for $\frac{6x-12}{6-3x}$. Then simplify the expression.

SOLUTION Solve $6-3x=0$ to find the excluded value.

$$6-3x=0$$
$$-3x=-6$$
$$x=2$$ The excluded value is 2.

Factor the numerator and denominator to simplify.

$$\frac{6x-12}{6-3x}=\frac{6(x-2)}{3(2-x)}$$ The factors $(x-2)$ and $(2-x)$ are opposite binomials.

$$=\frac{6(x-2)}{3(-1)(x-2)}$$ Factor -1 out of the binomial in the denominator.

$$=\frac{\overset{2}{\cancel{6}}(x-2)}{\underset{1}{\cancel{3}}(-1)(x-2)}=\frac{2}{-1}=-2$$

Example 4 Simplifying Rational Expressions with Trinomials

Identify any excluded values for $\frac{-2x^2-4x+30}{3x^2+21x+30}$. Then simplify the expression.

SOLUTION To find the excluded values, set the denominator equal to 0.

$$0=3x^2+21x+30$$
$$0=3(x^2+7x+10)$$
$$0=3(x+2)(x+5) \longrightarrow x+2=0 \text{ or } x+5=0$$
$$x=-2 \text{ or } x=-5$$

The excluded values are -2 and -5.

Now, factor and divide out common factors.

$$\frac{-2x^2-4x+30}{3x^2+21x+30}=\frac{-2(x^2+2x-15)}{3(x+2)(x+5)}=\frac{-2(\cancel{x+5})(x-3)}{3(x+2)(\cancel{x+5})}=\frac{-2(x-3)}{3(x+2)}$$

Example 5 Application: Sports

A basketball is a sphere. Write a simplified rational expression for the volume to surface area ratio of a basketball. Then find the volume to surface area ratio for a basketball whose diameter is 9 inches.

Hint

The radius is one-half of the diameter.

SOLUTION

For a sphere, $V=\frac{4}{3}\pi r^3$ and $SA=4\pi r^2$. The ratio is $\frac{\frac{4}{3}\pi r^3}{4\pi r^2}=\frac{\frac{4}{3}r}{4}=\frac{1}{3}r$.

Substitute $\frac{9}{2}$ for r: $\frac{1}{3}\left(\frac{9}{2}\right)=\frac{3}{2}$.

The volume to surface area ratio of a basketball with a 9-inch diameter is $\frac{3}{2}$.

Lesson Practice

Identify any excluded values. Then simplify the expression.

a. (Ex 1) $\dfrac{3p^8}{9p^3}$

b. (Ex 1) $\dfrac{a}{6a^4}$

c. (Ex 2) $\dfrac{x^2 + 7x}{6x + 42}$

d. (Ex 2) $\dfrac{-2x^2 + 2}{10x - 10}$

e. (Ex 3) $\dfrac{a^2 - 81}{81 - 9a}$

f. (Ex 4) $\dfrac{2x^2 - 4x - 6}{2x^3 - 2x}$

g. (Ex 3) The moon is a sphere. Write a simplified rational expression for the surface area to volume ratio of the moon. Then find the surface area to volume ratio for the moon given that its diameter is 2160 miles.

Practice Distributed and Integrated

***1.** (27) Convert $f(x) = (-x + 1)^2$ to standard form.

Identify any excluded values, then simplify each expression.

***2.** (28) $\dfrac{-2q^3}{16q}$

***3.** (28) $\dfrac{s + 1}{15(s + 1)^2}$

***4.** (28) $\dfrac{x^2}{x}$

***5.** (28) $\dfrac{4x^5}{12x}$

Use Cramer's Rule to solve the following systems of equations.

6. (16) $\begin{aligned} x + y &= 5 \\ x - y &= 5 \end{aligned}$

7. (16) $\begin{aligned} -5x + y &= 7 \\ x - 3y &= 7 \end{aligned}$

Simplify each of the following expressions.

8. (3) $x^3y^{-3}x^{-5}y^8y^7$

9. (3) $\dfrac{xx^{-2}y^5x^{-1}y^{-1}}{x^8y^{-3}y^2x^{-4}}$

10. (3) $\dfrac{2x^6y^{-4}}{24xy^{-1}}$

Change each of the following to percents.

11. (6) $\dfrac{3}{5}$

12. (6) 0.8

13. (6) 0.025

14. (10) **Carnival** The entry cost at the local fair is \$9.00. Each ride costs \$1.50. If Matthew has \$45, how many rides could he go on? Write an inequality to help you answer the problem.

15. (18) **Write** Write a situation explaining the problem: $\dfrac{20 \text{ hr}}{1} \times \dfrac{60 \text{ min}}{1 \text{ hr}} \times \dfrac{60 \text{ s}}{1 \text{ min}} = 72{,}000 \text{ s}$

***16.** (28) **Analyze** Compare and contrast the Quotient of Powers Property to the Product of Powers Property.

17. *(17)* **Multiple Choice** Solve $-3|4x| - 2 \geq 22$.

A $-2 \leq x \leq 2$ **B** $x \leq -2$ or $x \geq 2$

C All real numbers **D** No solution

***18.** *(26)* **Aviation** The Federal Aviation Administration tracks airline delays for airports around the country. During the first half of 2007, the flights delayed at Dallas-Fort Worth Airport averaged 4,500 per month. Use this information to write an equation in slope-intercept form that could be used to predict the total number of delays throughout the year. Let x equal the month of the year and y equal the cumulative delays.

***19.** *(25)* **Data Analysis** A chemist weighs samples obtained from a production run. The weights of the samples are 13 g, 14 g, 65 g, 11 g, 15 g, 14 g, 14 g, 12 g, 13 g, 15 g, 14 g, and 12 g.

a. Find the mean of the data.

b. Find the standard deviation.

c. Identify any outliers.

d. Describe how any outlier affects the mean and the standard deviation of this data set.

20. *(11)* **Packaging Technology** An engineer has a box in the shape of a cube. He wants to know how the volume is affected if he adds 2 units to each base edge and subtracts 3 units from the height. Does the volume increase, decrease, or remain the same? Explain your answer.

21. *(13)* **Multi-Step** Use $-7x + 14y = -84$.

a. Find the x-intercept.

b. Find the y-intercept.

22. *(24)* **Geometry** Two angles are complementary and the difference between the larger angle and twice the smaller angle is 30°. Write a system to represent the scenario and solve the system to find the measures of the angles.

23. *(16)* **Error Analysis** Two students solved the system of equations $\begin{matrix} 7 - 5y - 4x = 0 \\ 16 - 2y - 5x = 0 \end{matrix}$. Find the error or errors in these students' work and solve the system correctly.

Student A $\begin{matrix} 4x + 5y = 7 \\ 5x + 2y = 16 \end{matrix}$ $x = \dfrac{\begin{vmatrix} 7 & 5 \\ 16 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & 5 \\ 5 & 2 \end{vmatrix}} = \dfrac{-66}{-17} \approx 3.88$ and $y = \dfrac{\begin{vmatrix} 5 & 7 \\ 2 & 16 \end{vmatrix}}{\begin{vmatrix} 4 & 5 \\ 5 & 2 \end{vmatrix}}$

$= \dfrac{66}{-17} \approx -3.88$

Student B $\begin{matrix} -5y - 4x = -7 \\ -2y - 5x = -16 \end{matrix}$ $y = \dfrac{\begin{vmatrix} -7 & -4 \\ -16 & -5 \end{vmatrix}}{\begin{vmatrix} -5 & -4 \\ -2 & -5 \end{vmatrix}} = \dfrac{-29}{17} \approx -1.71$ and $x = \dfrac{\begin{vmatrix} -5 & -7 \\ -2 & -16 \end{vmatrix}}{\begin{vmatrix} -5 & -4 \\ -2 & -5 \end{vmatrix}}$

$= \dfrac{66}{17} \approx 3.88$

*24. (20) **Graphing Calculator** Given $f(x) = 2x - 2$ and $g(x) = x^2$, find $(f + g)(x)$. Graph $(f + g)(x)$ using a graphing calculator. Explain what the graph shows.

25. (15) **Multi-Step** Set up and solve a system to find an equation of the line through points (2, 5) and (4, −3). Put the equation in the form $y = mx + b$.

a. Substitute the point (2, 5) for (x, y) in the form $y = mx + b$ to get a linear equation in unknowns m and b.

b. Substitute the point (4, −3) into the form $y = mx + b$ to get another linear equation in m and b.

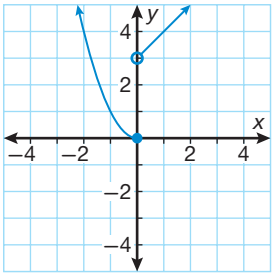
c. Solve the system for m and b.

d. What is the equation of the line?

26. (21) **Business School** This table shows the change in rank of two undergraduate business schools from 2006 to 2007. If the rankings continue to change at these constant rates, in what year will University of Florida and University of Wisconsin have the same rank?

School	2006	2007	Change
University of Florida	47	43	−4
University of Wisconsin	27	28	+1

27. (22) **Error Analysis** Courtney determines that the relation shown at right fails the vertical line test. Write a response to correct her decision.



*28. (27) **Verify** Verify that the roots of $f(x) = x^2 + 7x - 18$ are −9 and 2 by substituting −9 and 2 into the function.

*29. (25) **Analyze** Cassie's job is to analyze a set of 100 scores that range from 0 to 10 from a recent survey. The set of values includes exactly 6 scores that are zero. She decides not to include the 6 zero scores in the analysis, based on the false assumption that zero is not really a number since it has no value.

a. Is it possible to determine whether the mean will be greater than, less than, or equal to the value using all 100 scores?

b. Is it possible to determine whether the standard deviation will be greater than, less than, or equal to the value using all 100 scores? Explain your answers.

30. (23) **Artwork** Jacob would like to buy a frame for some artwork he painted. His painting is 20 feet by 16 feet. He wants an equal-sized border around the outside of the painting, and he wants to increase the amount of area that the painting covers by 50%.

a. Write an expression to solve for the width of the border around the painting.

b. Calculate what size frame he should buy.

LESSON
29 Solving Systems of Equations in Three Variables

Warm Up

1. (21) **Vocabulary** A system of simultaneous equations that have no common solution is called _____.

2. (2) Evaluate the expression $-2x - 5y$ when $x = -5$ and $y = -3$.

3. (24) Solve the system of equations using the elimination method: $\begin{aligned} x + y &= 5 \\ x - y &= 1 \end{aligned}$

New Concepts

Systems of equations are not limited to two equations and two variables. A linear equation in three variables is of the form $Ax + By + Cz = D$. Systems of equations in three variables require three equations to solve. The solution to a system of three equations in three variables is an **ordered triple** and is noted (x, y, z). Similar to systems of two equations, systems of three equations can be solved using elimination.

Hint

The order in which the equations are eliminated is not important. It is important that all three equations are used.

To solve a system of three equations in three variables
Step 1: Use two of the three equations to eliminate one of the variables.
Step 2: Use the remaining equation and one of the equations used in Step 1 to eliminate the same variable as in Step 1. From Steps 1 and 2, there is now a system of two equations in two variables.
Step 3: Solve the system of equations resulting from Steps 1 and 2.
Step 4: Substitute the values for the two variables found in Step 3 into one of the original equations to solve for the third variable.
Step 5: Check the solution by substituting the values for the three variables into each of the three original equations.

A solution to a system in three variables means that there is an ordered triple that satisfies all three equations. A solution is either a single point or a line. If the three planes do not share a common point, then there is no solution.

There are three possible results when solving systems of three equations: one solution, infinitely many solutions, and no solution.

Number of Solutions	Type of Solution	Graphic Representation
one	consistent/ independent	planes intersect at exactly one point
infinitely many	consistent/dependent	same plane or planes intersect in a line
none	inconsistent	parallel planes or planes that each intersect at exactly one of the other planes

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Example 1 System of Three Equations with One Solution

Solve the system of equations.

$$\begin{aligned} x + y + z &= 4 \\ 9x + 3y + z &= 0 \\ 4x + 2y + z &= 1 \end{aligned}$$

SOLUTION

Step 1: Eliminate one variable using the first and second equations.

$$\begin{array}{rr} -1(x + y + z) = -1(4) & -x - y - z = -4 \\ \underline{+9x + 3y + z = 0} & \underline{+9x + 3y + z = \quad 0} \\ & 8x + 2y \quad = -4 \end{array}$$

Step 2: Eliminate the same variable using the second and third equations.

$$\begin{array}{rr} -1(9x + 3y + z) = -1(0) & -9x - 3y - z = 0 \\ \underline{+4x + 2y + z = 1} & \underline{+4x + 2y + z = 1} \\ & -5x - y \quad = 1 \end{array}$$

Step 3: Solve the system of equations resulting from Steps 1 and 2.

$$\begin{array}{lll} 8x + 2y = -4 & 8x + 2y = -4 & 8x + 2y = -4 \\ 2(-5x - y) = 2(1) & \underline{+ \ -10x - 2y = \quad 2} & 8(1) + 2y = -4 \\ & -2x \quad = -2 & 8 + 2y - 8 = -4 - 8 \\ & \frac{-2x}{-2} = \frac{-2}{-2} & 2y = -12 \\ & x = 1 & \frac{2y}{2} = \frac{-12}{2} \\ & & y = -6 \end{array}$$

Step 4: Substitute the values of the two variables found in Step 3 into one of the original equations to solve for the third variable.

$$\begin{aligned} x + y + z &= 4 \\ (1) + (-6) + z &= 4 \\ -5 + z &= 4 \\ -5 + z + 5 &= 4 + 5 \\ z &= 9 \end{aligned}$$

Since $x = 1$, $y = -6$, and $z = 9$, the ordered triple is $(1, -6, 9)$.

Step 5: Check the ordered triple in each of the original equations.

$$\begin{array}{lll} x + y + z = 4 & 9x + 3y + z = 0 & 4x + 2y + z = 1 \\ (1) + (-6) + (9) \stackrel{?}{=} 4 & 9(1) + 3(-6) + (9) \stackrel{?}{=} 0 & 4(1) + 2(-6) + (9) \stackrel{?}{=} 1 \\ 4 = 4 \checkmark & 0 = 0 \checkmark & 1 = 1 \checkmark \end{array}$$

All three statements are true. Therefore, the single solution to the system of three equations is $(1, -6, 9)$.

Math Reasoning

Write How would you determine if a given ordered triple is a solution to a system of three equations in three variables?

Example 2 System of Three Equations with Infinitely Many Solutions

Solve the system of equations. $2x + 3y + 4z = 12$
$-6x - 12y - 8z = -56$
$4x + 6y + 8z = 24$

Hint

It does not matter which variable you eliminate in step 1 as long as you eliminate the same variable in step 2.

SOLUTION

Step 1: Eliminate one variable using the first and second equations.

$$\begin{array}{l} 2(2x + 3y + 4z) = 2(12) \\ \underline{-6x - 12y - 8z = -56} \end{array} \qquad \begin{array}{r} 4x + 6y + 8z = 24 \\ \underline{-6x - 12y - 8z = -56} \\ -2x - 6y = -32 \end{array}$$

Step 2: Eliminate the same variable using the second and third equations.

$$\begin{array}{r} -6x - 12y - 8z = -56 \\ \underline{+4x + 6y + 8z = 24} \\ -2x - 6y = -32 \end{array}$$

Step 3: Solve the system of equations resulting from Steps 1 and 2.

$$\begin{array}{l} -2x - 6y = -32 \\ -2x - 6y = -32 \end{array}$$

Use elimination to solve for the solution to the system of two equations.

$$\begin{array}{l} -1(-2x - 6y) = -1(32) \\ \underline{+ \ -2x - 6y = -32} \end{array} \qquad \begin{array}{r} 2x + 6y = 32 \\ \underline{+ \ -2x - 6y = -32} \\ 0 = 0 \end{array}$$

The result is an identity statement. Therefore, there are infinitely many solutions to the system of equations. Those solutions form a line and the equation of the line is $-2x - 6y = 32$.

Example 3 System of Three Equations with No Solutions

Solve the system of equations. $x + 2y - 3z = 4$
$2x + 4y - 6z = 3$
$-x + 5y + 3z = 1$

SOLUTION

Step 1: Eliminate one variable using the first and second equations.

$$\begin{array}{l} -2(x + 2y - 3z) = -2(4) \\ \underline{+2x + 4y - 6z = 3} \end{array} \qquad \begin{array}{r} -2x - 4y + 6z = -8 \\ \underline{+2x + 4y - 6z = 3} \\ 0 = -5 \end{array}$$

The result is a false statement. Therefore, there is no solution to the system of equations. Notice that the coefficients of the first and second equations are multiples of each other, but their constants are not. Therefore, these equations form parallel planes and do not share a common point.

A system that has at least one solution is **consistent**, so systems with one solution or infinitely many solutions are consistent.

A system of simultaneous equations that have no common solution is **inconsistent**.

Example 4 Types of Systems of Three Equations

a. Solve the system of linear equations. Determine whether the system is consistent or inconsistent.

$$\begin{aligned} -x - 5y + 3z &= 7 \\ 5x + y + 3z &= 9 \\ 2x + 10y - 6z &= -4 \end{aligned}$$

SOLUTION Notice that the coefficients of x for the first and third equations are multiples. The coefficients of y for these same two equations are also multiples, as are the coefficients of z. The constants for these two equations are *not* multiples.

$$\begin{aligned} -x - 5y + 3z &= 7 \\ 5x + y + 3z &= 9 \\ 2x + 10y - 6z &= -4 \end{aligned} \quad \longrightarrow \quad \begin{aligned} -x - 5y + 3z &= 7 \\ 5x + y + 3z &= 9 \\ -2(-x - 5y + 3z) &= -4 \end{aligned}$$

Combine the first and third equations to solve.

$$\begin{aligned} 2(-x - 5y + 3z) &= 2(7) \\ 2x + 10y - 6z &= -4 \end{aligned} \quad \longrightarrow \quad \begin{aligned} -2x - 10y + 6z &= 14 \\ 2x + 10y - 6z &= -4 \\ 0 &= 10 \end{aligned}$$

Because the coefficients of the variables are multiples for each of the variables and the constants are not multiples, there is no solution. This means the system is inconsistent.

Math Reasoning

Analyze Why must linear systems be independent, dependent, or inconsistent?

b. Solve the system of linear equations. Determine whether the system is consistent or inconsistent.

$$\begin{aligned} 5x - 9y - 6z &= 11 \\ -5x + 9y + 6z &= -11 \\ 2x - 4y - 3z &= 6 \end{aligned}$$

SOLUTION Notice that the first and second equations are multiples of each other, which means they are parallel planes.

Combine the second and third equations to solve.

$$\begin{aligned} 2(2x - 4y - 3z) &= 2(6) \\ (4x - 8y - 6z &= 12) \end{aligned} \quad \longrightarrow \quad \begin{aligned} -5x + 9y + 6z &= -11 \\ 4x - 8y - 6z &= 12 \\ -x + y &= 1 \end{aligned}$$

There are an infinite number of solutions that form a line. The equation of the line is $-x + y = 1$.

Caution

You cannot assume that equations that do not fit the criteria in Example 4 parts a and b have one solution. You may have to solve the system of equations to find the number of solutions.

Example 5 Application: Investing

A person invests $12,000 for one year; some is invested at 5%, some at 10%, and the remainder at 13% interest. The combined interest earned at the end of the year from these investments was $920. The amount invested at 10% is $4,000 less than the amount invested at 5% and 13% combined. Find the amount of money invested at each rate.

SOLUTION

Money is invested in three accounts and each account earns a different rate of interest: 5%, 10%, and 13%.

Define the variables: Let x = amount invested in 5% account, y = amount invested in 10% account, and z = amount invested in 13% account.

The total amount invested is $12,000. So,

$$x + y + z = 12000$$

At the end of the year, the total amount of interest earned from all three accounts was $920. The interest earned on an account for one year can be found by multiplying the interest rate by the amount invested.

$$0.05x + 0.10y + 0.13z = 920$$

The amount invested in the 10% account is $4,000 **less** than the total amount in the 5% and 13% accounts.

$$y = (x + z) - 4000 \text{ or } -x + y - z = -4000$$

Now we have a system of three linear equations with three variables.

Caution

Always organize your thoughts and write a plan before attempting to write a system of equations.

Write the system of equations, then solve.

$$\begin{aligned} x + y + z &= 12{,}000 \\ 0.05x + 0.1y + 0.13z &= 920 \\ -x + y - z &= -4{,}000 \end{aligned}$$

$$\begin{aligned} x + y + z &= 12000 \\ -x + y - z &= -4000 \\ \hline 2y &= 8000 \\ y &= 4000 \end{aligned}$$

Substitute the y-value of 4000 in the second and third equations.

$$\begin{aligned} -x + 4000 - z &= -4000 \\ -x - z &= -8000 \end{aligned} \qquad \begin{aligned} 0.05x + 400 + 0.13z &= 920 \\ 0.05x + 0.13z &= 520 \end{aligned}$$

Combine the two simplified equations to eliminate a variable.

$$\begin{aligned} -x - z &= -8000 \\ 20(0.05x + 0.13z) &= 20(520) \end{aligned}$$

$$\begin{aligned} -x - z &= -8000 \\ x + 2.6z &= 10400 \\ \hline 1.6z &= 2400 \\ z &= 1500 \end{aligned}$$

Substitute the z-value of 1500 to find x.

$$\begin{aligned} -x - 1500 &= -8000 \\ x &= 6500 \end{aligned}$$

The amount placed in each account is: $6,500 in the 5% account, $4,000 in the 10% account, and $1,500 in the 13% account.

Lesson Practice

Solve the systems of equations.

a. (Ex 1)
$$\begin{aligned} 6x - 3y + 3z &= 3 \\ 3x + 3y - z &= 5 \\ 5x + 2y - 2z &= 4 \end{aligned}$$

b. (Ex 2)
$$\begin{aligned} -3x - y - 4z &= 15 \\ 9x + 3y + 12z &= -45 \\ -6x + 2y - 8z &= -30 \end{aligned}$$

c. (Ex 3) Solve the system of equations.
$$\begin{aligned} 4x - 2y + z &= 2 \\ 2x + 4y - 3z &= 0 \\ -12x + 6y - 3z &= 6 \end{aligned}$$

How many solutions does the system of equations have and what type of system is it?

d. (Ex 4)
$$\begin{aligned} 4x + 9y - 22z &= 10 \\ 3x - y + 2z &= -3 \\ -15x + 5y - 10z &= -15 \end{aligned}$$

e. (Ex 4)
$$\begin{aligned} 5x - 3y + z &= 13 \\ -28x + 24y - 4z &= 20 \\ -7x + 6y - z &= 5 \end{aligned}$$

f. (Ex 5) A person invests \$5,400 for one year; some is invested at 12%, some at 15%, and the remainder at 18%. The combined interest earned at the end of the year from these investments was \$822. The amount invested at 12% is \$2,600 less then the amount invested at 15% and 18% combined. Find the amount of money invested at each rate.

Practice Distributed and Integrated

Solve each system of equations.

***1.** (29)
$$\begin{aligned} x - 5y + 2z &= 2 \\ 2x - y + z &= 1 \\ 7x - 3y + z &= 8 \end{aligned}$$

***2.** (29)
$$\begin{aligned} -4x + 2y - z &= 2 \\ 4x + 8y - 6z &= 0 \\ 4x - 2y + z &= 2 \end{aligned}$$

Factor the polynomials.

3. (23) $144x^2 - 1$

4. (23) $9x^2 - 25$

Given the functions $f(x) = 2x + 1$ and $g(x) = x^2$,

5. (20) find $(f - g)(4)$.

6. (20) find $(f + g)(-3)$.

Given the functions $f(x) = -x - 7$ and $g(x) = -(x^2 + 3)$,

7. (20) find $(g + f)(0)$.

8. (20) find $(g - f)(-1)$.

Solve each equation.

9. (17) $|x + 3| = 4$

10. (17) $|2x + 6| + 2 = 1$

11. (17) $|1 - 3x| + 5 = 6$

12. (19) **Write** Explain FOIL for the multiplication of any two binomials in the form $(a + b)(a + b)$.

Convert the following to feet per minute.

13. (18) 18 in./hour

14. (18) 144 in./min

***15.** (29) **Verify** Is the ordered triple $(-1, 2, -3)$ a solution of the system?

$$-x + y - z = 6$$
$$-x - 2y + 2z = -9$$
$$3x - y + 2z = -11$$

16. (13) Given the equation $6x + 3y = -9$, identify the slope and y-intercept.

17. (10) **Error Analysis** A student solved the inequality $-4x - 2 \geq 14$ and got the solution $x \geq -4$. Explain and correct the error.

***18.** (14) **Graphing Calculator** Calculate the determinant of $\begin{bmatrix} 4 & -2 & 1 \\ 3 & 0 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ using a graphing calculator.

***19.** (26) **Multiple Choice** Using the data in the table, what is the equation in standard form of the line that passes through the three points?

x	y
1	−5
2	−17.5
3	−30

A $y = -12.50x + 7.5$ **B** $y = 12.50x + 7.5$

C $125x + 10y = 75$ **D** $-125x + 10y = 75$

***20.** (23) **Coordinate Geometry** A rectangle is drawn on a coordinate grid with one side along the x-axis and one vertex at $(4 + v, 0)$. The area of the rectangle is $v^2 - 7v + 10$. Determine an expression for an adjacent vertex. Explain your answer.

21. (8) If x varies directly with y and the constant of variation is 3, what is the value of x when $y = 12$?

22. (22) **Justify** A man records his height every 5 years. Examine the table below. Does this table represent a continuous, discontinuous, and/or discrete function? Justify your answer.

Age	5	10	15	20
Height	36 in.	48 in.	60 in.	72 in.

23. (21) **Multi-Step** An online nut store sells nuts and raisins by the pound. Cora bought 4 pounds of nuts and 2 pounds of raisins for $37.20. Mark bought 2 pounds of nuts and 4 pounds of raisins for $26.40.

a. Write a system of equations that represents the price of the nuts, n, and the price of the raisins, r.

b. How much should a pound of nuts and a pound of raisins cost together? Explain.

*24. (24) **Write** Describe the steps used when solving a linear system by the elimination method.

25. (20) **Multiple Choice** Given $f(x) = -2x + 11$ and $g(x) = 6x - 1$, find $(f + g)(8)$.

A 42 **B** 52 **C** 74 **D** 76

26. (18) The Olympic record time for swimming the 100-meter freestyle is held by P. van den Hoogenband, who swam it in 47.84 seconds on September 19, 2000. What is his time in feet per second?

27. (17) **Error Analysis** Explain the error in the statement and provide numerical support: The transformation that occurs from $f(x) = |x|$ to $f(x) = |x - 3|$ results in a graph that shifts to the left 3 units.

*28. (29) **Population** The table shows the population in millions for the two most populated states. If the populations of the next three most populated states, New York, Florida, and Illinois, are added, the total population is 102.2 million. The difference between the populations of New York and Florida is 3 million. The difference between the populations of Florida and Illinois is 3.6 million. Find the populations of New York, Florida, and Illinois.

State	Population in Millions
California	33.9
Texas	20.9
New York	
Florida	
Illinois	

29. (16) Use Cramer's Rule to solve the system $\begin{aligned} x + y &= 2 \\ 2x - y &= 7 \end{aligned}$.

*30. (28) **Geometry** Suppose that the height of a square pyramid equals the length of one side of its base, s. Write the ratio of the area of the base to the volume of the pyramid as a simplified rational expression.

LESSON 30

Applying Transformations to the Parabola and Determining the Minimum or Maximum

Warm Up

1. (27) **Vocabulary** The graph of a quadratic function is a U-shaped curve called a _____.
2. (27) The parent quadratic function is _____.
3. (27) Write $y = 2(x - 3)^2 + 5$ in standard form.
4. (27) Where is the vertex of the graph of $y = x^2$ located?

New Concepts

A transformation is a change in the location or shape of a graph. One type of transformation is a **shift,** or slide.

Math Language

A shift can also be called a **translation.**

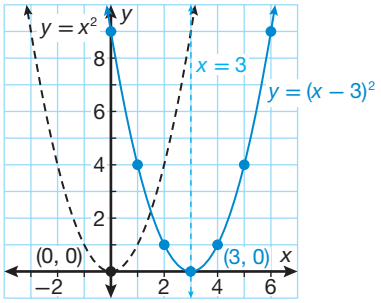
Example 1 Examining Shifts in Parabolas

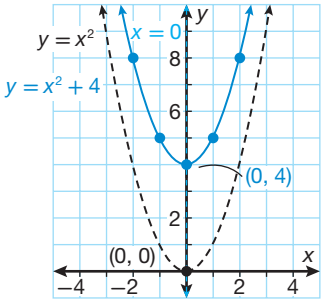
Graph $y = (x - 3)^2$ and $y = x^2 + 4$. For each, determine the vertex and the equation of the axis of symmetry, and explain how the graph shifted relative to the parent function $y = x^2$.

SOLUTION To help answer the question, the parent function is also graphed. Use $x = -\frac{b}{2a}$ to find the x-coordinate of the vertex and the axis of symmetry. Then make a table of values to find other points on the parabola.

x	0	1	2	3	4	5	6
y	9	4	1	0	1	4	9

x	−2	−1	0	1	2
y	8	5	4	5	8





The vertex of $y = (x - 3)^2$ is (3, 0) and the axis of symmetry is $x = 3$. Relative to the parent function, the graph shifted 3 units to the right. The vertex of $y = x^2 + 4$ is (0, 4) and the axis of symmetry is $x = 0$. Relative to the parent function, the graph shifted 4 units up.

The **vertex form of a quadratic function** is $f(x) = a(x - h)^2 + k$ where $a \neq 0$, and (h, k) represents the vertex of the parabola. The functions in Example 1, written in vertex form, show the values of a, h, and k: $y = 1(x - 3)^2 + 0$ and $y = 1(x - 0)^2 + 4$. Notice that, compared to the parent function, h indicates a horizontal shift and k indicates a vertical shift of the vertex of the parabola from the origin.

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The vertex form of a quadratic equation, the axis of symmetry and the reflective properties of a parabola can be used to sketch the graph of the parabola. Recall that the reflection of each point on the left side of a parabola is located on the right side of the parabola.

Justify

What are the advantages of using the vertex form versus the standard form of an equation?

Example 2 Using the Vertex Form and Symmetry to Graph a Quadratic Function

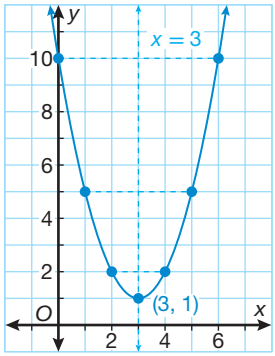
Graph $f(x) = (x - 3)^2 + 1$.

SOLUTION

Step 1: Identify the vertex (h, k) and plot it.

$$f(x) = (x - \boxed{3})^2 + \boxed{1}$$

The vertex is $(3, 1)$.



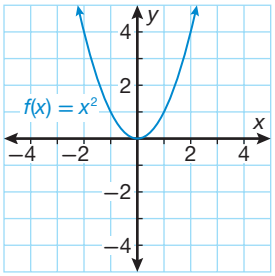
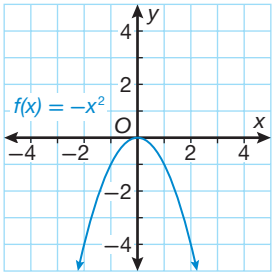
Step 2: Sketch the axis of symmetry. The axis of symmetry always passes through the vertex, so the equation of the line is $x = 3$.

Step 3: Plot a few points on one side of the axis of symmetry.

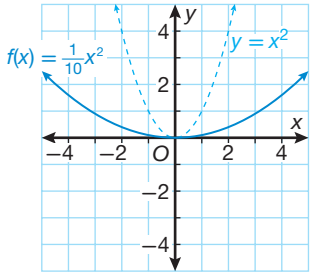
x	0	1	2
y	10	5	2

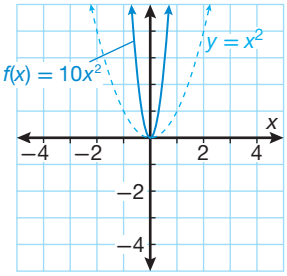
Find the reflection of each of those points across the axis of symmetry, then join the points with a smooth curve.

The sign of a in $f(x) = a(x - h)^2 + k$ tells whether the parabola opens upward or downward. When a is positive, the parabola opens upward, its vertex is a low point, and its y-coordinate is the **minimum value of the function.** When a is negative, it opens downward, its vertex is a high point, and its y-coordinate is the **maximum value of the function.**

Minimum and Maximum Values	
Opens Upward	**Opens Downward**
	
The domain is all real numbers. The range is all values greater than or equal to the minimum.	The domain is all real numbers. The range is all values less than or equal to the maximum.

The absolute value of a tells whether the parabola is vertically **compressed** or vertically **stretched**. When $0 < |a| < 1$, the parabola is vertically compressed, so it is wider than the parent function. When $|a| > 1$, the parabola is vertically stretched, so it is narrower than the parent function.





Example 3 Using the Value of *a* to Identify Graphs of Quadratic Functions

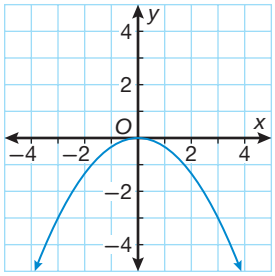
Match each function with its graph.

$$y = 3x^2, \quad y = -3x^2, \quad y = \frac{1}{3}x^2, \quad y = -\frac{1}{3}x^2$$

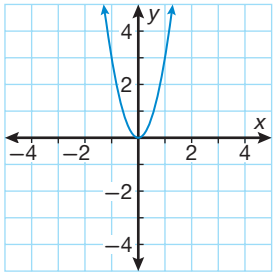
Math Reasoning

Analyze Which graphs are reflections of each other over the *x*-axis?

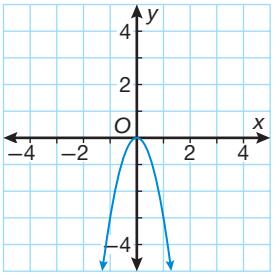
Graph A



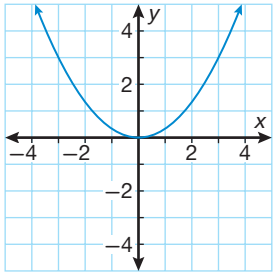
Graph B



Graph C



Graph D

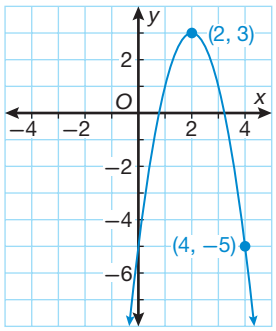


SOLUTION Graphs B and D both open upward, so the value of a in each is positive. Since Graph B is narrower, the absolute value of a is greater than that in Graph D, so Graph B shows $y = 3x^2$ and Graph D shows $y = \frac{1}{3}x^2$. Graphs A and C both open downward, so the value of a in each is negative. Since Graph C is narrower, the absolute value of a must be greater than in Graph A. Therefore, Graph A shows $y = -\frac{1}{3}x^2$ and Graph C shows $y = -3x^2$.

Given the coordinates of the vertex of a parabola and another point on the parabola, the equation of the parabola can be determined using the vertex form.

Example 4 Finding the Equation of a Parabola

Find the equation of the parabola shown.



SOLUTION Substitute the coordinates of the vertex for h and k into the vertex form of a quadratic function.

$$y = a(x - 2)^2 + 3$$

Substitute the coordinates of the other point labeled for x and y, then solve for a.

$$-5 = a(4 - 2)^2 + 3$$

$$-5 = 4a + 3$$

$$-8 = 4a$$

$$-2 = a$$

→ Substitute the value of a into the equation: $y = -2(x - 2)^2 + 3$.

Caution

It is easy to assume that there is a minimum when the value of a is negative. Think about the shape of the graph before answering.

Example 5 Application: Investing

The value of a stock, in dollars, over the past x weeks can be approximated by the function $f(x) = -0.185(x - 34)^2 + 220$. Does the graph of this function have a minimum or maximum? What is this value? What does it represent?

SOLUTION The value of a is negative, so the graph opens downward and has a maximum. The maximum is \$220, which occurs when $x = 34$. So, the maximum represents the value of the stock over 34 weeks.

Lesson Practice

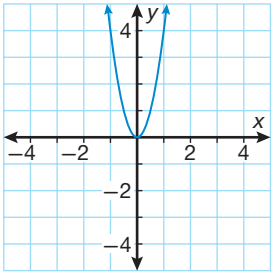
a. *(Ex 1)* Graph $y = (x + 5)^2$ and $y = x^2 - 2$. For each, determine the vertex and axis of symmetry, and explain how the graph is shifted relative to the parent function $y = x^2$.

b. *(Ex 2)* Graph $f(x) = (x + 4)^2 + 3$.

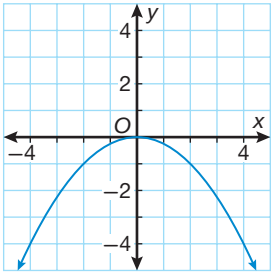
c. (Ex 3) Match each function with its graph.

$$y = 0.25x^2, \quad y = 4x^2, \quad y = -0.25x^2, \quad y = -4x^2$$

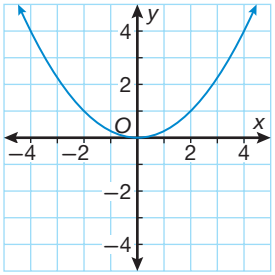
Graph A



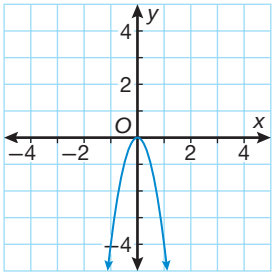
Graph B



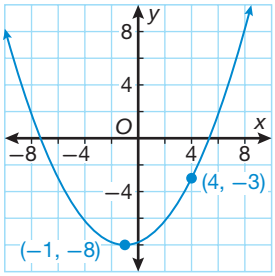
Graph C



Graph D



d. (Ex 4) Find the equation of the parabola shown.



e. (Ex 5) The value of stock, in dollars, over the past x weeks can be approximated by the function $f(x) = 0.445(x - 18)^2 + 39.50$. Does the graph of this function have a minimum or maximum? What is this value? What does it represent?

Practice Distributed and Integrated

Determine the vertex and axis of symmetry of each function.

***1.** (30) $y = x^2 + 3$

***2.** (30) $y = 2(x - 1)^2 + 3$

***3.** (30) $y = -(x + 2)^2$

***4.** (30) $y = 3x^2 + 5$

***5.** (29) Solve the system of equations:

$$\begin{aligned} x + 3y - z &= 10 \\ 2x + 5y + z &= 9 \\ -3x + y - 3z &= 8 \end{aligned}$$

6. (27) Determine the domain and range of the function $f(x) = 2x^2$.

Find each product.

7. (19) $2x^2(2x - 4)$

8. (19) $\frac{4x^2}{3x^3}(x^3 + 9)$

Determine whether each of the following are functions.

9. (4) $y^2 = x + 5$

10. (4) $y + 3x^3 = 2x + 1$

11. (4) $y^2 - x^2 = 3$

***12.** (30) **Tall Structures** The height of an object, in meters, dropped from the roof of the Empire State Building after x seconds can be approximated by the function $y = -5x^2 + 381$, whereas the height of an object dropped from the Sears Tower can be approximated by $y = -5x^2 + 442$. Without graphing, tell how the graphs of each function compare to the parent quadratic function and to each other.

***13.** (29) **Estimate** Use
$$-5x - z = -4 - 9y$$
$$15x - 12 + 4z = 6y$$
$$20x - 36y = -4z + 16$$

a. Rewrite the equations by lining up the variables.

b. Estimate the number of solutions. Explain your answer.

c. Solve the system of equations to verify your estimate.

***14.** (28) **Travel** The London Eye is a large Ferris wheel in England. Write the circumference to area ratio of the Eye as a simplified rational expression. Then find the ratio given that the diameter of the Eye is 135 meters.

***15.** (15) **Graphing Calculator** Solve the system of equations using a graphing calculator.

$$5y = 7 - 2x$$
$$6x + 3y = 9$$

16. (14) **Multi-Step** **a.** Calculate the determinant of $\begin{bmatrix} 0 & -2 & 3 \\ 1 & 4 & 2 \\ -3 & -1 & 1 \end{bmatrix}$.

b. Write the result of interchanging the first two rows of the matrix above.

c. Compare the determinants of the matrices in parts a and b. Explain what you see.

17. (25) **Model** Each week, Damien records his car's miles per gallon. Over a period of 10 weeks, he records:18, 17, 19, 18, 18, 25, 29, 30, 26, 19. Make a box-and-whisker plot of his data.

18. (24) **Justify** Which property of equality allows one equation to be added to another equation? Explain.

19. (23) **Error Analysis** Explain the error in the student's work below.

$$3x^2 - x - 4 = 0$$
$$2x = 4$$
$$x = 2$$

20. (21) **Analyze** Describe how a graph of 2 linear equations looks when the system has:

a. exactly one solution.

b. an infinite number of solutions.

c. no solution.

21. (17) **Multiple Choice** Solve $|2x + 1| < 4$.

A $-\frac{5}{2} < x < \frac{3}{2}$

B $\frac{3}{2} < x < -\frac{5}{2}$

C $x < -\frac{5}{2}$ or $x > \frac{3}{2}$

D $x < \frac{3}{2}$

22. (22) **Mailing Weights** A customer visits the post office to weigh a package. The cost of mailing a package depends on its weight. Examine the table. Does this table represent a continuous, discontinuous, and/or discrete function? Justify your answer.

Weight	Cost
$0 \le x < 5$	\$2.00
$5 \le x < 10$	\$4.00
$10 \le x < 15$	\$6.00

23. (19) **Basketball** The length of a regulation high school basketball court is represented by the expression $(x + 24)$ feet. The width of a regulation high school basketball court is represented by the expression $(x - 10)$ feet.

a. Write an expression that represents the area of a regulation high school basketball court.

b. Evaluate the area of the court if $x = 60$.

24. (18) **Track and Field** Katie ran two miles to get ready for Thursday's meet. How many feet did Katie run?

25. (17) **Coordinate Geometry** A triangle on a coordinate grid has vertices at $(3, 0)$, $(-5, 0)$, and $(0, -6)$. Write the distance from each vertex to the origin as an absolute value expression. Then simplify the expression.

26. (11) **Multi-Step** Given the polynomial functions $f(x) = x^3 - x$, $g(x) = x^2 + 1$, and $h(x) = x^2 - 1$:

a. Write the polynomial function $g(x) + h(x)$.

b. Write the composite function $f(g(x) + h(x))$ in standard form.

27. (17) **Multiple Choice** Solve $-|2x + 9| \le -3$.

A $-6 \le x \le -3$

B $x \le -6$ or $x \ge -3$

C All real numbers

D No solution

***28.** (30) **Error Analysis** Two students are asked to find the axis of symmetry for the parabola of the function $f(x) = -(x + 4)^2 + 1$. Student A said the line was $x = 4$ and Student B said the line was $x = -4$. Which student is incorrect? Explain the error.

29. (27) **Justify** Identify the domain and range of $f(x) = 2x^2 - 7$. Explain how you know.

30. (26) **Geometry** The Mendez family is going to start a vegetable garden by their house and is going to enclose it within a fence on three sides. If the total amount of fencing they purchase is 300 feet, write an equation in standard form for finding all possible lengths and widths for the garden.

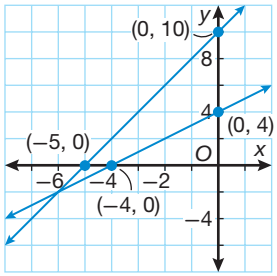
INVESTIGATION

3

Applying the Substitution Method to a System in Three Variables

The same method of substitution, which has been used to find a solution to a system of equation in two variables, can be applied to a system of equations in three variables. Use the following steps to scrutinize the method. Then apply the method to a system of equations in three variables.

The system consisting of the equations $2x - y = -10$ and $-x + y = 4$ is graphed at the right. Solve the system, using the substitution method, by following the steps.



1. The first step, when solving a system using the substitution method, is choosing an equation and solving it for a variable. Solve for y in the second equation.

2. Explain how the resulting expression is used in the second step of the substitution method. Record the equation found after completing this step.

3. **Generalize** Notice that the equation is written in one variable. Explain how forming a one-variable equation contributes to finding a solution to the system.

4. **Verify** Find the values of x and y. Explain how the values are determined.

5. **Explain** How is the solution of the system checked?

Hint

It is not necessary to use the first equation when beginning the substitution method. Choose the equation that is easiest to solve for a variable.

In order to solve a two-variable system, it is necessary to have at least two equations. In the same way, to solve a three-variable system there must be at least three equations.

Use method to solve the system:

$$x - 3y + 2z = 11$$
$$-x + 4y + 3z = 5$$
$$2x - 2y - 4z = 2$$

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6. Apply the first step of the substitution method, found in Step 1, by solving the first equation for x.

7. Substituting is the next step. The equation produced by substituting the resulting expression from Step 1 into the second equation is $-(3y - 2z + 11) + 4y + 3z = 5$. Write this equation in simplified form and draw a box around it.

8. Substitute the same expression from Step 1 into the third equation. Write the equation formed, simplify it, and draw a box around it.

For an equation to be solvable, the equation must be written in one variable. Therefore, the process of substitution must be repeated until this situation arises.

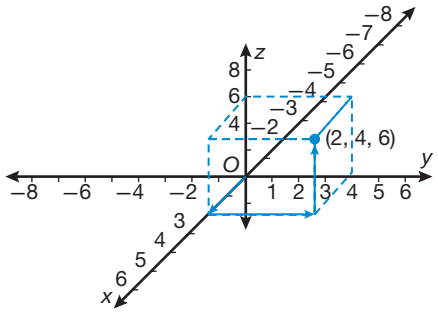
9. How many variables are there in the equations that were boxed? What are the variables?

10. Apply the substitution method for solving a system of equations in two variables to solve the smaller system of equations formed by the two boxed-equations. What are the values of y and z?

11. **Generalize** Find the value of x. How is the third variable of a system of equations in three variables determined?

The solution to this system is a point with three coordinates. Points with three coordinates are graphed on a three-dimensional coordinate system. A **three-dimensional coordinate system** is a space that is divided into eight regions by an x-axis, a y-axis, and a z-axis.

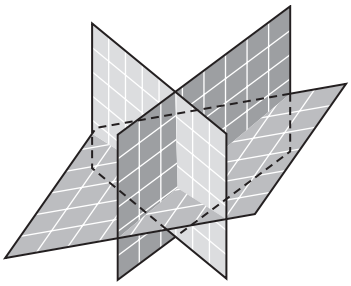
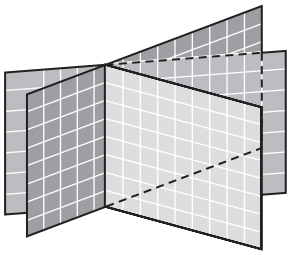
The ***z*-axis** is the third axis in a three-dimensional coordinate plane and is typically drawn as a vertical line. The y-axis is drawn as a horizontal line and the x-axis is drawn as if it is going into the page, as shown in the graph. Coordinates in a three-dimensional plane are called ordered triples and are written in the form (x, y, z). The arrows in the graph show how to plot the point (2, 4, 6).



An ordered triple is one type of solution to a system of equations in three variables. The graph of an equation in three variables is a plane and is written as $Ax + By + Cz = D$, where A, B, and C are nonzero real numbers. If a system of equations in three variables has a single ordered triple as a solution, then

- the coordinates of that point satisfies all three equations and
- the point is the intersection of the three planes.

In a system of equations in three variables, the intersection of all three planes is the solution to the system.

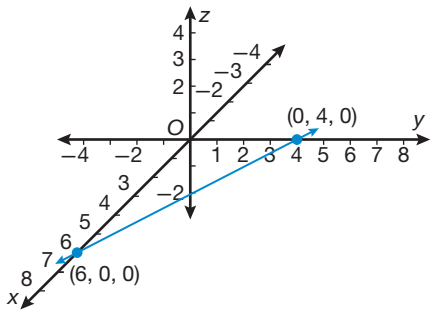
One Solution	Infinitely Many Solutions
	
The planes intersect at a single point.	The planes intersect in a line.

To graph the equation $2x + 3y + 6z = 12$, use the intercept method that is used to graph a linear equation.

12. **Explain** If setting $y = 0$ and solving for x is the process for determining the x-intercept of a linear equation, then what is the process for determining the x-intercept for the equation $2x + 3y + 6z = 12$?

13. **Formulate** What are the x-, y-, and z-intercepts for the equation $2x + 3y + 6z = 12$? Write the answer in ordered triples.

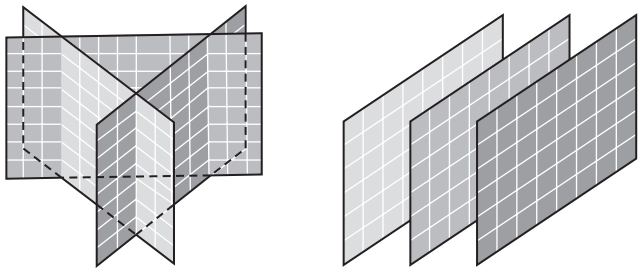
14. **Model** Copy and complete the drawing by plotting the other intercept. Then draw the two other lines through the other points to form a triangle. Shade the triangle to give the illusion of a plane.



Caution

The plane formed by the shaded triangle is not limited to the triangle. However, the shaded triangle is contained in the plane.

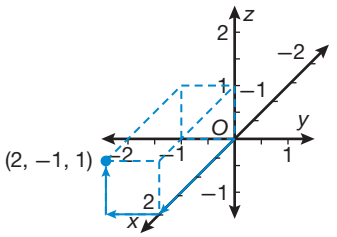
A system of equations in three variables may not have a solution. This occurs when there is no point that lies in all three planes. These systems may have equations that form three planes in the following ways:

No Solution

No point lies in all three planes.

Investigation Practice

a. **Multi-Step** Solve the system using the substitution method. If the solution is a single point, then graph the coordinate using arrows on a three-dimensional coordinate system.

$$x + 4y - 5z = 7$$
$$3x + 2y + 3z = 7$$
$$2x + y + 5z = 8$$



b. **Multi-Step** Find the intercepts of the equation $2x - y + 3z = 6$. Graph the intercepts of the equation on a graph and shade the plane that is formed by the intercepts.

LESSON 31

Multiplying and Dividing Rational Expressions

Warm Up

1. (28) **Vocabulary** A rational expression is _____ if its denominator equals zero.

2. (SB) Multiply. $\frac{3}{10} \cdot \frac{5}{9}$

3. (SB) Divide. $\frac{2}{5} \div \frac{1}{20}$

4. (2) Evaluate $4xy^3$ for $x = 3$, $y = -2$.

New Concepts

To multiply rational expressions, multiply the numerators and multiply the denominators. To simplify a rational expression, factor and then divide out all common factors. To evaluate an algebraic expression for given values of the variables, substitute the given values, and then simplify the numerical expression.

Recall that a rational expression is undefined if its denominator equals zero. In this lesson, assume that all rational expressions are defined unless otherwise specified.

Example 1 Multiplying Rational Expressions

a. Multiply, and then evaluate for $a = 4$, $b = 5$: $\frac{5ab^2}{6b^2} \cdot \frac{12a^2b^2}{3ab^3}$.

SOLUTION Multiply:

$$\frac{5ab^2}{6b^2} \cdot \frac{12a^2b^2}{3ab^3} = \frac{60a^3b^4}{18ab^5}$$ Multiply numerators and denominators.

$$= \frac{\cancel{6} \cdot 10 \cdot \cancel{a} \cdot a^2 \cdot \cancel{b^4}}{3 \cdot \cancel{6} \cdot \cancel{a} \cdot b \cdot \cancel{b^4}}$$ Factor. Then divide out common factors.

$$= \frac{10a^2}{3b}$$ Write the algebraic expression in simplified form.

Evaluate the expression for $a = 4$, $b = 5$:

$$\frac{10a^2}{3b} = \frac{10 \cdot 4^2}{3 \cdot 5}$$ Substitute 4 for a and 5 for b.

$$= \frac{2 \cdot \cancel{5} \cdot 4^2}{3 \cdot \cancel{5}}$$ Factor as needed to divide out all common factors.

$$= \frac{32}{3}$$ Write the numerical expression in simplified form.

The product is the expression $\frac{10a^2}{3b}$. Its value is $\frac{32}{3}$ for the given values of a and b.

Hint

Another way to simplify $\frac{10 \cdot 4^2}{3 \cdot 5}$ is to multiply, divide, and then simplify.

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Caution

To identify values that make an expression undefined, use the given expression, not the simplified result.

b. Multiply, and then identify all values of x that make the expression undefined: $\frac{15x - 3x^2}{x^2 - 9} \cdot \frac{x^2 - 8x + 15}{x^2 - 10x + 25}$.

SOLUTION

$$\frac{15x - 3x^2}{x^2 - 9} \cdot \frac{x^2 - 8x + 15}{x^2 - 10x + 25}$$

$$= \frac{3x(5 - x)}{(x - 3)(x + 3)} \cdot \frac{(x - 3)(x - 5)}{(x - 5)(x - 5)}$$ Factor.

$$= \frac{3x(-1)\cancel{(x - 5)}\cancel{(x - 3)}\cancel{(x - 5)}}{\cancel{(x - 3)}(x + 3)\cancel{(x - 5)}\cancel{(x - 5)}}$$ Rewrite $(5 - x)$ as $(-1)(x - 5)$ so that the common factor $(x - 5)$ appears. Then multiply. Then divide out common factors.

$$= \frac{-3x}{(x + 3)}$$

Identify values of x that make the *given* expression undefined:

The given expression is $\frac{15x - 3x^2}{x^2 - 9} \cdot \frac{x^2 - 8x + 15}{x^2 - 10x + 25}$, which is equivalent to $\frac{3x(5 - x)}{(x - 3)(x + 3)} \cdot \frac{(x - 3)(x - 5)}{(x - 5)(x - 5)}$. The factors in the denominator are $(x - 3)$, $(x + 3)$, and $(x - 5)$. These factors are zero when x is 3, -3, and 5, respectively. So, the values of x that make the given expression undefined are 3, -3, and 5.

c. Multiply $\frac{5x + 10}{3x^2 + 2x - 1} \cdot \frac{9x^2 - 1}{x + 2} \cdot \frac{x + 1}{15x + 5}$.

SOLUTION

$$\frac{5x + 10}{3x^2 + 2x - 1} \cdot \frac{9x^2 - 1}{x + 2} \cdot \frac{x + 1}{15x + 5}$$

$$= \frac{\cancel{5(x + 2)}}{\cancel{(3x - 1)(x + 1)}} \cdot \frac{\cancel{(3x + 1)(3x - 1)}}{\cancel{x + 2}} \cdot \frac{\cancel{x + 1}}{\cancel{5(3x + 1)}}$$ Factor. Divide by the common factors and then multiply.

$$= 1$$

d. Multiply: $\frac{x + 4}{x^2 - 4x} \cdot (2x^2 - 9x + 4)$

SOLUTION

$$\frac{x + 4}{x^2 - 4x} \cdot \frac{2x^2 - 9x + 4}{1}$$ Rewrite the polynomial as a rational expression with a denominator of 1. Then factor, multiply, and divide out common factors.

$$= \frac{x + 4}{x\cancel{(x - 4)}} \cdot \frac{(2x - 1)\cancel{(x - 4)}}{1}$$

$$= \frac{(x + 4)(2x - 1)}{x}$$

Dividing by a Rational Expression

The **reciprocal** of a real number $\frac{c}{d}$ is $\frac{d}{c}$, provided that $c \neq 0$ and $d \neq 0$.

To divide by a rational expression, multiply by its reciprocal: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, where b, c, and d are all nonzero.

Example 2 Dividing Rational Expressions

Hint

One quick way to check simplification is to substitute values for the variables in the original expression and evaluate. They should be the same.

a. Divide, and then evaluate for $x = -2$, $y = 3$: $\frac{12x^3}{5y} \div \frac{4x^5}{5x^3y^2}$.

SOLUTION

$$\frac{12x^3}{5y} \div \frac{4x^5}{5x^3y^2} = \frac{12x^3}{5y} \cdot \frac{5x^3y^2}{4x^5}$$ Multiply by the reciprocal.

$$= \frac{60x^6y^2}{20x^5y}$$ Multiply numerators and denominators.

$$= \frac{3 \cdot \cancel{20} \cdot x \cdot \cancel{x^5} \cdot \cancel{y} \cdot y}{\cancel{20} \cdot \cancel{x^5} \cdot \cancel{y}}$$ Factor. Then divide out common factors.

$$= \frac{3xy}{1}$$ Write the algebraic expression in simplified form.

$$= 3xy$$

Evaluate for $x = -2$, $y = 3$:

$3xy = 3(-2)(3)$ Substitute -2 for x and 3 for y, and then simplify.

$= -18$

The quotient is the expression $3xy$. Its value is -18 for the given values of x and y.

Caution

Do not divide out terms of polynomials.

$\frac{x+1}{x-3} \neq \frac{1}{-3}$ (unless $x = 0$).

b. Divide, and then identify all values of x that make the expression undefined: $\frac{x+1}{x-2} \div \frac{x^2-3x}{x^2-2x}$.

SOLUTION Divide:

$$\frac{x+1}{x-2} \div \frac{x^2-3x}{x^2-2x} = \frac{x+1}{x-2} \cdot \frac{x^2-2x}{x^2-3x}$$ Multiply by the reciprocal.

$$= \frac{x+1}{x-2} \cdot \frac{x(x-2)}{x(x-3)}$$ Factor.

$$= \frac{(x+1)\cancel{x(x-2)}}{\cancel{(x-2)}\cancel{x}(x-3)}$$ Divide out common factors.

$$= \frac{x+1}{x-3}$$

Identify values of x that make the *given* expression undefined:

The given expression in factored form is $\frac{x+1}{x-2} \div \frac{x(x-3)}{x(x-2)}$. In a division expression of the form $\frac{a}{b} \div \frac{c}{d}$, b, c, and d must all be nonzero. So, in this case, $x-2$, $x(x-3)$, and $x(x-2)$ must all be nonzero. Therefore, the factors $(x-2)$, x, and $(x-3)$ must all be nonzero. So the values of x that make the given division expression undefined are 2, 0, and 3.

Example 3 Simplifying a Rational Expression Containing Multiplication and Division

Simplify: $\dfrac{45}{x^2+2x-15} \cdot \dfrac{2x^2+10x}{5x^2y} \div \dfrac{1}{x-3}$

SOLUTION

$$\frac{45}{x^2+2x-15} \cdot \frac{2x^2+10x}{5x^2y} \div \frac{1}{x-3}$$

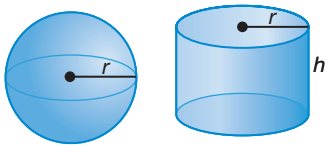
$$= \frac{45}{x^2+2x-15} \cdot \frac{2x^2+10x}{5x^2y} \cdot \frac{x-3}{1}$$ Multiply by the reciprocal.

$$= \frac{\cancel{5} \cdot 9}{\cancel{(x+5)}\cancel{(x-3)}} \cdot \frac{2 \cdot \cancel{x} \cdot \cancel{(x+5)}}{\cancel{5} \cdot \cancel{x} \cdot x \cdot y} \cdot \frac{\cancel{(x-3)}}{1}$$ Factor and divide out common factors.

$$= \frac{18}{xy}$$

Example 4 Application: Determining a Dimension

An engineer is designing two storage tanks. One tank is to be a sphere and the other a cylinder. He wants the tanks to have the same volume and the same radius. What must be the height of the cylinder, in terms of the radius?



SOLUTION

$$V_{sphere} = V_{cylinder}$$

$$\frac{4\pi}{3}r^3 = \pi r^2 h$$

$$\frac{1}{\pi r^2} \cdot \frac{4\pi}{3}r^3 = \frac{1}{\pi r^2} \cdot \pi r^2 h$$ Multiply both sides by the reciprocal of πr^2.

$$\frac{1}{\pi r^2} \cdot \frac{4\pi}{3}r^3 = h$$

$$\frac{1}{\cancel{\pi} \cdot r^2} \cdot \frac{4 \cdot \cancel{\pi} \cdot r \cdot \cancel{r^2}}{3} = h$$ Factor and divide out common factors.

$$\frac{4r}{3} = h$$

Check Substitute $\frac{4r}{3}$ for h in the expression for the volume of a cylinder. The result is the expression for the volume of a sphere, so the answer is correct:

$$V_{cylinder} = \pi r^2 h = \pi r^2 \cdot \frac{4r}{3} = \frac{4\pi r^3}{3} = V_{sphere}$$

Math Reasoning

Analyze Suppose the engineer wanted the tanks to have the same surface area. What must be the height of the cylinder in terms of the radius?

Lesson Practice

a. *(Ex 1)* Multiply, and then evaluate for $x = 5$: $\frac{4x^2 - 9}{x + 1} \cdot \frac{x^2 + x}{4x^2 + 12x + 9}$.

b. *(Ex 1)* Multiply, and then identify all values of x that make the expression undefined: $\frac{x^2 + 5x - 14}{x^2 + x - 6} \cdot \frac{x^2 - x - 12}{5x^2 - 20x}$.

c. *(Ex 1)* Multiply: $\frac{x^2 - 4}{7x^2 + 14x - 56} \cdot \frac{7x - 21}{x + 4} \cdot \frac{x + 4}{x^2 - x - 6}$.

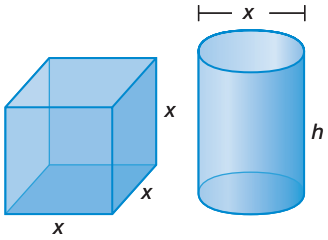
d. *(Ex 1)* Multiply: $\frac{3x - 9}{6x^2 + 12x} \cdot (x^2 - 4x - 12)$.

e. *(Ex 2)* Divide, and then evaluate for $x = -1$, $y = -6$: $\frac{3x^2y^5}{7xy^8} \div \frac{9x^8y^2}{14xy^6}$

f. *(Ex 2)* Divide, and then identify all values of x that make the expression undefined: $\frac{1}{x^2 + 2x} \div \frac{2x - 1}{x^3 + 7x^2 + 10x}$

g. *(Ex 3)* Simplify: $\frac{16 - x^2}{4xy} \cdot \frac{x^2y}{x - 4} \div \frac{x + 4}{4}$

h. *(Ex 4)* A cube and a cylinder are to have the same volume. The diameter x of the cylinder is equal to the length of the edge of the cube. What must be the height of the cylinder, in terms of x?



Practice Distributed and Integrated

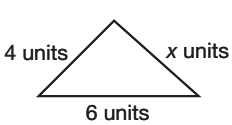
1. *(9)* **Generalize** Given the dimensions of four matrices, A: 2×4, B: 2×2, C: 4×2, D: 4×4, determine all of the possible multiplication pairs and the dimensions of each product matrix.

***2.** *(31)* If $f(x) = \frac{x^2 - 100}{x^2 + 20x + 100}$ and $g(x) = \frac{10x + 100}{3x - 30}$, what is $f(x) \cdot g(x)$ in simplest form?

***3.** *(28)* **Engineering** The Channel Tunnel is a tunnel that runs under the English Channel. It is cylindrical in shape. Write the volume to lateral area ratio as a simplified rational expression. Then find the ratio given that the tunnel has an inner diameter of 7.6 meters. (Hint: The lateral area of a cylinder is the same as the surface area minus the area of the bases.)

4. *(8)* When mixing concentrated fertilizer for a garden, the amount of fertilizer varies directly with the amount of water. If 2 cups of fertilizer are used in 16 gallons of water, how much fertilizer is used in 48 gallons of water?

5. *(10)* **Geometry** Write a compound inequality using the fact that the sum of the lengths of any two sides of a triangle is greater than the length of the third side (Triangle Inequality) using the diagram. (Hint: Remember that length cannot be negative.)

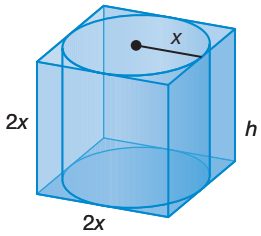


Use the description to write the vertex form of a quadratic equation.

***6.** (30) The parent function is shifted 4 units right and 6 units down.

***7.** (30) The parent function is reflected across the x-axis, and then shifted 3 units up.

8. (11) **Woodworking** A lathe operator has a piece of wood in the shape of a square prism with dimensions $2x$, $2x$, and h. He wants to make the largest possible cylinder by turning the wood on the lathe and cutting it down. The cylinder will have radius x and length h. Write a polynomial that represents the volume of wood he will cut off to make the cylinder.



Multiply.

***9.** (31) $\frac{3a^3b^2}{5a^5} \cdot \frac{10a^3b}{7a^4b^2}$

***10.** (31) $\frac{2xy}{3y^{11}} \cdot \frac{6x^2y}{3xy}$

11. (23) **Verify** Use FOIL to verify that $a = c + d$ and $b = cd$ in the formula:

$$x^2 + ax + b = (x + c)(x + d).$$

12. (13) **Mountains** Mt. Elbert in Colorado has an elevation of 14,433 ft and from the center, it runs horizontally approximately 22,105 ft. Find the approximate slope of Mt. Elbert, round your answer to the thousandths place.

13. (11) **Multiple Choice** Which polynomial is in standard form?

A $\frac{1}{2}x^3 - x$ **B** $\frac{1}{3} - \frac{1}{3}x^3$ **C** $4x^2 + 3x^3 + 2x^4$ **D** $x^4 + x^5$

14. (18) Convert 87 ft^2 to square inches.

15. (10) **Analyze** Find the largest two consecutive even integers with a total sum less than or equal to 130.

16. (14) **Error Analysis** Explain the error in the student's work. Then find the correct determinant. $\begin{vmatrix} -2 & 6 \\ 3 & -1 \end{vmatrix} = (6)(3) - (-2)(-1) = 18 - 2 = 16$

***17.** (15) **Graphing Calculator** Solve the system of equations using a graphing calculator.

$$2x - 3y = 4$$
$$4x + y = 1$$

18. (17) **Multi-Step** $|ax + b| \leq c$, where $a > 0$ and $c > 0$.

a. Solve the inequality for x in terms of a, b, and c.

b. Apply this general solution to solve $|2x + 3| \leq 5$.

19. *(13)* **Coordinate Geometry** Graph the pentagon formed by these five equations on a coordinate plane.

$$x = 3 \qquad x = -3 \qquad y = -4 \qquad y = \frac{2}{3}x + 3 \qquad y = -\frac{2}{3}x + 3$$

a. Is the pentagon regular?

b. What are the coordinates of the vertices of the pentagon?

***20.** *(30)* **Error Analysis** A student said the graph of the function $y = (x - 6)^2$ is a shift of the parent function 6 units to the left because h is negative. Explain and correct the error in the student's thinking.

21. *(19)* **Multiple Choice** Multiply $(x + 11)(x - 8)$.

A $x^2 - 19x + 88$ **B** $x^2 + 3x - 88$ **C** $x^2 - 3x - 88$ **D** $x^2 + 19x - 88$

Identify the following equations as direct or inverse variations.

22. *(12)* $y = 3x$

23. *(12)* $y = \frac{20}{x}$

24. *(21)* **Analyze** Describe how a graph of 2 linear equations looks when the system has:

a. exactly one solution.

b. an infinite number of solutions.

c. no solution.

***25.** *(31)* **Write** Describe how to find the following product in simplest form: $\frac{a^2 - 49}{a - 7} \cdot \frac{a}{2a^2 + 14a}$.

***26.** *(28)* **Multi-Step** **a.** Simplify $\frac{4x^2 - 13x + 3}{9 - 6x + x^2}$.

b. Evaluate the simplified expression for $x = 9$.

27. *(24)* **Money** A parking meter contains 157 coins worth \$26.95. If the meter only accepts quarters and dimes, how many of each type of coin are in the meter?

Given $f(x) = 5x - 9$ and $g(x) = 9x + 8$.

28. *(20)* Find $(f \cdot g)(3)$.

29. *(20)* Find $\left(\frac{f}{g}\right)(2)$.

30. *(Inv 3)* Find the intercepts of the equation $10x + 3y - 8z = 12$.

LESSON 32

Solving Linear Systems with Matrix Inverses

Warm Up

1. (5) **Vocabulary** A _____ is a rectangular array of elements.
2. (9) Multiply the matrix by the scalar. $-2[1 \;\; 3 \;\; -4]$
3. (9) Multiply the matrices. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 4 \\ 1 & 5 \end{bmatrix}$

New Concepts

Exploration **Exploring Matrix Inverses**

1. Find products AB and BA for $A = \begin{bmatrix} -2 & 0 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -0.5 & 0 \\ 2.5 & 1 \end{bmatrix}$.
2. Find products PQ and QP for $P = \begin{bmatrix} 8 & 5 \\ 6 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & -2.5 \\ -3 & 4 \end{bmatrix}$.
3. Find products ST and TS for $S = \begin{bmatrix} 3 & 1 \\ 11 & 4 \end{bmatrix}$ and $T = \begin{bmatrix} 4 & -1 \\ -11 & 3 \end{bmatrix}$.
4. What do you notice about the products of the matrices in Problems 1–3?
5. Explain what is special about the matrix you found to be the product in Problems 1–3.

Hint

Recall that an identity matrix has ones on the diagonal and zeros everywhere else.

If A is any $n \times n$ matrix and I is the $n \times n$ identity matrix, then $AI = IA = A$. The **multiplicative inverse of a square matrix,** if it exists, is a matrix such that the product of it and another matrix forms an identity matrix. The inverse of matrix A is notated A^{-1}. So, two matrices A and B are inverses of each other if $A \cdot B = B \cdot A = I$. Likewise, $A \cdot A^{-1} = A^{-1} \cdot A = I$.

Example 1 **Verifying Inverses**

Hint

Matrix multiplication is not commutative. Check the product in both orders to verify that the matrices are inverses.

Determine whether the matrices are inverses.

a. $P = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & -3 \\ -0.5 & 1 \end{bmatrix}$

SOLUTION

Determine whether $P \cdot Q = I$ and $Q \cdot P = I$.

$$P \cdot Q = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Q \cdot P = \begin{bmatrix} 2 & -3 \\ -0.5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4-3 & 12-12 \\ -1+1 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$P \cdot Q = I$ and $Q \cdot P = I$, so P and Q are inverses.

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b. $R = \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$

SOLUTION Determine whether $R \cdot S = I$ and $S \cdot R = I$.

$$R \cdot S = \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 12-4 & 3+6 \\ 20-2 & 5+3 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 18 & 8 \end{bmatrix} \neq I$$

$R \cdot S \neq I$, so R and S are not inverses.

Reading Math

If A is the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then these all represent the determinant of A:

- det A
- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- $|A|$
- $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Recall, that to find the determinant of a 2×2 matrix, use the following method.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

An $n \times n$ matrix A has an inverse if and only if $\det A \neq 0$. The inverse of matrix A is denoted A^{-1}. If A does not have an inverse, then A is called a **singular matrix.**

Inverse of a 2×2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - cb \neq 0$, then the inverse of A is:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 2 Finding Inverses

Find the inverse of each matrix, if it exists.

a. $T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$

SOLUTION First find the determinant:
$|T| = (3)(2) - (-1)(4) = 6 + 4 = 10.$

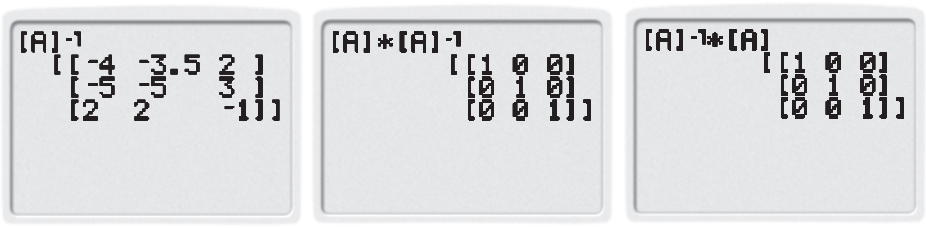
$$T^{-1} = \frac{1}{10}\begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0.2 & -0.4 \\ 0.1 & 0.3 \end{bmatrix}$$

b. $Y = \begin{bmatrix} 2 & 4 \\ 5 & 10 \end{bmatrix}$

SOLUTION Find the determinant: $|Y| = (2)(10) - (5)(4) = 20 - 20 = 0$.
The inverse of Y does not exist because $|Y| = 0$.

c. $A = \begin{bmatrix} -2 & 1 & -1 \\ 2 & 0 & 4 \\ 0 & 2 & 5 \end{bmatrix}$

SOLUTION Use a graphing calculator to find the inverse matrix. Then use the calculator to verify that $AA^{-1} = I$ and $A^{-1}A = I$.



Graphing Calculator Tip

For help in inputting matrices and finding inverses with the graphing calculator, see the lab on page 27.

A **matrix equation** represents a system of equations in matrix form where the variable is a matrix.

Using an Inverse to Solve a Matrix Equation

If matrix A has an inverse and the matrix equation $AX = B$ has a solution, then the solution is $X = A^{-1}B$.

$$AX = B$$
$$A^{-1}AX = A^{-1}B \quad \text{Multiply both sides by } A^{-1}.$$
$$IX = A^{-1}B \quad A^{-1}A = I \text{ by definition of inverse.}$$
$$X = A^{-1}B \quad IX = X \text{ by definition of identity.}$$

Example 3 **Solving a Matrix Equation**

Solve for matrix X. $\underbrace{\begin{bmatrix} 12 & 7 \\ 5 & 3 \end{bmatrix}}_{A} X = \underbrace{\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}}_{B}$

Caution

To solve $AX = B$, you must multiply both sides of the equation by A^{-1} from the left.

$(A^{-1})\,AX = (A^{-1})B$

SOLUTION

$$X = A^{-1}B = \begin{bmatrix} 3 & -7 \\ -5 & 12 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -15 & -17 \\ 26 & 29 \end{bmatrix}$$

Check $$AX = \begin{bmatrix} 12 & 7 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} -15 & -17 \\ 26 & 29 \end{bmatrix} = \begin{bmatrix} -180 + 182 & -204 + 203 \\ -75 + 78 & -85 + 87 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = B$$

When the matrix equation $AX = B$ is used to represent a system of equations, A is called the **matrix of coefficients,** X is called the **matrix of variables,** and B is called the **matrix of constants.**

Example 4 **Using an Inverse Matrix to Solve a Linear System**

Solve the system. $3x + 2y + z = 1$
$x - y - 3z = 12$
$-x + y + 4z = -15$

SOLUTION

Write the system as a matrix equation. $\underbrace{\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & -3 \\ -1 & 1 & 4 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 12 \\ -15 \end{bmatrix}}_{B}$

Solve the equation. $$X = A^{-1}B = \begin{bmatrix} 0.2 & 1.4 & 1 \\ 0.2 & -2.6 & -2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 12 \\ -15 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

The solution to the system is $x = 2$, $y = -1$, $z = -3$.

Hint

Remember
1 kilogram = 1000 grams

Example 5 Application: Analyzing Nutrition and Cost

A 460-gram package of trail mix costs \$4.42 and contains 59 grams of protein. The mix contains peanuts (\$12 per kilogram, 25% protein), raisins (\$5.50 per kilogram, 3% protein), and almonds (\$14 per kilogram, 13% protein). How much of each ingredient is in the trail mix?

SOLUTION Write a system.

$$p + r + a = 460 \quad \text{masses of ingredients, using grams}$$
$$0.25p + 0.03r + 0.13a = 59 \quad \text{masses of protein in the ingredients}$$
$$0.012p + 0.0055r + 0.014a = 4.42 \quad \text{costs of ingredients, using grams}$$

Write the matrix equation. $\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0.25 & 0.03 & 0.13 \\ 0.012 & 0.0055 & 0.014 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} p \\ r \\ a \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 460 \\ 59 \\ 4.42 \end{bmatrix}}_{B}$

Solve the equation. $X = A^{-1}B = \begin{bmatrix} 160 \\ 200 \\ 100 \end{bmatrix}$

The solution to the system is $p = 160$, $r = 200$, $a = 100$. The trail mix contains 160 grams of peanuts, 200 grams of raisins, and 100 grams of almonds.

Check The total mass of ingredients is $160 + 200 + 100 = 460$ grams. The total mass of protein is $(0.25)(160) + (0.03)(200) + (0.13)(100) = 59$ grams.

The total cost is $(0.012)(160) + (0.0055)(200) + (0.014)(100) = \$4.42$.

Lesson Practice

a. *(Ex 1)* Determine whether $Q = \begin{bmatrix} -1 & 2 \\ 1.5 & -2.5 \end{bmatrix}$ and $W = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ are inverses.

b. *(Ex 1)* Determine whether $A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{4} & \frac{1}{3} \end{bmatrix}$ are inverses.

c. *(Ex 2)* Find the inverse of $A = \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$, if it exists.

d. *(Ex 2)* Find the inverse of $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, if it exists.

e. *(Ex 2)* Use a calculator to find and verify the inverse of $A = \begin{bmatrix} 2 & 1 & -2 \\ 5 & 3 & 0 \\ 4 & 3 & 8 \end{bmatrix}$.

f. *(Ex 3)* Solve for matrix X. $\underbrace{\begin{bmatrix} 3 & 7 \\ 1 & 4 \end{bmatrix}}_{A}X = \underbrace{\begin{bmatrix} -1 & -8 \\ -3 & -24 \end{bmatrix}}_{B}$

(Ex 4) **g.** Solve the system. $-5x - y + z = -16$
$3x + 2y - z = 12$
$x + y + 2z = -10$

(Ex 5) **h.** A 350-gram entrée contains black beans (3% fat and 23% protein), chicken breast (20% fat and 80% protein), and rice (2% fat and 8% protein). The entrée has 24.7 grams of fat and 115.3 grams of protein. How much of each ingredient is in the entrée?

Practice Distributed and Integrated

(29) ***1.** **Verify** Is the ordered triple $(5, -3, -1)$ a solution of the system:
$-2x - y - 2z = 5$
$x - 4y + 3z = 14$?
$-x + 4y - 3z = -6$

(25) **2.** **Multi-Step** A data set consists of the following numbers: $-4, -7, 12, -2, 8, 3, -1, 2, 4, -52$

a. Find the mean of the data. **b.** Find the standard deviation.

c. Identify any outliers.

d. Describe how any outlier affects the mean and the standard deviation of this data set.

(10) **3.** Solve the inequality: $3t + 4 < 2(t + 3) + t$. Show your work.

(31) ***4.** **Multiple Choice** Which shows all the values of x that make the division expression $\frac{x^2 - 1}{x} \div \frac{3 - x}{2x + 2}$ undefined?

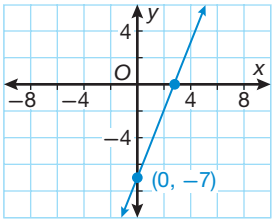
A $0, 3, -1$ **B** $0, -3, 1$ **C** $2, 3, -1$ **D** $2, -3, 1$

(11) **5.** **Analyze** If the degree of every term of a polynomial is even and all the coefficients are positive (including the constant term), can the polynomial have a negative value? Explain.

(32) ***6.** Write the matrix equation for the system and solve.

$$3x - y = 5 \text{ and } y = 2x - 4$$

(13) **7.** **Multiple Choice** Choose the letter that best represents the equation of the graph:



A $y = \frac{2}{5}x - 7$ **B** $y = -\frac{5}{2}x - 7$

C $y = \frac{5}{2}x - 7$ **D** $y = -\frac{2}{5}x - 7$

(15) **8.** **Verify** Verify that $(1.3, -2.7)$ is a solution of $\begin{matrix} 5.3x - 0.8y = 9.05 \\ -3.0x + 1.4y = -7.68 \end{matrix}$.

*9. (31) Divide. $\frac{36x^5y^{10}}{7y^4} \div \frac{12y^4}{21x^2}$

10. (27) **Recreational Planning** A playground that is to be rectangular in shape has an area modeled by the function $f(x) = -x^2 + 12x$ where x represent the length of one side of the playground in meters.

a. What value of x gives the playground the greatest possible area?

b. What is the greatest possible area?

*11. (30) **Graphing Calculator** An object dropped from a height 40 feet above the ground after x seconds is represented by $y = -16x^2 + 40$. When the object is dropped from 10 feet above the ground, the function becomes $y = -16x^2 + 10$.

a. Graph the functions on a graphing calculator and describe the transformation that changes the graph of the first function to the second.

b. For which quadrant or quadrants does the graph make sense? Why?

Write matrix equations for the systems of equations

*12. (32)
$$\begin{aligned} 2x + 4y + z &= 16 \\ x - 5y + z &= -5 \\ x + y + z &= 7 \end{aligned}$$

*13. (32)
$$\begin{aligned} 3x + 4z &= 13 \\ 15y - 2z &= 20 \\ 6z &= 6 \end{aligned}$$

14. (27) Written in vertex form, what are the values of a, h, and k in $f(x) = -2\left(x - \frac{1}{2}\right)^2 - 5$?

15. (11) **Write** Describe how to rewrite the polynomial $x + 4x + 12 - 5x^2 - x$ as an equivalent polynomial in standard form.

*16. (32) **Nutrition** A sandwich consists of Swiss cheese, ham, and rye bread. The sandwich contains 37.4 grams of carbohydrates, 46 grams of fat, and 36.6 grams of protein. How much of each ingredient is in the sandwich?

	Swiss Cheese	Ham	Rye Bread
Carbohydrates	6%	9%	77%
Fat	64%	48%	11%
Protein	30%	43%	12%

17. (28) **Error Analysis** Explain and correct the error a student made below.

$$\frac{10 - 2r}{r^2 - 10r + 25} = \frac{2(5 - r)}{(r - 5)(r - 5)} = \frac{2(\cancel{r - 5})}{(\cancel{r - 5})(r - 5)} = \frac{2}{r - 5}$$

18. (12) **Geometry** For each equation identify the kind of variation it represents.

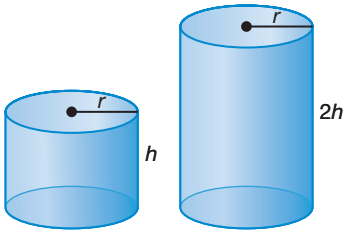
a. Area of a circle = π(radius)^2 **b.** Area of a rectangle = length × width

*19. (31) **Multi-Step** A container manufacturer is considering doubling the height of a cylindrical container.

a. Find the ratio of surface area to volume for the small container.

b. Find the ratio of surface area to volume for the large container.

c. A smaller ratio of surface area to volume results in a more economical container. Which container is more economical? Explain.



20. (17) **Exercise** For a 20-year old to workout within his cardio range, his heart rate should be within 25 beats per minute of 155 bpm.

a. Write an absolute value inequality to determine the lowest and highest heart rate at which he will be in his cardio range.

b. Solve the inequality.

21. (3) Simplify $\dfrac{x^2x^0x^{-1}(x^{-2})^2yx^{-3}}{(x^2y)^{-3}xyx^{-2}x^2}$.

22. (19) **Basketball** The length of a regulation high school basketball court is represented by the expression $(x + 24)$ feet. The width of a regulation high school basketball court is represented by the expression $(x - 10)$ feet.

a. Write an expression that represents the area of a regulation high school basketball court.

b. Evaluate the area of the court if $x = 60$.

***23.** (32) **Analyze** Suppose matrix A has an inverse and the matrix equation $AX = B$ has a solution. Explain why $X = A^{-1}B$ is the solution, but $X = BA^{-1}$ is not the solution.

24. (20) Given $f(x) = -7x + 4$ and $g(x) = 9x + 2$, represent $(f + g)(x)$ geometrically.

25. (28) **Probability** A mathematician has a number of red, white, and blue marbles. The number of red marbles is the same as the number of blue marbles and the number of white marbles is the square of the number of red marbles. If the mathematician puts all the marbles in a bag and randomly chooses one marble, what simplified rational expression represents the probability of pulling out one red marble?

Solve.

26. (24)
$$\frac{3}{8}x - \frac{1}{3}y = -9$$
$$2x + 4y = 4$$

27. (7)
$$\frac{-3 - x}{2} - \frac{x}{2} = 7$$

***28.** (29)
$$9x = y - 3z - 21$$
$$4y - 6z = 8x + 10$$
$$56x - 28y = 2 - 42z$$

a. Write the equations in standard form.

b. Estimate the number of solutions. Explain your answer.

c. Solve the system of equations to verify your estimate.

29. (23) Solve $x^2 - 5x - 36 = 0$

30. (14) **Error Analysis** Explain the error in the work below. Then find the correct determinant.

$$\begin{vmatrix} 1 & 2 & -2 \\ -1 & 4 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 5 - 6 + 0 = -1$$

LESSON 33

Applying Counting Principles

Warm Up

1. (25) **Vocabulary** The middle number in an ordered list of numbers is called the _____.

2. (25) Find the mean of this data set: 12, 20, 20, 30.

3. (25) Find the median of this data set: 10, 20, 40, 30, 32, 38.

New Concepts

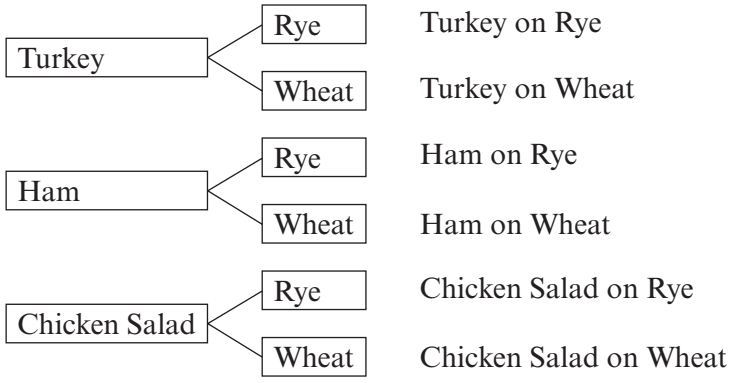
In counting theory, an **experiment** is any process that results in one or more **outcomes.** If a process is repeated one or more times, each time it is performed is sometimes called a **trial,** and the experiment consists of all the trials.

Example 1 Using a Tree Diagram

A cafeteria offers turkey, ham, and chicken salad sandwiches. The bread choices are rye and wheat. How many different sandwiches can be ordered?

SOLUTION

Ordering a sandwich can be considered a trial of an experiment. Each different sandwich is an outcome. Make a tree diagram to show all possible outcomes.



There are 6 different sandwiches that can be ordered.

Math Language

A **tree diagram** is a branching diagram that shows all possible combinations or outcomes of an experiment.

The **sample space** for an experiment is the set of all possible outcomes. A tree diagram was used in Example 1 to show a sample space. An **event** is any subset of a sample space, so an event is any outcome or set of outcomes. If you need only to count outcomes rather than actually show them, one method you can sometimes use is to apply the **Addition Counting Principle.**

Addition Counting Principle
Suppose a trial can result in any of n_1 outcomes from one category, any of n_2 outcomes from another category, and so on. If there are k different categories of outcomes, then the total number of outcomes that can result is $n_1 + n_2 + \ldots + n_k$.

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Two events in a sample space are **mutually exclusive** if they have no outcomes in common. To find the number of outcomes in an event that consists of mutually exclusive events, use the Addition Counting Principle.

Example 2 Using the Addition Counting Principle

Find the number of outcomes in each event.

a. Draw an ace or a face card by drawing a card at random from a standard deck.

SOLUTION

The events *draw an ace* and *draw a face card* are mutually exclusive. There are 4 aces. There are 12 face cards: 4 jacks, 4 queens, and 4 kings.

$4 + 12 = 16$

There are 16 outcomes in the event *draw an ace or a face card.*

b. Choose a prime number or a multiple of 6 or a multiple of 10 by choosing a number at random from the whole numbers 1 through 20.

SOLUTION

The three events *choose a prime number, choose a multiple of 6*, and *choose a multiple of 10* are mutually exclusive. There are 8 prime numbers: 2, 3, 5, 7, 11, 13, 17, and 19. There are 3 multiples of 6: 6, 12, and 18. There are 2 multiples of 10: 10 and 20.

$8 + 3 + 2 = 13$

There are 13 outcomes in the event *choose a prime number or a multiple of 6 or a multiple of 10.*

Math Language

An event that is the union or intersection of two events is a **compound event.** The event *draw an ace or a face card* is a compound event.

The **Fundamental Counting Principle** uses multiplication instead of addition.

Fundamental Counting Principle

Suppose k items are to be chosen. If there are n_1 ways to choose the first item, n_2 ways to choose the second item, and so on, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways to choose all k items.

Example 3 Using the Fundamental Counting Principle

A student is choosing a three-letter password to use for email.

a. How many passwords are possible if letters may be repeated?

SOLUTION

Because letters may be repeated, there are 26 choices for each letter.

$26 \cdot 26 \cdot 26 = 17{,}576$

There are 17,576 possible passwords.

b. How many passwords are possible if letters may not be repeated?

SOLUTION

Because letters may not be repeated, there are 26 choices for the first letter, 25 choices for the second letter, and 24 choices for the third letter.

$26 \cdot 25 \cdot 24 = 15{,}600$

There are 15,600 possible passwords.

Two events are **independent** if the probability of one event is not affected by whether or not the other event occurs. Two events are **dependent** if the probability of one event is affected by whether or not the other event occurs.

Example 4 Comparing Independent and Dependent Events

The letters A through E are written on 5 index cards. A card is chosen at random and then another card is chosen at random. Determine whether the events in each case are independent or dependent.

a. Choose A and then choose B, if the first card is replaced before the second card is chosen.

SOLUTION

The events are independent because there is a 1 in 5 probability of choosing B second, regardless of what is chosen first.

b. Choose A and then choose B, if the first card is not replaced before the second card is chosen.

SOLUTION

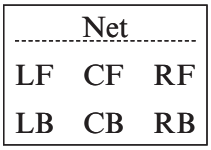
The events are dependent. If A is chosen first, there is a 1 in 4 probability of choosing B second. But if A is not chosen first, then it is possible that B is chosen first, in which case there is a zero probability of choosing B second.

Math Reasoning

Analyze How is finding the probability of independent events different than finding the probability of dependent events?

Example 5 Application: Sports

There are 6 positions in volleyball: left front, center front, right front, left back, center back, and right back. How many ways can 6 players be placed in the 6 positions?



SOLUTION

Use the Fundamental Counting Principle.

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

There are 720 ways to place 6 players in the 6 positions.

Check The method makes sense because any of the 6 players can be LF. Then any of the 5 remaining players can be CF, and so on: any of 4 can be RF, any of 3 can be LB, either of 2 can be CB, and the last player would be RB.

Lesson Practice

a. (Ex 1) A student has four choices for a T-shirt: red, white, blue, and black. He has 2 choices for pants: jeans or khakis. How many different outfits can he make, using one T-shirt and one pair of pants? Make a tree diagram to show all possible outcomes.

b. (Ex 2) A card is drawn at random from a standard 52-card deck. How many outcomes are in the event *draw a heart or a club*?

c. (Ex 2) A number is chosen at random from the whole numbers 21 through 40. How many outcomes are in the event *choose a perfect square or a multiple of 8 or a prime number*? List all the outcomes.

d. (Ex 3) A student is choosing a four-digit password to use for her bank account. How many passwords are possible if digits may be repeated?

e. (Ex 3) A student is choosing a four-digit password to use for her bank account. How many passwords are possible if digits may not be repeated?

There are five tiles in a bag, labeled W, O, R, D, and S. A tile is chosen at random and then another tile is chosen at random. Determine whether the following events are independent or dependent. (Ex 4)

f. Choose W and then choose S, if the first tile is replaced before the second tile is chosen.

g. Choose W and then choose S, if the first tile is not replaced before the second tile is chosen.

h. (Ex 5) There are 9 players in a starting baseball lineup. How many ways can batting positions 1, 2, and 3 be filled, choosing from 9 players?

Practice Distributed and Integrated

***1.** (30) **Geometry** The function representing the area of a rectangle whose length is 12 units longer than its width is $f(x) = (x + 6)^2 - 36$ where x is the width of the rectangle. Graph this function. For which quadrant or quadrants does the graph make sense? Why?

2. (10) Solve and graph the compound inequality $4 < 2(x - 2) \leq 16$.

Use the letters A-B-C-D-E-F-G.

***3.** (33) How many seven-letter passwords are possible if letters can be repeated?

***4.** (33) How many seven-letter passwords are possible if letters cannot be repeated?

5. (26) Find the equation of the line through $(-3, -5)$ and $(4, -5)$.

6. (Inv 2) **Airplanes** As an airplane ascends after takeoff, its altitude increases at a rate of 45 ft/s while its distance on the ground from the airport increases at 210 ft/s. Write parametric equations to model the location of the plane.

***7.** (32) Write the matrix equation that represents the system of equations.

$$p + q + r = 6$$
$$0.4p + 0.5q + 2r = 5.7$$
$$2.2p + 3.4q + r = 12$$

8. (20) **Error Analysis** Two students evaluated $(f - g)(7)$, given $f(x) = 5x + 8$ and $g(x) = 9x + 2$. Who is correct? Explain the error.

Student A	Student B
$(f - g)(7)$ $= 5(7) + 8 - (9(7) + 2)$ $= 35 + 8 - 63 - 2$ $= -22$	$(f - g)(7)$ $= 5(7) + 8 - (9(7) + 2)$ $= 35 + 8 - 63 + 2$ $= -2$

9. (11, 19) Given the polynomial functions $f(x) = \frac{1}{2}x^2 - 3$, $g(x) = 2x^2$, and $h(x) = 4x$,

a. write the sum $(f + g + h)(x)$ in standard form.

b. write the product $(f \cdot g \cdot h)(x)$ in standard form.

***10.** (31) **Justify** Is multiplication of rational expressions commutative? Is division of rational expressions commutative? Justify your answers.

11. (31) **Multiply** Assume all expressions are defined. $\frac{2x^4y^5}{3x^2} \cdot \frac{15x^2}{8x^3y^2}$

***12.** (33) Using the numbers 2-4-6 and the letters P-A-R-K, how many one-digit, two-letter passwords can be made, if repeats are allowed and the password starts with a digit?

13. (14) **Multi-Step** $\begin{vmatrix} x & 2 \\ -6 & 4 \end{vmatrix} = 8$

a. Write the determinant as an equation.

b. Solve for x.

c. Check your answer by substituting for x and finding the determinant with a calculator.

***14.** (32) Find the inverse of $M = \begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix}$, if it exists.

Write each equation in standard form.

15. (26) $y = 1.5x + 3$

16. (26) $y = -0.05x + 20$

17. (16) **Ticket Prices** A zoo charges \$10.95 for each child's admission and \$14.95 for each adult admission. If a busload of 45 people attend, and together they pay \$516.75, how many adults and how many children are admitted?

a. Set up equations relating the number of children and adults to the total admissions. Let c represent the number of children and a the number of adults.

b. Use Cramer's Rule to solve the system of equations.

18. (21) **Write** Without solving the system, can you tell how many solutions this system of equations has? Explain.

$$y + x = 2$$
$$y + 1 = -x$$

19. (19) Simplify: $\dfrac{(xm^{-2})^0x^0m^0}{xx^2m^0(2x)^{-2}}$

***20.** (33) **Write** Describe how to determine the number of different arrangements of the letters M, A, T, and H.

21. (23) **Multiple Choice** Factor the expression $-12x^4 + 6x^2 + 18x$.

A $-6x(2x^3 - x - 3)$ **B** $-6(2x^4 - x^2 - 3)$

C $-6x(2x^3 + x + 3)$ **D** $-6(2x^4 + x^2 + 3)$

***22.** (27) **Graphing Calculator** Use a graphing calculator to determine the y-intercept of the quadratic function $y = x^2 + x - 2$.

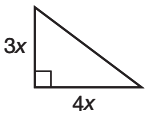
***23.** (33) **Sports** There were 12 players on the United States team that competed in the 2006 Women's Basketball (under 19) World Championship tournament. How many ways can 5 starters be chosen from 12 players?

24. (16) **Motion** A speed boat leaves a nearby shore of a lake and moves at 25 knots toward the opposite shore, 2 nautical miles away. At the same time, a pleasure boat leaves the opposite shore and moves at 10 knots toward the speed boat. When will the boats meet?

a. Set up a system of equations representing the position of each boat relative to the nearby shore.

b. Use Cramer's Rule to solve the system.

***25.** (31) **Measurement** Write a rational expression in simplest form for the ratio of area to perimeter for the right triangle.



***26.** (Inv 3) Find the intercepts of the equation $40 + 8y - 2z = -5x$.

27. (25) **Analyze** The first quartile of some test scores is 55 and the third quartile is 74.

a. Is it possible for the median of the test scores to be 77? Explain why or why not.

b. The maximum score is 93 and the range of the test scores is 55. What is the minimum test score?

c. What is the range of test scores that will place a student in the middle 50% of the class?

28. (24) **Multi-Step** **a.** Write an equation in two variables to represent that the sum of two numbers is 73.

b. Write an equation with the same variables to represent that twice the second number minus the first number is 2.

c. Write and solve a linear system to find the two numbers.

29. (18) Convert 61 yd^2 to square inches

***30.** (32) What is the inverse of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix}$?

LESSON 34

Graphing Linear Equations II

Warm Up

1. (13) **Vocabulary** The expression $\frac{\text{rise}}{\text{run}}$ is the _____ of a line.
2. (13) What is the y-intercept of the line whose equation is $y = 2x + 5$?
3. (7) One point on the line with equation $-x + 3y = 4$ is (___, 2).

New Concepts

A relation is a set of ordered pairs, usually denoted (x, y). A function is a relation in which there is exactly one value of y for every value of x. To determine whether a graph is the graph of a function, you can use the vertical line test.

A function with a constant rate of change is called a **linear function,** and its graph is a line. Every non-vertical line represents a linear function.

Example 1 Identifying Graphs of Linear Functions

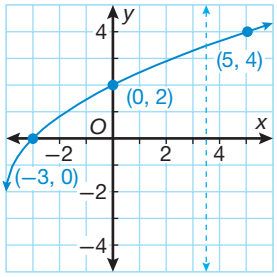
Determine whether each graph represents a linear function.

a. Rate of change from $(-3, 0)$ to $(0, 2)$:

$$\frac{2-0}{0-(-3)} = \frac{2}{3}$$

Rate of change from $(0, 2)$ to $(5, 4)$:

$$\frac{4-2}{5-0} = \frac{2}{5}$$



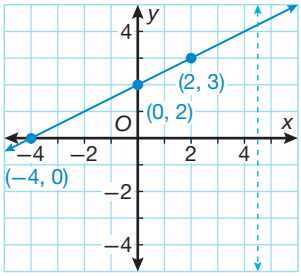
SOLUTION No vertical line intersects the graph in more than one point, so it represents a function. However, it is not a linear function because it does not have a constant rate of change.

b. Rate of change from $(-4, 0)$ to $(0, 2)$:

$$\frac{2-0}{0-(-4)} = \frac{2}{4} = \frac{1}{2}$$

Rate of change from $(0, 2)$ to $(2, 3)$:

$$\frac{3-2}{2-0} = \frac{1}{2}$$



SOLUTION No vertical line intersects the graph in more than one point, so it represents a function. And, it is a linear function because it has a constant rate of change.

Hint

To be sure that a graph represents a linear function, you either need to know that it is a line, or you need to know its equation.

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A linear function can be represented by a linear equation. There are three forms of a linear equation that are commonly used.

Forms of a Linear Equation		
Standard Form	$Ax + By = C$	A and B are not both zero.
Slope-Intercept Form	$y = mx + b$	m is the slope and b is the y-intercept.
Point-Slope Form	$y - y_1 = m(x - x_1)$	m is the slope and (x_1, y_1) is a point on the line.

Example 2 Graphing Linear Equations, Given Standard Form

Graph each linear equation.

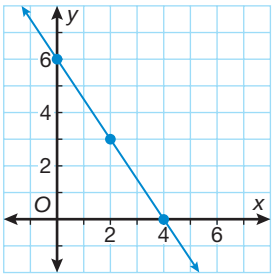
Hint

It is easy to compute with zeros, so it is often helpful to include one or both intercepts when making a table of values.

a. $3x + 2y = 12$

SOLUTION Make a table of ordered pairs, plot the ordered pairs, and draw a line through them.

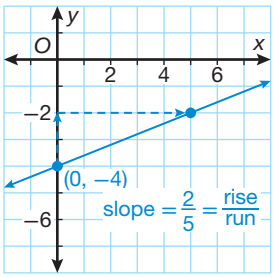
x	0	4	2
y	6	0	3



b. $-2x + 5y = -20$

SOLUTION Solve the equation for y to get slope-intercept form.

$$-2x + 5y = -20$$
$$5y = 2x - 20$$
$$y = \frac{2}{5}x - 4$$



The y-intercept is -4, so the line crosses the y-axis at $(0, -4)$. Plot $(0, -4)$. Then use the slope: from $(0, -4)$, count 2 up and 5 right to plot another point on the line. Draw the line through the points.

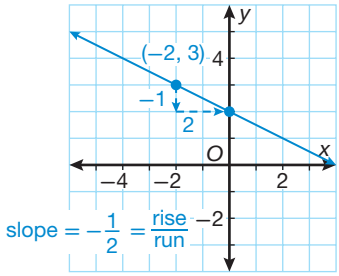
Example 3 Graphing a Linear Equation, Given Point-Slope Form

Hint

There are different ways to count using slope. For example, $-\frac{1}{2} = \frac{2}{-4} = \frac{\text{rise}}{\text{run}}$, so in Example 3, you can count 2 up and 4 left to plot another point on the line.

Graph $y - 3 = -\frac{1}{2}(x + 2)$.

SOLUTION $y - 3 = -\frac{1}{2}(x + 2)$ is in the point-slope form $y - y_1 = m(x - x_1)$. The slope is $m = -\frac{1}{2}$ and a point on the line is $(x_1, y_1) = (-2, 3)$. Plot $(-2, 3)$ and then count 1 down and 2 right to plot another point on the line. Draw the line through the points.



The most basic linear function is $y = x$. It is called the *parent function* of all linear functions. All other linear functions are *transformations* of $y = x$. Reflections, shifts, stretches, and compressions are types of transformations.

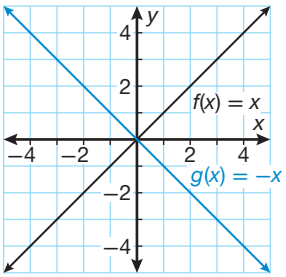
Transformations of $f(x) = x$
$-f(x)$ is a reflection over the x-axis.
$f(x) + c$ is a vertical shift c units up if c is positive.
$f(x) + c$ is a vertical shift c units down if c is negative.
$c \cdot f(x)$ is a vertical stretch by a factor of c if $c > 1$.
$c \cdot f(x)$ is a vertical compression by a factor of c if $0 < c < 1$.

Example 4 Graphing Transformations of $f(x) = x$

Graph each transformation of $f(x) = x$.

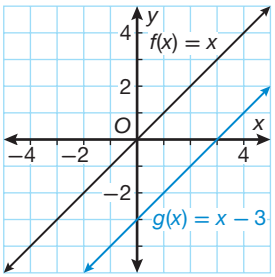
a. $g(x) = -x$

SOLUTION $g(x) = -x = -f(x)$, which is the reflection of $f(x)$ over the x-axis.



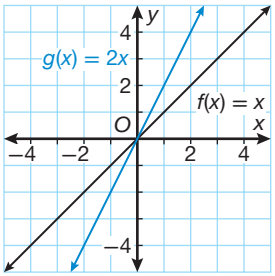
b. $g(x) = x - 3$

SOLUTION $g(x) = x - 3 = f(x) - 3$, which is a vertical shift of $f(x)$ down 3 units.



c. $g(x) = 2x$

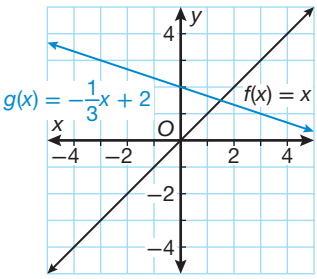
SOLUTION $g(x) = 2x = 2 \cdot f(x)$, which is a vertical stretch of $f(x)$ by a factor of 2.



d. $g(x) = -\frac{1}{3}x + 2$

SOLUTION $g(x) = -\frac{1}{3}x + 2 = -\frac{1}{3}f(x) + 2$, which consists of three transformations of $f(x)$:

- a vertical compression by a factor of $\frac{1}{3}$
- a reflection over the x-axis
- a vertical shift 2 units up



Graphing Calculator Tip

To enter $g(x) = -\frac{1}{3}x + 2$, use these key strokes: Y=, then (−) 1 ÷ 3 X,T,θ,n + 2, then GRAPH Adjust the viewing window as needed.

If a line has a slope of zero, then its equation $y = mx + b$ is $y = 0x + b$, or $y = b$.

Horizontal and Vertical Lines

Let b be any constant.

$y = b$ is a linear function whose graph is a **horizontal line** with a slope of zero.

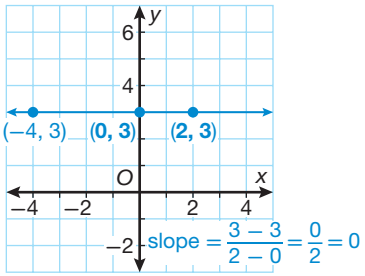
$x = b$ is not a function; its graph is a vertical line with an undefined slope.

Example 5 Graphing Horizontal and Vertical Lines

a. Graph $y = 3$.

SOLUTION Make a table of ordered pairs. Use 3 for y and any number for x.

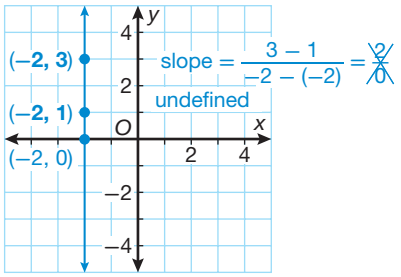
x	-4	0	2
y	3	3	3



b. Graph $x = -2$.

SOLUTION Make a table of ordered pairs. Use -2 for x and any number for y.

x	-2	-2	-2
y	3	1	0

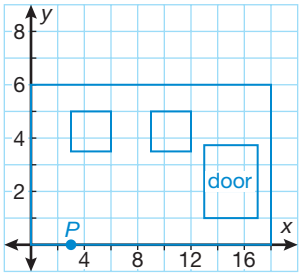


Math Language

An **undefined slope** is also called **no slope.** Remember that slope is a number. A zero in a denominator indicates no slope because there is no number that has zero in the denominator.

Example 6 Application: Wheelchair Accessibility

The Americans with Disabilities Act specifies that the maximum slope for a wheelchair ramp should be 1:12. A builder needs to install a wheelchair ramp from point P to the entrance door shown in the diagram. Graph a line that can be used to represent the ramp. Write an equation of the line in slope-intercept form.



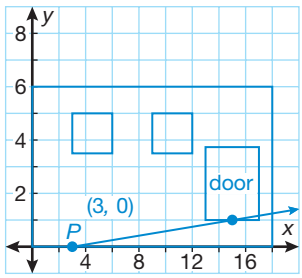
SOLUTION Draw a line through point P with a rise of 1 and a run of 12. Use the given point and given slope to write an equation of the line in point-slope form. Then solve for y to obtain slope-intercept form.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{1}{12}(x - 3)$$

$$y = \frac{1}{12}(x - 3)$$

$$y = \frac{1}{12}x - \frac{1}{4}$$



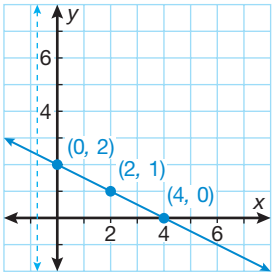
Check The graph is correct because the rise is 1, the run is 12, and it passes through $P(3, 0)$. The equation $y = \frac{1}{12}x - \frac{1}{4}$ is correct because its slope is $\frac{1}{12}$ and the ordered pair $(3, 0)$ is a solution.

Lesson Practice

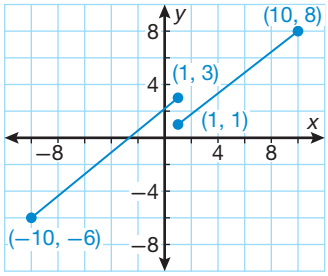
Determine whether the graph represents a linear function.

(Ex 1)

a.



b.



c. (Ex 2) Graph $2x - 4y = 8$ using a table.

d. (Ex 2) Graph $2x - 5y = 5$ by solving for y to obtain slope-intercept form.

e. (Ex 3) Graph $y + 1 = \frac{3}{4}(x - 4)$.

Graph by transforming the parent function $f(x) = x$.

(Ex 4)

f. $g(x) = -6x$

g. $g(x) = x + 5$

h. $g(x) = 4x - 7$

i. $g(x) = -3x + 1$

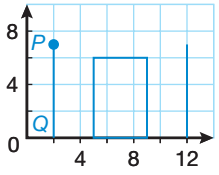
Graph.

(Ex 5)

j. $x = 4$

k. $y = -5$

l. (Ex 6) The diagram shows two walls and a door of a shed. $\overline{PQ}$ represents a wall. Graph a line through point P to represent a roof with a slope of $\frac{3}{5}$. Write an equation of the line in slope-intercept form.



Practice Distributed and Integrated

1. (7) Berries Chuck and Nikita picked 173 quarts of berries. How many did each pick if Chuck picked 11 more quarts than Nikita?

2. (23) Factor: $-2ab + abx + abx^2$.

Evaluate.

3. (2) $x^2 - y^2(x - y)$ if $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

4. (2) $ax - a(a - x)$ if $a = -\frac{1}{2}$ and $x = \frac{1}{4}$.

5. (21) Use substitution to solve: $\begin{aligned} x + 2y &= 5 \\ 3x - y &= 7 \end{aligned}$

6. (19) Expand: $\frac{4x^{-2}y^{-2}}{z^2}\left(\frac{3x^2y^2z^2}{4} + \frac{2x^0y^{-2}}{z^2y^2}\right)$.

Simplify.

7. (3) $\frac{(0.0003 \times 10^8)(6000)}{(0.006 \times 10^{15})(2000 \times 10^5)}$

8. (3) $\frac{3m}{x} - \frac{2x^{-1}}{m^0m^{-1}} + \frac{5x^2m^2}{x^3m}$

9. (26) Write the equation $y = \frac{2}{3}x + \frac{4}{5}$ in standard form.

10. (5) Solve for C: $[2 \quad -29 \quad 5] + C = [31 \quad -18 \quad 14]$.

***11.** (34) **Multiple Choice** Which statement is true about the graph of $y = 3x + 2$?

A The rate of change between any two points on the graph is 2.

B The rate of change between any two points on the graph is 3.

C The rate of change decreases along the graph from left to right.

D The rate of change increases along the graph from left to right.

***12.** (31) **Analyze** Given the functions $f(x) = \frac{3x-2}{x+2}$, $g(x) = \frac{10x^2+20x}{3x^2-2x}$, and $h(x) = 10$, compare the function product $f(x) \cdot g(x)$ and the constant function $h(x) = 10$. Describe how they are related.

13. (13) **Population** The table lists the population estimates, in hundreds of thousands, for Vermont from 2000 to 2002.

Year	2000	2001	2002
Population	6.1	6.13	6.16

a. Write the data above as ordered pairs where x is the number of years after 2000 and y is the population in hundreds of thousands.

b. Estimate Use slope and y-intercept to find the equation of the line representing this data.

14. (24) **a.** Multiply the first equation in $\begin{aligned} 0.2x + 0.5y &= -1 \\ x - y &= 16 \end{aligned}$ by 10. How does this make solving the system easier?

b. Solve the system.

***15.** (30) **Graphing Calculator** Use a graphing calculator to graph $y = (x + 2)^2$ and $y = x^2$ on the same coordinate axes. How do the graphs compare?

16. (15) **River Current** A motor boat makes a 175-kilometer trip down the Mississippi River to New Orleans in 3.3 hours. But it takes an hour longer for the boat to return upstream to its starting point. How fast is the river's current?

***17.** (34) **Probability** A circle has a radius of 4 and center (0, 0). Suppose a point is chosen at random inside the circle. What is the probability that the chosen point is in quadrant I and below the line $y = x$?

18. (21) **Geometry** You are given the equation of a line on a coordinate grid. You are then given three other equations of lines that combine with the first to form a rectangle. How many of the three lines have an inconsistent solution with the original line? How many of the three lines have a consistent solution with the original line?

19. (32) **Multi-Step** Find the inverse of matrix A. Then prove your answer is correct. Do not use a calculator. $A = \begin{bmatrix} 3 & 8 \\ 2 & 4 \end{bmatrix}$

20. (19) **Multiple Choice** Multiply $(x + 7)(x - 9)$

A $x^2 - 16x + 63$ **B** $x^2 - 2x + 63$ **C** $x^2 - 2x - 63$ **D** $x^2 + 2x - 63$

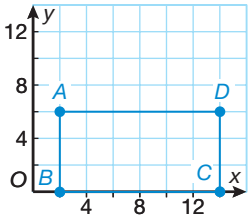
21. (7) **Verify** Is $x = 4$ the solution to the equation $6x - 4 = 20$?

22. (25) **Shipping Boxes** The table shows the measurements in inches of the side lengths of various shipping boxes available from the United States Postal Service.

Box	Measurement 1	Measurement 2	Measurement 3
1	11.875	3.375	13.625
2	11	8.5	5.5
3	7.5	5.125	14.375

Which measurement has the greatest range of side lengths? What is that range?

***23.** (34) **Roofing** The minimum slope (pitch) that is normally recommended for an asphalt shingle roof is $\frac{1}{3}$. Rectangle $ABCD$ represents the front wall of a building. Graph a line through point A to represent a roof with a slope of $\frac{1}{3}$. Write an equation of the line in slope-intercept form.



24. (33) **Passwords** A voicemail system password is 1 letter followed by a 3-digit number less than 600. How many different voicemail passwords are possible?

25. (15) **Analyze** Explain the difficulty you encounter when trying to use Cramer's Rule to solve the system $\begin{aligned} 4x - 12y &= -4 \\ -6x + 18y &= 6 \end{aligned}$

26. (33) **Error Analysis** A student is choosing a four-letter password. She wants to know how many passwords are possible if letters may not be repeated. To find out, she adds: $26 + 25 + 24 + 23 = 98$. What is the error? Find the correct answer.

27. (23) **Analyze** Is $x^2 + 7x + 12$ a basic quadratic polynomial, a perfect square trinomial, or a difference of two squares? Explain how you can tell by looking at the polynomial.

***28.** (34) **Write** Describe how to obtain the graph of $y = \frac{3}{4}x - 3$ by transforming the graph of $y = x$.

29. (15) **Error Analysis** A student solved the linear system $\begin{aligned} x + y &= 3 \\ x - y &= 1 \end{aligned}$ and got 2 as the answer. Explain and correct the student's error.

30. (23) Use $x^2 + 16x + 64$. Rewrite the expression in the form of a perfect square trinomial.

LESSON 35

Solving Quadratic Equations I

Warm Up

1. **Vocabulary** A parabola is a graph of a _____ function. (27)
2. Factor $8x^2 + 4x$ completely. (23)
3. Factor $16x^2 - 20x - 6$ completely. (23)

New Concepts

Every quadratic function can be written in the standard form $f(x) = ax^2 + bx + c$, where $a \neq 0$. The zeros of a function $f(x)$ are the x-values that make $f(x) = 0$. To find the zeros of a quadratic function, solve the related equation $ax^2 + bx + c = 0$. Many quadratic equations can be solved by factoring and applying the Zero Product Property, which says that if a product is zero, then at least one of its factors is zero.

Hint

The Zero Product Property applies to any number of factors. For example, if $abc = 0$, then $a = 0$ or $b = 0$ or $c = 0$.

Zero Product Property
Let a and b be real numbers. If $ab = 0$, then $a = 0$ or $b = 0$.

Example 1 Finding the Zeros of Quadratic Functions

Find the zeros of each quadratic function.

a. $f(x) = 4x^2 - 25$

SOLUTION

Hint

The function $f(x) = 4x^2 - 25$ is in standard form $f(x) = ax^2 + bx + c$ with $b = 0$.

$4x^2 - 25 = 0$	Write the related quadratic equation.
$(2x + 5)(2x - 5) = 0$	Factor.
$2x + 5 = 0$ or $2x - 5 = 0$	Zero Product Property
$2x = -5$ $\quad$ $2x = 5$	Solve each equation.
$x = -\frac{5}{2}$ $\quad$ $x = \frac{5}{2}$	

The zeros are $-\frac{5}{2}$ and $\frac{5}{2}$.

b. $f(x) = 2x^2 + x$

SOLUTION

$2x^2 + x = 0$	Write the related quadratic equation.
$x(2x + 1) = 0$	Factor.
$x = 0$ or $2x + 1 = 0$	Zero Product Property
$2x = -1$	
$x = -\frac{1}{2}$	

The zeros are 0 and $-\frac{1}{2}$.

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The **roots of an equation** are the solutions to the equation.

Example 2 Solving Quadratic Equations

Find the roots of the quadratic equation.

$x^2 = -2x + 15$

SOLUTION

$x^2 = -2x + 15$	Write the given equation.
$x^2 + 2x - 15 = 0$	Add $2x - 15$ to both sides to get zero on one side.
$(x + 5)(x - 3) = 0$	Factor.
$x + 5 = 0$ or $x - 3 = 0$	Zero Product Property
$x = -5$ $\quad$ $x = 3$	Solve each equation.

The roots are -5 and 3.

Math Language

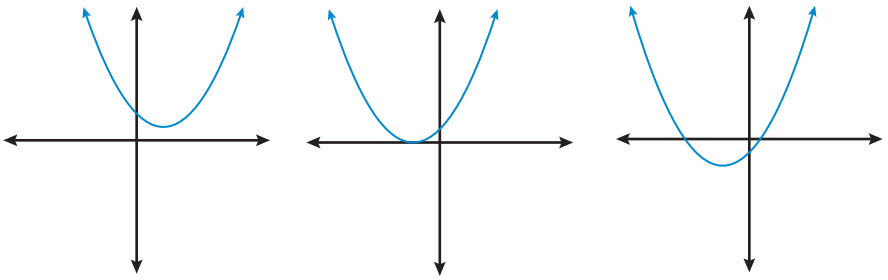
Functions have zeros or *x*-intercepts.

Equations have solutions or roots.

The graph of a quadratic function shows how many zeros the function has.

Zeros of a Quadratic Function

A quadratic function has 0, 1, or 2 real zeros.



No Real Zeros | One Real Zero | Two Real Zeros

When a quadratic function has exactly one real zero, that zero is a **double root** of the related equation.

Math Reasoning

Generalize When a quadratic function has exactly one real zero, at how many points does the graph intersect the *x*-axis?

Example 3 Comparing Roots, Zeros, and *x*-Intercepts

Find the roots of each equation. Graph the related function and describe the relationship between the roots, the zeros, and the x-intercepts.

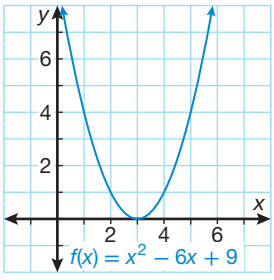
a. $x^2 - 6x + 9 = 0$

SOLUTION

$x^2 - 6x + 9 = 0$

$(x - 3)(x - 3) = 0$

$x = 3$



3 is the double root of the equation, the only zero of the related function $f(x) = x^2 - 6x + 9$, and the only x-intercept of the graph.

Graphing Calculator Tip

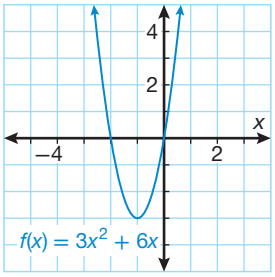
Verify your graph with a graphing calculator. To enter $f(x) = 3x^2 + 6x$, use these key strokes:

3 | X,T,θ,n | x^2

+ | 6 | X,T,θ,n

Adjust the viewing window as needed.

b. $3x^2 + 6x = 0$



SOLUTION

$$3x^2 + 6x = 0$$

$$3x(x + 2) = 0$$

$$3x = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 0 \quad \text{or} \quad x = -2$$

0 and -2 are the roots of the equation, the zeros of the related function $f(x) = 3x^2 + 6x$, and the x-intercepts of the graph.

Example 4 Writing a Quadratic Function, Given its Zeros

Write a quadratic function that has zeros $-\frac{5}{3}$ and 1.

SOLUTION Reverse the process of solving a quadratic equation.

$\left(x + \frac{5}{3}\right)(x - 1) = 0$ — Write the related equation in factored form.

$3\left(x + \frac{5}{3}\right)(x - 1) = 3(0)$ — Multiply both sides by 3 to eliminate the fraction.

$(3x + 5)(x - 1) = 0$ — Distribute the 3.

$3x^2 + 2x - 5 = 0$ — Multiply the binomials.

The quadratic function $f(x) = 3x^2 + 2x - 5$ has zeros $-\frac{5}{3}$ and 1.

Hint

Velocity is a vector quantity, which means it has both magnitude and direction. If an object is thrown straight down at a speed of 20 feet per second, its initial velocity is -20 feet per second.

Example 5 Application: Vertical Motion

The height of a free-falling object is given by the function $h(t) = -16t^2 + v_0t + h_0$, where h_0 is the initial height in feet, v_0 is the initial velocity in feet per second, and $h(t)$ is the height in feet at time t seconds. How long does it take for an object to hit the ground after it is thrown straight down at a speed of 20 feet per second from a height of 84 feet?

SOLUTION

$h(t) = -16t^2 + v_0t + h_0$

$0 = -16t^2 - 20t + 84$ — Substitute given values for v_0 and h_0.

$0 = -4(4t^2 + 5t - 21)$ — Factor out -4.

$0 = -4(4t - 7)(t + 3)$ — Factor the trinomial.

$4t - 7 = 0 \quad \text{or} \quad t + 3 = 0$ — Use the Zero Product Property.

$t = 1.75 \quad \text{or} \quad t = -3$ — Ignore $t = -3$. Time cannot be negative.

It takes the object 1.75 seconds to hit the ground.

Lesson Practice

Find the zeros of each quadratic function.
(Ex 1)

a. $f(x) = 25x^2 - 1$.

b. $f(x) = x^2 - 7x$.

c. Find the roots of $3x^2 - x = 2$.

Find the roots of each equation. Graph the related function and describe the relationship between the roots, the zeros, and the *x*-intercepts.
(Ex 3)

d. $x^2 + 8x + 16 = 0$

e. $f(x) = x^2 + 2x - 8$

f. Write a quadratic function that has zeros -5 and $\frac{3}{4}$.
(Ex 4)

g. (Ex 5) The height of a free-falling object is given by the function $h(t) = -16t^2 + v_0t + h_0$, where h_0 is the initial height in feet, v_0 is the initial velocity in feet per second, and $h(t)$ is the height in feet at time t seconds. How long does it take for an object to hit the ground after it is thrown straight *up* at a speed of 8 feet per second from a height of 80 feet?

Practice Distributed and Integrated

Simplify.

1. (3) $\dfrac{(b^2c^{-2})^{-3}c^{-3}}{(b^2c^0b^{-2})^4}$

2. (3) $\dfrac{(2x^2ya)^{-3}ya^3}{x^2y(ay)^{-2}y}$

Solve.

3. (7) $3(-2x - 3) - 2^2 = -(-3x - 5) - 2$

4. (23) $x^2 + 26x - 56 = 0$

5. (26) Find the equation of the line that passes through $(-3, 4)$ and $(3, -2)$.

Use elimination to solve.

6. (24)
$$\begin{aligned} 4x - 3y &= -1 \\ 2x + 5y &= 19 \end{aligned}$$

7. (24)
$$\begin{aligned} 5x + \frac{1}{2}y &= -4 \\ -x + y &= 3 \end{aligned}$$

8. (5) Solve for a and b: $\begin{bmatrix} a \\ 64 \end{bmatrix} = \begin{bmatrix} 102 \\ a - b \end{bmatrix}$.

9. (20) Find $(f \cdot g)(x)$, given $f(x) = 4x - 3$ and $g(x) = 8x + 9$.

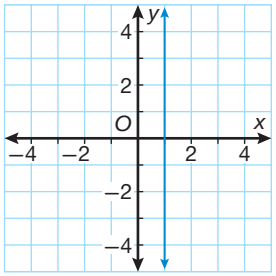
10. (22) **Formulate** Describe a plan for determining if a function is continuous.

***11.** (35) **Multiple Choice** What are the zeros of the function $f(x) = x^2 - 5x + 6$?

A 0 and 6 **B** 0 and -6 **C** 2 and 3 **D** 2 and -3

12. (32) **Probability** Suppose you roll a pair of number cubes twice. What is the probability of rolling a sum greater than 10 on both rolls?

*13. (34) **Error Analysis** A student described this line as having no slope, or zero slope. What is wrong with this description?



14. (11) If a polynomial of degree 3 is added to a polynomial of degree 4, what are all possible values for the degree of the result?

15. (31) **Analyze** The product $\frac{x^2-4}{x^2-3x-4} \cdot \frac{x-4}{x+2}$ in simplest form is $\frac{x-2}{x+1}$, as shown:

$\frac{x^2-4}{x^2-3x-4} \cdot \frac{x-4}{x+2} = \frac{\cancel{(x+2)}\,(x-2)}{(x+1)\,\cancel{(x-4)}} \cdot \frac{\cancel{(x-4)}}{\cancel{(x+2)}} = \frac{x-2}{x+1}$. When 4 is substituted for x in $\frac{x-2}{x+1}$, the value is $\frac{2}{5}$. Explain why the value of $\frac{x^2-4}{x^2-3x-4} \cdot \frac{x-4}{x+2}$ is not $\frac{2}{5}$ when $x = 4$.

16. (23) **Write** A square is drawn on a coordinate grid with one side along the x-axis and one vertex at $(a - 1, 0)$. The area of the square is $a^2 - 8a + 16$. Determine an expression for an adjacent vertex. Explain.

*17. (34) **Justify** Explain why this paragraph describes a method of graphing the equation: $y + 1 = \frac{1}{4}(x - 2)$: Plot the ordered pair $(2, -1)$. Then from that point, count 1 unit up and 4 units to the right to plot another point. Then draw a line through the two plotted points.

18. (10) Solve and graph the compound inequality: $4 < 2(x - 2) \leq 16$.

19. (31) **Average Velocity** If an object is thrown straight down with initial velocity 8 feet per second, the number of feet it travels in t seconds is given by the expression $16t^2 + 8t$ (neglecting air resistance).

a. Write a rational expression for the average velocity of the object before it hits the ground over the time interval from 0 seconds to t seconds. Then simplify the expression. (Average velocity equals distance traveled divided by elapsed time.)

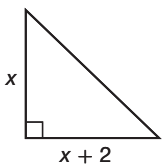
b. Evaluate the expression to find the average velocity of the object over the time interval from 0 seconds to 2.5 seconds.

20. (24) **Chemistry** How many liters of a 15% acid solution should be mixed with a 35% acid solution to make 7 liters of a 20% acid solution?

*21. (35) **Write** State all possible numbers of real zeros a quadratic function can have. Explain what information the number of real zeros provides about the point(s) of intersection of the graph and the x-axis.

22. (12) Is the equation $y = \sqrt{2}x$ a direct or inverse variation?

*23. (35) **Geometry** The area of the right triangle is 24 square units. Find the value of x.



*24. (34) **Currency Exchange** During 2006, the average exchange rate for euros and U.S. dollars was \$1.2563 per euro. This relationship is given by the function $y = 1.2563x$, where y represents dollars and x represents euros. Describe how to transform the graph of the parent function $y = x$ to obtain the graph of $y = 1.2563x$.

25. (22) **Multiple Choice** An example of a set of numbers that can be listed is:

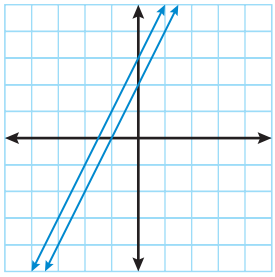
A real numbers **B** integers

C rational numbers **D** irrational numbers

***26.** (26) **Graphing Calculator** When you're traveling in a car that comes to an abrupt halt, the car continues forward even though the brakes have been pressed. The distance traveled is called the "stopping distance." The table below shows the stopping distance at three different speeds for a car traveling on wet pavement. Use your graphing calculator and the linear regression capabilities to write an equation in slope-intercept form for the stopping distance.

Initial Traveling Speed (mile/hour)	Stopping Distance (feet)
35	53.389
49	114.159
62	202.678

27. (21) **Coordinate Geometry** Which set of equations could be represented by this graph?



A $\begin{aligned} y &= 2x + 4 \\ 2y - 2x &= 8 \end{aligned}$

B $\begin{aligned} y &= 2x + 4 \\ 12 - 3y &= -6x \end{aligned}$

C $\begin{aligned} 2y &= 4x - 12 \\ y &= 2x + 6 \end{aligned}$

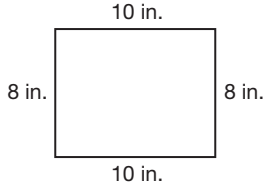
D $\begin{aligned} -4x &= 8 - 2y \\ y &= 2x + 6 \end{aligned}$

28. (32) **Nutrition** A spaghetti entrée consists of spaghetti, ground beef, and tomato sauce. The entrée contains 118.68 grams of carbohydrates, 40.12 grams of fat, and 41.2 grams of protein. How much of each ingredient is in the entrée? To find the answer, write a system of equations and use an inverse matrix to solve it. Use the table below.

	Spaghetti	Ground Beef	Tomato Sauce
Carbohydrates	81%	0%	87%
Fat	5%	60%	3%
Protein	14%	40%	10%

***29.** (35) **Flags** There is no required size for a United States flag, but the ratio of width to length that is recommended in most cases is 1 : 1.9. Write and solve a quadratic equation to find the length and width of a United States flag that has an area of 17.1 square feet.

30. (18) What is the perimeter of this rectangle in feet?



LESSON 36

Using Parallel and Perpendicular Lines

Warm Up

1. (I3) **Vocabulary** If you know the slope of a line and the coordinates of a point on the line, you can use the _____ formula to write the equation of the line.

2. (I3) Find the slope of the line that crosses (1, 4) and (5, 9).

3. (I3) Write the equation $3x - 7y = 28$ in slope-intercept form.

New Concepts

From geometry you know that parallel lines do not intersect and that perpendicular lines meet at right angles. When parallel and perpendicular lines are graphed on an coordinate plane, the equations of these lines have special properties.

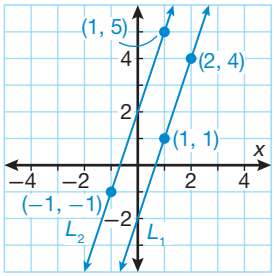
Parallel lines have identical slopes and different y-intercepts. Perpendicular lines have slopes that are reciprocals of each other and have opposite signs. The product of the slopes of two perpendicular lines is -1.

Example 1 Graphs of Parallel and Perpendicular Lines

Hint

For parallel lines: $m_1 = m_2$. For perpendicular lines: $m_1 m_2 = -1$

a. Show that L1 and L2 are parallel.



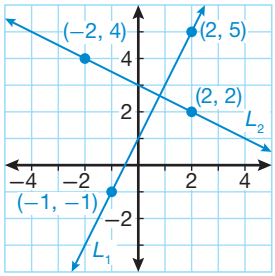
SOLUTION

Use the slope formula to find the slope of each line.

L_1	L_2
$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$	$m_2 = \frac{y_2 - y_1}{x_2 - x_1}$
$m_1 = \frac{4 - 1}{2 - 1}$	$m_2 = \frac{5 - (-1)}{1 - (-1)}$
$m_1 = 3$	$m_2 = 3$

Since the slopes are identical and the lines do not have the same y-intercepts, the lines are parallel.

b. Show that L_1 and L_2 are perpendicular.



SOLUTION

Use the slope formula to find the slope of each line.

L_1	L_2
$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$	$m_2 = \frac{y_2 - y_1}{x_2 - x_1}$
$m_1 = \frac{5 - (-1)}{2 - (-1)}$	$m_2 = \frac{4 - 2}{2 - 2}$
$m_1 = 2$	$m_2 = -0.5$

Since $2(-0.5) = -1$, the lines are perpendicular.

Math Reasoning

Generalize For a linear equation in standard form, $Ax + By = C$, what is the slope-intercept form in terms of the coefficients A, B, and C?

Example 2 Writing Equations of Parallel and Perpendicular Lines

a. Find the equation of the line parallel to $y = 3x + 4$ that crosses $(2, 3)$.

SOLUTION

Step 1:

The equation of the line parallel to $y = 3x + 4$ has the same slope: $m = 3$

Step 2:

Use the point-slope formula to find the equation of the line.

$$m = \frac{y - y_1}{x - x_1}$$

$$3 = \frac{y - 3}{x - 2}$$ Substitute the coordinate point $(2, 3)$ for (x_1, y_1) and 3 for the slope m.

$$3(x - 2) = y - 3$$

$$y = 3x - 6 + 3$$ Solve for y by multiplying each side by $(x - 2)$, then adding 3 to both sides.

$$y = 3x - 3$$

b. Find the equation of the line that passes through the point $(-1, 3)$ and is perpendicular to the line $y = -0.5x - 1$.

SOLUTION

Step 1:

Find the slope that is the opposite reciprocal of -0.5.

$$m = -\left(-\frac{1}{0.5}\right) = 2$$

Step 2:

Use the point-slope formula to find the equation of the line.

$$m = \frac{y - y_1}{x - x_1}$$

$$2 = \frac{y - 3}{x - (-1)}$$ Substitute the coordinate point $(-1, 3)$ for (x_1, y_1) and 2 for the slope m.

$$2\big(x - (-1)\big) = y - 3$$

$$y = 2x + 2 + 3$$ Solve for y by multiplying each side by $\big(x - (-1)\big)$, then adding 3 to both sides.

$$y = 2x + 5$$

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Example 3 Determining if Two Lines Are Perpendicular or Parallel

a. Given two equations, determine if they are perpendicular or parallel.

$$5x + 3y = 27 \qquad 15x + 9y = -72$$

SOLUTION

Write each equation in slope-intercept form.

$$5x + 3y = 27 \qquad 15x + 9y = -72$$
$$3y = -5x + 27 \qquad 9y = -15x - 72$$
$$y = -\frac{5}{3}x + 9 \qquad y = -\frac{5}{3}x - 9$$

Because the slopes are equal, the lines are parallel.

b. The points A, B, and C are collinear. The points D, E, and F are also collinear. Compare the equations of the lines through those sets of points.

	A	*B*	*C*	*D*	*E*	*F*
x	1	3	4	2	4	6
y	0.5	5.5	8	0.7	−0.1	−0.9

Math Reasoning

Formulate Use the point-slope formula to find the equations of the two lines in Example 3. Use the coordinates for C and F.

SOLUTION

Step 1:

Use the slope formula with points A and B and D and E.

$$m_1 = \frac{0.5 - 5.5}{1 - 3} \qquad m_2 = \frac{0.7 - (-0.1)}{2 - 4}$$
$$m_1 = \frac{-5}{-2} \qquad m_2 = \frac{0.8}{-2}$$
$$m_1 = \frac{5}{2} \qquad m_2 = -0.4$$

Step 2:

To test if the lines are perpendicular, multiply the two slopes.

$$m_1 m_2 = \left(\frac{5}{2}\right)(-0.4)$$
$$= -\frac{2}{2}$$
$$= -1$$

Because the product of the slopes is -1, the lines are perpendicular.

Reading Math

$L_1 \parallel L_2$ is read "L_1 is parallel to L_2".

$L_1 \perp L_2$ is read "L_1 is perpendicular to L_2".

Example 4 Finding the Equations of Horizontal and Vertical Lines

a. Find the equation of the line perpendicular to the graph of $y = 2$ that passes through the point $(4, -1)$.

SOLUTION

The graph of $y = 2$ is a horizontal line. The equation of a line perpendicular to $y = 2$ is a vertical line of the form $x = c$ for some constant c.

Use the x-coordinate of the point $(4, -1)$ to find the equation.

The equation $x = 4$ is perpendicular to the graph of $y = 2$ and passes through the point $(4, -1)$.

b. Find the equation of the line perpendicular to the graph of $x = 15$ that passes through the point $(7, -1)$.

SOLUTION

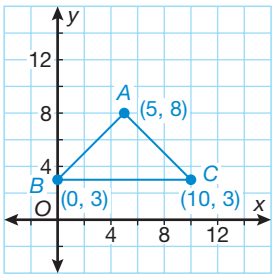
The graph of $x = 15$ is a vertical line. The equation of a line perpendicular to $x = 15$ is a horizontal line of the form $y = c$ for some constant c.

Use the y-coordinate of the point $(7, -1)$ to find the equation.

The equation $y = -1$ is perpendicular to the graph of $x = 15$ and passes through the point $(7, -1)$.

Example 5 Application: Right Triangles

Geometry Use what you have learned about the properties of perpendicular lines to prove that ΔABC is a right triangle.



SOLUTION

If ΔABC is a right triangle, then $\angle BAC$ is a right angle and $\overline{AB} \perp \overline{AC}$. Use the slope formula with points A and B and with A and C to see if $\overline{AB}$ and $\overline{AC}$ are perpendicular.

$$m\overline{AB} = \frac{8 - 3}{5 - 0} \qquad m\overline{AC} = \frac{8 - 3}{5 - 10}$$

$$m\overline{AB} = \frac{5}{5} \qquad m\overline{AC} = \frac{5}{-5}$$

$$m\overline{AB} = 1 \qquad m\overline{AC} = -1$$

Since the product of the slopes is -1, then $\angle BAC$ is a right angle. ΔABC is a right triangle.

Lesson Practice

a. (Ex 1) The points (1, −4) and (2, −1) are collinear. The points (3, 19) and (5, 25) are also collinear. What can you say about the lines that cross these points?

b. (Ex 1) The points (1, 16.88) and (3, 16.63) are collinear. The points (2, 24) and (4, 40) are also collinear. What can you say about the lines that cross these points?

c. (Ex 2) Write the equation of the line that is parallel to the graph of $y = -13x + 10$ and crosses the point (0, 15).

d. (Ex 2) Write the equation of the line that is perpendicular to the graph of $y = 25x + 99$ and crosses the point (1, 37).

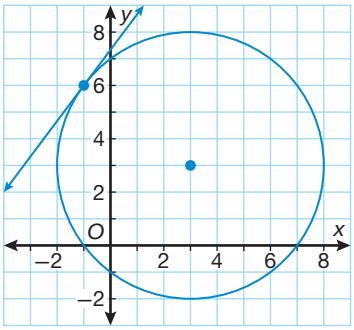
e. (Ex 3) What is the relationship between the graphs of $y = -27x + 15$ and $y = -27x + 5$.

f. (Ex 3) What is the relationship between the graphs of $y = -0.125x + 10$ and $y = 8x + 1$.

g. (Ex 4) Find the equation of the line perpendicular to the graph of $y = 5$ that passes through the point (21, 1).

h. (Ex 4) Find the equation of the line perpendicular to the graph of $x = 7$ that passes through the point (−3, 5).

i. (Ex 5) The graph of $y = \frac{4}{3}x + \frac{22}{3}$ is tangent to the circle at (−1, 6). The center of the circle is at (3, 3). Use what you know about perpendicular lines to show that the tangent is perpendicular to the diameter at that point.



Practice Distributed and Integrated

Use the slope formula to find the slope of each line.

***1.** (26) (−3, 7)(1, 0)

***2.** (26) (2, −5)(0, −6)

Determine the kind of variation, if any, for each equation.

3. (12) $y = mx + b$, for $m, b \neq 0$

4. (12) $y = mx + b$, for $m \neq 0$ and any value of b

Simplify. Write answers with all exponential expressions in the numerator.

5. (3) $\dfrac{xx^2(x^0y^{-1})^2}{x^2x^{-5}(y^2)^5}$

6. (3) $\dfrac{m^2p^0(m^{-2}p)^2}{m^{-2}p^{-1}(m^{-3}p^2)^3}$

Solve.

7. (27) $x^2 + 2 = 4x + 7$

8. (24) $\begin{aligned} 12x - 5y &= 24 \\ 20y - 4x &= 3 \end{aligned}$

9. (26) Write the equation in standard form: $y = \frac{1}{11}x + \frac{1}{7}$.

10. (28) **Analyze** Explain how to create a rational expression with four excluded values. Then give an example.

11. (23) **Error Analysis** Carmen factored the polynomial $x^2 + 5x - 6$ into $(x - 5)(x - 1)$. Describe her error.

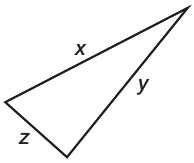
12. (22) **Verify** that the graph $y = 3x - 6$ is a continuous function.

***13.** (14) **Graphing Calculator** Calculate the value of x in the matrix below. Substitute the value of x and use a graphing calculator to verify your answer. $\begin{vmatrix} -4 & 3 \\ 5 & x \end{vmatrix} = -x$

14. (31) **Multiply** Assume that all expressions are defined. $\frac{x}{15} \cdot \frac{x^7}{2x} \cdot \frac{20}{x^4}$

***15.** (35) **Olympic Swimming Pool Size** An Olympic swimming pool has an area of 1250 square meters, and its length is 25 meters more than its width. Write and solve a quadratic equation to find the length and width of an Olympic swimming pool.

16. (32) **Geometry** In the triangle, x is 5 greater than y, and y is 21 less than 3 times z, and the perimeter is 180. What is the value of each variable? To find the answer, write a system of equations and use an inverse matrix to solve it.



17. (29) **Data Analysis** The table represents the number of ice creams sold at an ice cream shop for the months of January (01), May (05), and October (10).

Month	Sales
01	124
05	4,476
10	3,796

Let x be the month and y be the sales. Use the data in the table to create a system of three equations to find a quadratic equation of the form $y = ax^2 + bx + c$ that models the data.

***18.** (34) **Cab Fares** The normal fares for taxicab rides in New York City in the year 2006 satisfied a linear function. The table shows the fares for rides of several different distances.

Number of Miles (x)	0.2	0.4	1	5
Fare in Dollars (y)	2.90	3.30	4.50	12.50

Graph the function, state the rate of change, and describe what it means.

19. (Inv 2) A helicopter takes off with a horizontal speed of 5 ft/s and a vertical speed of 20 ft/s. Write parametric equations to show the motion of the helicopter.

*20. (36) **Error Analysis** Two students are finding the equation of the line parallel to $y = 5x - 3$ that crosses $(1, 4)$ but get different results. Which student made the mistake? Explain.

Student A	Student B
$5 = \frac{y-4}{x-1}$	$5 = \frac{y-1}{x-4}$
$5x - 5 = y - 4$	$5x - 20 = y - 1$
$y = 5x - 1$	$y = 5x - 19$

21. (25) **Multiple Choice** Which data set gives a median of 8 and a mode of 4?

A {4, 5, 6, 8, 8, 9, 12} **B** {4, 4, 8, 9, 10}

C {2, 3, 4, 8, 8, 9} **D** {2, 3, 4, 4, 8, 8}

*22. (36) **Write** Is the triangle defined by vertices $(-4, 6)$, $(4, 4)$, and $(1, -8)$ a right triangle? How do you know?

23. (30) **Small Business** The profits, for x weeks, in one year for a small business can be modeled by the function $y = 0.45(x - 22)^2 - 840.50$. Does the graph of the function have a maximum or a minimum? What is that value? During what week did it occur?

*24. (35) Find the roots of $x^2 + 7x + 6 = 0$.

25. (27) **Justify** In $f(x) = ax^2 + bx + c$, why is a the only constant that is not allowed to be zero?

*26. (33) **Multiple Choice** There are 8 runners in a race. How many ways can runners finish in first, second, and third place?

A 21 **B** 24 **C** 336 **D** 512

27. (14) **Multi-Step** The matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ represents a parallelogram, with vertices at $(0, 0)$, (a, b), (c, d), and $(a + c, b + d)$.

a. Graph this parallelogram in the first quadrant, putting (a, b) clockwise from (c, d).

b. To find the area of this parallelogram, draw a rectangle around it and subtract off the areas of triangles and rectangles.

c. Show that the area of the parallelogram equals the determinant of the matrix.

*28. (33) **Women's Basketball** There are 12 teams in the Atlantic Coast Conference of college women's basketball. How many ways can teams finish in first, second, and third place?

29. (30) **Write** Describe how symmetry can be used to graph $y = (x - 2)^2 + 2$.

30. (31) **Multiple Choice** What is the value of $\frac{x^2 + x}{x^2 - 9} \cdot \frac{3x - x^2}{3x + 3}$ if $x = 6$?

A $-\frac{4}{3}$ **B** $-\frac{4}{9}$ **C** $\frac{4}{9}$ **D** $\frac{4}{3}$

LESSON 37

Adding and Subtracting Rational Expressions

Warm Up

1. (1) **Vocabulary** In the expression $\frac{a}{b}$, if a and b are integers and $b \neq 0$, then $\frac{a}{b}$ is a _____ number.

2. (1) Add the fractions. $\frac{2}{7} + \frac{4}{5}$

3. (23) Factor the expression. $12x^2 - 28x + 15$

New Concepts

Both rational numbers and rational expressions are expressed as ratios.

Rational Number	Rational Expression
$\frac{a}{b}$, for integers a and b, $b \neq 0$	$\frac{f(x)}{g(x)}$, for polynomials $f(x)$ and $g(x)$, $g(x) \neq 0$, $g(x)$ of degree ≥ 1

Recall that a rational expression is undefined for values of the variable that cause the denominator to equal zero. In this lesson assume that all rational expressions are defined unless otherwise indicated.

Example 1 Combining Expressions with Like Denominators

a. Simplify the following expression.

$$\frac{1}{2x} + \frac{3}{2x}$$

SOLUTION

When two rational expressions have the same denominator, add the numerators.

$$\frac{1}{2x} + \frac{3}{2x} = \frac{4}{2x} = \frac{2}{x}$$

b. Simplify the following expression.

$$\frac{x}{x^2 - 2x + 1} - \frac{1}{x^2 - 2x + 1}$$

SOLUTION

When two rational expressions have the same denominator, subtract the numerators.

$$\frac{x}{x^2 - 2x + 1} - \frac{1}{x^2 - 2x + 1} = \frac{x - 1}{x^2 - 2x + 1}$$

Note that the denominator can be factored. $\frac{x - 1}{x^2 - 2x + 1} = \frac{x - 1}{(x - 1)^2}$

Both the numerator and the denominator have an $(x - 1)$ term in common, which can be further simplified. $\frac{x - 1}{(x - 1)^2} = \frac{1}{x - 1}$

Math Reasoning

Write If $f(x)$ is a polynomial of degree 1, and not equal to zero, explain why $\frac{1}{f(x)}$ is a rational expression.

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c. Simplify the following expression.

$$\frac{2x}{x^2-4x+3}-\frac{x+2}{x^2-4x+3}+\frac{-1}{x^2-4x+3}$$

SOLUTION

When two or more rational expressions have the same denominator, add or subtract the numerators. With binomial expressions in the numerator, distribute the sign to both terms.

$$\frac{2x}{x^2-4x+3}-\frac{x+2}{x^2-4x+3}+\frac{-1}{x^2-4x+3}=\frac{2x-x-2-1}{x^2-4x+3}$$

$$=\frac{x-3}{x^2-4x+3}$$

Note that the denominator can be factored. $\frac{x-3}{x^2-4x+3}=\frac{x-3}{(x-1)(x-3)}$

Both the numerator and the denominator have an $(x-3)$ term in common, which can be further simplified. $\frac{x-3}{(x-1)(x-3)}=\frac{1}{x-1}$

Example 2 Finding the LCD of Two Rational Expressions

a. Find the least common denominator of the two rational expressions.

$$\frac{1}{2x} \qquad \frac{1}{7x}$$

Math Language

The **least common denominator** is the least common multiple of the denominators of the fractions.

SOLUTION

Finding the LCD of the two rational expressions, in this case, means finding the LCM of $2x$ and $7x$.

$2x$ $4x$ $6x$ $8x$ $10x$ $12x$ $14x$

$7x$ $14x$

The LCD is $14x$, resulting in the rational expressions $\frac{7}{14x}$ and $\frac{2}{14x}$.

b. Find the least common denominator of the two rational expressions.

$$\frac{1}{x^2-4} \qquad \frac{1}{x^2+5x+6}$$

SOLUTION

With more complicated rational expressions, follow these steps.

Step 1: Write each term in factored form.

$$\frac{1}{(x-2)(x+2)} \qquad \frac{1}{(x+2)(x+3)}$$

Step 2: Multiply by the *least* number of factors they do not have in common:

$$\frac{(x+3)}{(x-2)(x+2)(x+3)} \qquad \frac{(x-2)}{(x-2)(x+2)(x+3)}$$

Example 3 Undefined Values of a Rational Expression

Find any values of x for which the following expression is undefined:

$$\frac{1}{2x^2 + 4x - 6}$$

SOLUTION

A rational expression $\frac{f(x)}{g(x)}$ is undefined when $g(x) = 0$. Follow these steps to find these undefined values:

Step 1: Write each term in factored form.

$$\frac{1}{(2x - 2)(x + 3)}$$

Step 2: Determine the values where each factor is zero.

$2x - 2 = 0$ $\qquad$ $x + 3 = 0$

$x = 1$ $\qquad$ $x = -3$

So, the rational expression is undefined for $x = 1$ and $x = 3$.

Math Reasoning

Analyze Suppose that $\frac{1}{ax^2 + bx + c}$ has no restrictions on x. What can you can conclude about this expression?

Example 4 Combining Expressions with Unlike Denominators

a. Add the two rational expressions.

$$\frac{1}{3x^3} + \frac{1}{6x^2}$$

SOLUTION

Find the least common denominator by finding the LCM.

$$\frac{1 \cdot 2}{3x^3 \cdot 2} + \frac{1 \cdot x}{6x^2 \cdot x}$$

$$= \frac{2 + x}{6x^3}$$

b. Subtract the two rational expressions.

$$\frac{1}{x^2 - 1} - \frac{1}{x^2 + 3x + 2}$$

SOLUTION

Factor each denominator to find the LCM.

$$\frac{1}{(x - 1)(x + 1)} - \frac{1}{(x + 1)(x + 2)}$$

$$= \frac{(x + 2)}{(x - 1)(x + 1)(x + 2)} - \frac{(x - 1)}{(x + 1)(x + 2)(x - 1)}$$

$$= \frac{3}{(x - 1)(x + 1)(x + 2)}$$

Example 5 Application: Travel Time

On a roundtrip by car, it took 2 hours longer on the return trip than it did on the first leg of the trip. How much faster did the car go on the first part of the trip?

SOLUTION

Use the equation for speed:

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

First part of the trip	Return trip
$s_1 = \frac{d}{t}$	$s_2 = \frac{d}{t+2}$

$$s_1 - s_2 = \frac{d}{t} - \frac{d}{t+2}$$

$$= \frac{d(t+2)}{t(t+2)} - \frac{dt}{t(t+2)} = \frac{2d}{t(t+2)}$$

Math Language

The **equation for speed** is a variation of the equation for distance: $d = rt$, where r is the rate, t is the time, and d is the distance. Solving the equation for rate gives $r = d \div t$.

Lesson Practice

a. *(Ex 1)* Simplify the expression. $\frac{7}{5x^3} + \frac{3}{5x^3}$

b. *(Ex 1)* Simplify the expression. $\frac{x}{x^2-1} - \frac{1}{x^2-1}$

c. *(Ex 1)* Simplify. $\frac{4x}{3x^2+19x-14} + \frac{1}{3x^2+19x-14} - \frac{x+3}{3x^2+19x-14}$

d. *(Ex 2)* Find the LCD: $\frac{1}{6x^3}$ $\frac{1}{7x^2}$

e. *(Ex 2)* Find the LCD: $\frac{1}{7x^2+32x-15}$ $\frac{1}{2x^2+3x-35}$

f. *(Ex 3)* For what values of x is the denominator zero? $\frac{1}{7x^2-66x+27}$

g. *(Ex 4)* Add the two rational expressions. $\frac{1}{5x^4} + \frac{1}{7x^5}$

h. *(Ex 4)* Subtract the two rational expressions.

$$\frac{1}{8x^2-22x+15} - \frac{1}{2x^2+13x-24}$$

i. *(Ex 5)* On a canoe trip, it took 45 minutes longer going upstream than it did downstream. How much slower was the upstream part of the trip, in miles per hour?

Practice Distributed and Integrated

1. (24) Use elimination to solve the system $\begin{matrix} 5x + 2y = 70 \\ 3x - 2y = 10 \end{matrix}$.

2. (26) Find the equation of the line that has a slope of $-\frac{3}{8}$ and passes through (4, 4).

***3.** (37) Add: $\frac{k^2}{2p} + c - \frac{4}{p^2c}$.

***4.** (36) Find the equation of the line that passes through the point (3, 5) and is parallel to the line $y = \frac{1}{6}x - 2$.

5. (20) Expand: $\frac{x^2y^{-2}}{z^2}\left(\frac{z^2}{y^2\,(2x^{-2})^{-1}} - \frac{4x^2\,y^0}{z^{-2}}\right)$.

6. (9) Multiply: $\begin{bmatrix} 4 & -1 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

Factor completely.

7. (23) $x^2 + x - 6$

8. (23) $x^3 - 9x$

***9.** (37) Find the difference. $\dfrac{1}{1 + \dfrac{1}{x+1}} - \dfrac{1}{1 + \dfrac{1}{x-1}}$

10. (13) Given $f(x) = 2x - 7$ and $g(x) = -5x + 8$, graph $f(x)$ and $g(x)$ on the same grid.

11. (17) **Write** Determine the four resulting inequalities that are used to solve $|x| > |2 + 7x|$.

12. (27) **Multi-Step** **a.** Tell how to determine the values of a, b, and c for any quadratic function written in standard form.

b. Write a quadratic equation in standard form where $a = -1$, $b = -2$, and $c = 3$.

***13.** (37) **Aviation** Because of the jet stream, on a roundtrip flight from Washington, D.C. to California, the first part of the trip will take half an hour longer than the return trip. How much faster is the return trip?

14. (31) **Error Analysis** A student multiplied rational expressions as follows:

$$\frac{x^2 - 1}{x^2 + 2x + 1} \cdot \frac{x + 1}{x - 3} = \frac{-1}{2x + 1} \cdot \frac{1}{-3} = \frac{1}{3(2x + 1)}.$$

What is the error? Find the correct product.

15. (15) **Multiple Choice** Which is the solution to the linear system $\begin{matrix} y = x \\ y = 3x - 2 \end{matrix}$?

A (3, 3) **B** (2, 4) **C** (1, 1) **D** There is no solution.

16. *(24)* **Error Analysis** Find and correct the error a student made below.

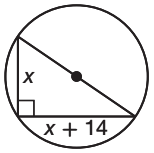
$$\begin{array}{l} 2x + 2y = 6 \\ -2x + y = -12 \end{array} \longrightarrow \begin{array}{r} 2x + 2y = 6 \\ +\ -2x + y = -12 \\ \hline 2y = -6 \\ y = -3 \end{array} \longrightarrow \begin{array}{r} -2x + y = -12 \\ -2x - 3 = -12 \\ -2x = -9 \\ x = 4.5 \end{array}$$

The solution is (4.5, –3).

***17.** *(36)* **Economics** The exchange rate from dollars to different currencies is shown in the table. A company charges a fee of $3 for amounts up to $50 and $5 for amounts above $50. Write the equations for exchanging Euros to dollars. What do the graphs of the equations have in common?

Currency	US $
1 euro =	1.377
1 Canadian dollar =	0.9472
1 British pound =	2.0284

18. *(35)* **Probability** The right triangle is inscribed in the circle. The area of the triangle is 120 square units. Suppose a point is chosen at random inside the circle. What is the probability that the chosen point is inside the triangle?



19. *(34)* **Multi-Step** **a.** Graph the equations $y = \frac{1}{2}x - 2$ and $y = \frac{1}{2}x + 3$.

b. What is the slope of each line?

c. State a property of lines and their slopes that is illustrated by the graphs.

***20.** *(37)* **Write** What must be true if the LCM of rational expressions $\frac{f(x)}{g(x)}$ and $\frac{h(x)}{i(x)}$ is $g(x)i(x)$?

***21.** *(14)* **Graphing Calculator** Use a graphing calculator to solve.

$$\det \begin{bmatrix} 2 & -3 & 5 \\ 2 & 0 & -1 \\ 2 & 3 & 1 \end{bmatrix}$$

22. *(33)* **Analyze** A card is chosen from a standard 52-card deck. Consider these four events: *Choose a red card. Choose a face card. Choose the jack of spades. Choose a black jack.* How many pairs of mutually exclusive events can be made from the events? List all the pairs.

23. *(19)* **Basketball** The length of a regulation high school basketball court is represented by the expression $(x + 24)$ feet. The width of a regulation high school basketball court is represented by the expression $(x - 10)$ feet.

a. Write an expression that represents the area of a regulation high school basketball court.

b. Evaluate the area of the court if $x = 60$.

***24.** *(36)* **Justify** Suppose that $y_1 = m_1x + b_1$ and $y_2 = m_2x + b_2$ are each perpendicular to $y_3 = m_3x + b_3$. Justify the statement $y_1 \parallel y_2$.

25. *(21)* **Verify** Solve the system of equations by solving the first equation for y then substituting. Verify your answer by solving the second equation for y, then substituting.

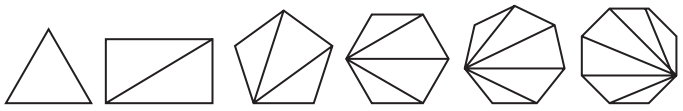
$2y - 3x = -10$

$4x = 7 - y$

26. *(32)* **Item Costs** An office worker ordered breakfast items for coworkers as shown below. Monday: 6 muffins, 6 bagels, and 6 fruit cups; total bill $24.00
Tuesday: 4 muffins, 12 bagels, and 4 fruit cups; total bill $22.80
Wednesday: 8 muffins, 12 bagels, and 10 fruit cups; total bill $39.30

What was the cost of each item? To find the answer, write a system of equations and use an inverse matrix to solve it.

27. *(22)* **Geometry** Examine the polygons given below. Each polygon has been divided up into triangles. Complete the table. Graph the function. State the domain and range. Determine if the graph is continuous, discontinuous, and/or discrete.



Number of sides of the polygon	Number of triangles

28. *(31)* **Multi-Step** Given the rational functions $f(x) = \frac{2x-1}{x+5}$ and $g(x) = \frac{14x-7}{14x^2+70x}$:

a. Find the function quotient $f(x) \div g(x)$ in simplest form.

b. Find the value of $f(x) \div g(x)$ if $x = 10$.

***29.** *(37)* What is the sum of $\frac{1}{x^2+x-72}$ and $\frac{1}{x^2-x-56}$?

A $\frac{2}{x^2+16x+63}$ **B** $\frac{2}{x^2+x-72}$

C $\frac{2}{x^2-x-56}$ **D** $\frac{2x+16}{x^3+8x^2-65x-504}$

30. *(16)* **Construction** A contractor is replacing the tile in two bathrooms. For one bathroom, she purchased 416 large tiles and 256 small tiles at a cost of $233.60. For the other bathroom, she purchased 400 large tiles and 512 small tiles at a cost of $251.20. How much does each size tile cost?

LESSON
38

Dividing Polynomials Using Long Division

Warm Up

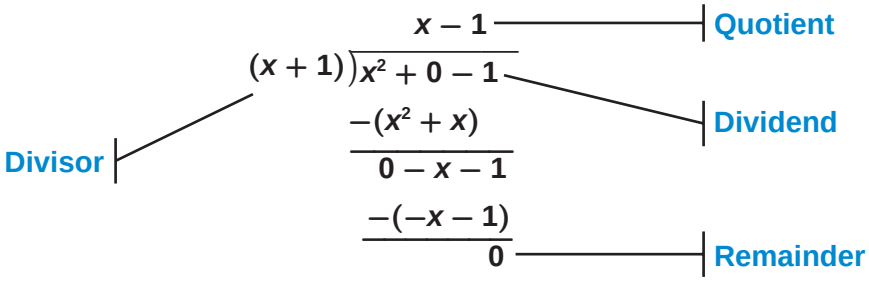
(10) **1.** **Vocabulary** An expression that contains the symbols $<$ or $>$ is called a(n) _____.

(39) **2.** Multiply: $(x + 3)(x + 7)$.

(4) **3.** Determine $f(-5)$ if $f(x) = 2x^2 - 3x$.

New Concepts

Polynomial long division is a similar process to integer long division: The quotient can have a remainder of zero or not.



Math Reasoning

Analyze Suppose m and n are two different integers. If m is a prime number and $n > 1$, how do you know that $\frac{m}{n}$ has a nonzero remainder?

Example 1 Dividing a Polynomial by a Monomial

Divide: $(12x^3 - 6x^2 + 5x + 4)$ by $3x$.

SOLUTION

Use polynomial long division.

$$
\begin{array}{r}
4x^2 - 2x + 1 \\
3x\overline{)\,12x^3 - 6x^2 + 5x + 4} \\
-(12x^3) \\
\hline
0 - 6x^2 + 5x + 4 \\
-(-6x^2) \\
\hline
0 + 5x + 4 \\
-(3x) \\
\hline
2x + 4
\end{array}
$$

The quotient and remainder can be written as:

$$(4x^2 - 2x + 1) + \frac{2x + 4}{3x}$$

Check

$$\left(4x^2 - 2x + 1 + \frac{2x + 4}{3x}\right)(3x) = (12x^3 - 6x^2 + 3x) + (2x + 4)$$

$$= 12x^3 - 6x^2 + 5x + 4 \quad \checkmark$$

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Example 2 Dividing a Polynomial by a Linear Polynomial

Divide: $(x^4 + 2x^3 - 13x^2 - 38x - 24)$ by $(x + 4)$.

Hint

Distribute the -1 to each of the numbers when subtracting the product of the quotient and the divisor.

SOLUTION

$$
\begin{array}{r}
x^3 - 2x^2 - 5x - 18 \\
(x+4)\overline{)\,x^4 + 2x^3 - 13x^2 - 38x - 24} \\
\underline{-(x^4 + 4x^3)} \\
0 - 2x^3 - 13x^2 - 38x - 24 \\
\underline{-(-2x^3 - 8x^2)} \\
0 - 5x^2 - 38x - 24 \\
\underline{-(-5x^2 - 20x)} \\
0 - 18x - 24 \\
\underline{-(-18x - 72)} \\
0 + 48
\end{array}
$$

The quotient and remainder are

$$(x^3 - 2x^2 - 5x - 18) + \frac{48}{x+4}$$

Check

$$\left(x^3 - 2x^2 - 5x - 18 + \frac{48}{x+4}\right)(x+4) = (x^4 + 2x^3 - 13x^2 - 38x - 72) + 48$$

$$= x^4 + 2x^3 - 13x^2 - 38x - 24 \quad \checkmark$$

Example 3 Dividing a Polynomial by a Lower-Degree Polynomial

Divide: $(4x^4 - x^3 - 11x - 484)$ by $(x^2 + 11)$.

SOLUTION

Even though there is no quadratic term in the dividend, include a place for it.

$$
\begin{array}{r}
4x^2 - x - 44 \\
(x^2+11)\overline{)\,4x^4 - x^3 + 0x^2 - 11x - 484} \\
\underline{-(4x^4 \quad + 44x^2)} \\
0 - x^3 - 44x^2 - 11x - 484 \\
\underline{-(-x^3 \quad - 11x)} \\
0 \quad - 44x^2 + 0 \quad - 484 \\
\underline{-(-44x^2 \quad - 484)} \\
0
\end{array}
$$

The quotient is $(4x^2 - x - 44)$. There is no remainder.

Example 4 Testing if One Polynomial Is a Factor of Another

a. Is $(x + 2)$ a factor of $2x^3 - x^2 - 7x + 6$?

SOLUTION

Use polynomial long division.

$$
\begin{array}{r}
2x^2 - 5x + 3 \\
(x + 2)\overline{)\,2x^3 - x^2 - 7x + 6} \\
-(2x^3 + 4x^2) \\
\hline
0 - 5x^2 - 7x + 6 \\
-(-5x^2 - 10x) \\
\hline
0 + 3x + 6 \\
-(3x + 6) \\
\hline
0
\end{array}
$$

Because the remainder is zero, $(x + 2)$ is a factor.

b. Is $(x + 3)$ a factor of $6x^3 - 6x^2 - 6x + 6$?

SOLUTION

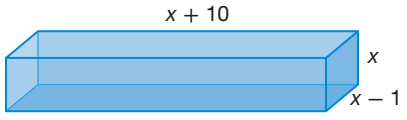
Use polynomial long division.

$$
\begin{array}{r}
6x^2 - 24x + 66 \\
(x + 3)\overline{)\,6x^3 - 6x^2 - 6x + 6} \\
-(6x^3 + 18x^2) \\
\hline
0 - 24x^2 - 6x + 6 \\
-(-24x^2 - 72x) \\
\hline
66x + 6 \\
(66x + 198) \\
\hline
204
\end{array}
$$

Because the remainder is not zero, $(x + 3)$ is not a factor.

Example 5 Application

What is the ratio of the volume to the surface area for the rectangular prism?



SOLUTION

$$\frac{\text{volume}}{\text{surface area}} = \frac{x(x + 10)(x - 1)}{2x(x + 10) + 2(x - 1)(x + 10) + 2x(x - 1)}$$

$$= \frac{x(x + 10)(x - 1)}{6x^2 + 36x - 20}$$

Since the numerator and denominator do not share any factors, the ratio has a nonzero remainder.

Math Reasoning

Verify Show that the ratio of the volume to the surface area has no common factors.

Lesson Practice

a. (Ex 1) Divide $12x^3 - 9x^2 - 237x + 40$ by $3x$.

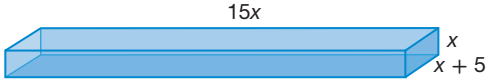
b. (Ex 2) Divide $6x^4 - 23x^3 - 34x^2 + 207x - 180$ by $x - 3$.

c. (Ex 3) Divide $2x^4 + 7x^3 - 241x^2 - 192x + 3024$ by $x^2 - 5x - 36$.

d. (Ex 4) Is $x - 6$ a factor of $2x^3 + x^2 - 177x + 594$?

e. (Ex 4) Is $x + 27$ a factor of $7x^3 + 178x^2 - 107x - 78$?

f. (Ex 5) What is the ratio of the volume to surface area in the figure below? Does this expression have a nonzero remainder?



Practice Distributed and Integrated

Use the descriptions below to write the vertex form of a quadratic equation.

1. (30) The parent function is shifted 10 units left.

2. (30) The parent function is shifted 1 unit right and 3 units down.

***3.** (38) Divide $x^3 - 10x^2 - 69x + 30$ by $(x + 5)$.

4. (32) Given $A = \begin{bmatrix} 4 & -2 \\ 8 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, solve $AX = B$.

A number cube is rolled. Determine if the following events are mutually exclusive.

5. (33) *Roll an odd number* or *roll a 6*

6. (33) *Roll an even number* or *roll a number less than 3*

***7.** (37) Find the difference. $\frac{1}{x+3} - \frac{1}{x+2} - \frac{1}{x+1}$

8. (16) Use Cramer's Rule to solve $\begin{aligned} 3x - 12y &= -15 \\ x + 2y &= 7 \end{aligned}$.

Find the determinants of the following matrices.

9. (14) What is $|A|$ if $A = \begin{bmatrix} 2 & -12 \\ 7 & 13 \end{bmatrix}$?

10. (14) What is $|B|$ if $B = \begin{bmatrix} -1 & -24 \\ 0.25 & 0.5 \end{bmatrix}$?

***11.** (38) What are the slope and y-intercept of $3x + 10y = -1$?

12. (12) **Geometry** As the diameter (d) of circle increases in size, the circumference (C) increases. Likewise, as the diameter decreases in size, so does the circumference. The constant of variation between C and d is π. Describe the kind of variation between the circumference and diameter in a circle. Write the equation.

*13. (35) **Write** State all the possible numbers of y-intercepts a quadratic function graph can have. Explain.

*14. (38) **Justify** Consider the polynomial $x^2 + ax + bx + ab$. What is the remainder when this polynomial is divided by $x + a$? How do you know?

15. (29) **Hourly Pay** An employee makes three different rates of pay based on the type of work. The rates of pay are \$16 per hour, \$24 per hour, and \$32 per hour. The combined number of hours worked at the rate of \$16 per hour and \$24 per hour was 10 times the hours worked at the rate of \$32 per hour. If in one week the employee earned \$808 for working 44 hours, how many hours were worked at each rate of pay?

*16. (38) **Error Analysis** Two students are dividing $-14x^3 + 46x^2 + 44x - 16$ by $x - 2$ but get different results. Which student made the mistake?

Student A

$$
\begin{array}{r}
-14x^2 + 18x + 80 \\
(x-2)\overline{)-14x^3 + 46x^2 + 44x - 16} \\
\underline{-(-14x^3 + 28x^2)} \\
0 + 18x^2 + 44x - 16 \\
\underline{-(18x^2 - 36x)} \\
0 + 80x - 16 \\
\underline{-(80x - 160)} \\
144
\end{array}
$$

Student B

$$
\begin{array}{r}
-14x^2 + 18x - 8 \\
(x-2)\overline{)-14x^3 + 46x^2 + 44x - 16} \\
\underline{-(-14x^3 + 28x^2)} \\
0 + 18x^2 + 44x - 16 \\
\underline{-(18x^2 - 36x)} \\
0 + 8x - 16 \\
\underline{-(8x - 16)} \\
0
\end{array}
$$

*17. (27) **Graphing Calculator** Use a graphing calculator to determine the minimum of the quadratic equation $f(x) = x^2 - 4x - 2$.

*18. (36) **Multiple Choice** Which is the equation of the line perpendicular to $y = 4x + 10$ that crosses the (1, 1)?

A $y = 4x + \frac{5}{4}$ **B** $y = -\frac{1}{4}x + \frac{5}{4}$ **C** $y = -4x + \frac{5}{4}$ **D** $y = \frac{1}{4}x + \frac{5}{4}$

19. (34) **Multi-Step** Graph the equations $y = \frac{3}{2}x - 1$ and $y = -\frac{2}{3}x + 2$ on the same coordinate system.

a. What are the slopes of the lines?

b. What is the product of the slopes?

c. State a property of lines and their slopes that is illustrated by the graphs.

20. (28) **Geography** Crater Lake in southern Oregon lies inside a volcanic basin and is nearly circular in shape. Write the area to circumference ratio of the lake as a simplified rational expression. Then find the ratio given that the radius of the lake is about 3 miles.

21. (33) **Error Analysis** A card is drawn at random from a standard 52-card deck. How many outcomes are in the event *draw an ace or a spade*? A student solved this problem by reasoning as follows: There are 4 aces and 13 spades, so there are $4 + 13 = 17$ outcomes in the event. What is the error? Find the correct answer.

22. (24) **Multi-Step** **a.** Rewrite the system $\begin{aligned} y &= 5 - 7x \\ 10 &= 14x + 2y \end{aligned}$ so that both equations are in standard form.

b. By inspection, what do you think the solution is? Why?

c. Solve and classify the system.

23. (32) **Analyze** Students at a high school were asked if they visited a museum, public library, or historical site during the last month. The table shows data about the results of the survey. How many students in each grade were surveyed? To find the answer, write a system of equations and use an inverse matrix to solve it.

	Ninth Graders	Tenth Graders	Eleventh Graders	Total
Museum	20%	30%	25%	143
Public Library	40%	50%	45%	258
Historical Site	20%	20%	35%	142

24. (30) **Generalize** Describe shifts, stretches, and compressions of parabolas and discuss how those changes appear in the equations of parabolas.

25. (23) **Write** Explain in words how the Zero Product Property is useful in solving polynomial equations.

***26.** (37) **Aviation** Because of a strong head wind the return trip on an airplane takes 1.5 hours longer than the first part of the trip. How much slower is the return trip?

27. (27) **Justify** Graph $f(x) = -x^2 + 3$. Tell why the range of the function is the set of real numbers less than or equal to 3.

***28.** (38) **Multiple Choice** Which of the following is a factor of $3x^3 + 8x^2 - 31x + 20$?

A $(x + 2)$ **B** $(3x + 4)$ **C** $(x - 1)$ **D** $(x + 6)$

29. (26) **Medicine** The Centers for Disease Control has developed growth charts that pediatricians use to see if a child's height is within a certain average. The chart below shows data for a boy's height in the 75th percentile. The three points are on a line. Use the data to write the equation of the line in slope-intercept form.

Age (years)	Height (inches)
3	38.5
4	41.5
5	44.5

***30.** (37) **Probability** Two six-sided number cubes have labels $\frac{1}{x}$, $\frac{1}{2x}$, $\frac{1}{3x}$, $\frac{1}{4x}$, $\frac{1}{5x}$, and $\frac{1}{6x}$. If both number cubes are rolled, what is the probability that the sum of the expressions is $\frac{1}{x}$?

LESSON
39

Graphing Linear Inequalities in Two Variables

Warm Up

1. (13) **Vocabulary** The equation $y = \frac{1}{4}x - 2$ is written in _____ form.
2. (13) Find the slope of the line that passes through (9, 3) and (−7, 5).
3. (13) The slope of a vertical line is _____.
4. (13) True or False: The y-intercept of $2x - y = -4$ is 4.

New Concepts

When two expressions have the same value, they can be joined by an equal sign. If two expressions do not have the same value, they can be joined by one of the inequality signs shown below.

Less than	Greater than	Less than or equal to	Greater than or equal to
$<$	$>$	$\leq$	$\geq$

A **linear inequality in two variables** relates two variables, often x and y, with an inequality sign. A solution (x, y) makes the inequality true when the values of x and y are substituted into the inequality.

Example 1 Determining if an Ordered Pair is a Solution of a Linear Inequality

Determine if each point is a solution of the inequality.

a. $y > -3x + 5, (2, 9)$

SOLUTION

$9 \stackrel{?}{>} -3(2) + 5$

$9 > -1$ True; (2, 9) is a solution.

b. $3y - x < -18, (-3, -7)$

SOLUTION

$3(-7) - (-3) \stackrel{?}{<} -18$

$-18 < -18$ False; (−3, −7) is not a solution.

c. $-2y \leq 1 - x, (-3, 4)$

SOLUTION

$-2(4) \stackrel{?}{\leq} 1 - (-3)$

$-8 \leq 4$ True; (−3, 4) is a solution.

Caution

A number can not be less than (or greater than) itself. In Example 2b, the point would be a solution if the inequality symbol were ≤ rather than <.

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Exploration Finding Solutions of Linear Inequalities on the Coordinate Plane

Graph $y = \frac{4}{3}x - 2$. Then graph each of the following points and tell if the point is *above the line, on the line,* or *below the line.*

$(6, 2), (-3, -6), (1, 4), (-5, -5), (0, -2), (4, -1)$

Which of the points satisfies $y < \frac{4}{3}x - 2$? How are these solutions related to the graph?

Which satisfy $y \geq \frac{4}{3}x - 2$? How are these solutions related to the graph?

Reading Math

A dashed line indicates that the points on the line are not solutions of the inequality. A solid line indicates that they are solutions.

The graph of a linear inequality in two variables is the set of all points that satisfy the inequality. To graph a linear inequality, first graph the related linear equation (just suppose the inequality sign is an equal sign). Make the line dashed when the symbol is $<$ or $>$ and solid when the symbol is $\leq$ or $\geq$. This line is the **boundary line.** It separates the plane into two **half-planes.** Shade the half-plane that includes the solutions of the inequality.

To determine which half-plane contains the solutions of the inequality, use a test point. If the test point satisfies the inequality, then all the points in the half-plane that contains the test point will also satisfy the inequality. The test point should not be a point on the boundary line. The point $(0, 0)$ makes a good test point when it is not on the boundary.

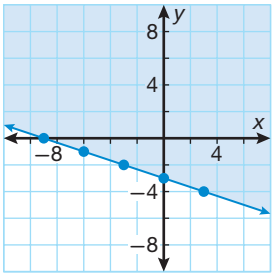
Example 2 Using a Table of Values to Graph a Linear Inequality in Two Variables

Graph $3y + x \geq -9$ by making a table of values.

SOLUTION

Step 1: Find ordered pairs that satisfy the equation $3y + x = -9$. Use the x- and y-intercepts as well as other points.

x	0	−9	−6	−3	3
y	−3	0	−1	−2	−4



Step 2: Connect the points with a solid line.

Step 3: Use the test point $(0, 0)$ to determine which half-plane to shade.

$$3y + x \geq -9$$

$$3(0) + 0 \stackrel{?}{\geq} -9$$

$$0 \geq -9 \quad \text{True}$$

The point $(0, 0)$ is a solution. Shade the half-plane that contains $(0, 0)$.

When the inequality is written in slope-intercept form, you can choose which half-plane to shade by looking at the direction of the inequality sign. Shade above the line for $>$ and $\geq$, and below the line for $<$ and $\leq$.

Example 3 Using Slope-Intercept Form to Graph a Linear Inequality in Two Variables

a. Graph $5y + x < 20$ using slope-intercept form.

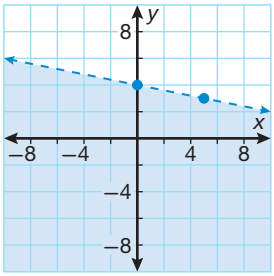
SOLUTION

Step 1: Write the inequality in slope-intercept form.

$$5y + x < 20$$
$$5y < -x + 20$$
$$y < -\frac{1}{5}x + 4$$

Step 2: Graph $y = -\frac{1}{5}x + 4$ by plotting $(0, 4)$ and a second point by moving 1 unit down and 5 units right. Connect the points with a dashed line to indicate that the points on the line are not solutions.

Step 3: Shade below the line because the inequality symbol is $<$.



The test point $(0, 0)$ verifies this.

$$5y + x < 20$$
$$5(0) + 0 \stackrel{?}{<} 20$$
$$0 < 20 \quad \text{True}$$

b. Graph $-2y + 10x < -2$ using slope-intercept form.

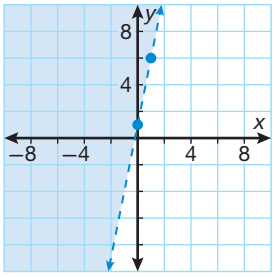
SOLUTION

Step 1: Write the inequality in slope-intercept form. Don't forget to switch the direction of the inequality sign when multiplying or dividing by a negative number.

$$-2y + 10x < -2$$
$$-2y < -10x - 2$$
$$y > 5x + 1$$

Step 2: Plot $(0, 1)$ and find a second point by moving 5 units up and 1 unit right. Connect the points with a dashed line.

Step 3: Shade above the line because the inequality symbol is $>$ in slope-intercept form.



The test point $(0, 0)$ verifies this.

$$-2y + 10x < -2$$
$$-2(0) + 10(0) \stackrel{?}{<} -2$$
$$0 < -2 \quad \text{False}$$

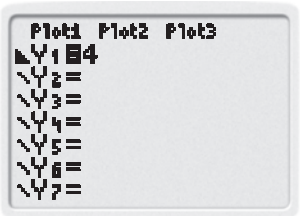
Hint

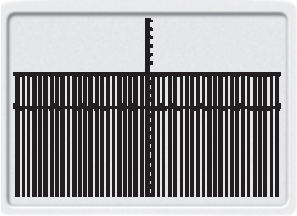
Look at the *y*-axis to help decide which half-plane is above the line.

Example 4 Using Technology to Graph a Linear Inequality in Two Variables

a. Graph $y \leq 4$ on a graphing calculator.

SOLUTION In the **[Y=]** editor, after y, enter 4. To indicate shading below the line, use the arrow key to highlight the symbol to the left of Y_1. Press enter until the symbol shows the lower left portion of the square shaded, then graph.

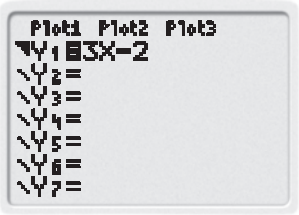


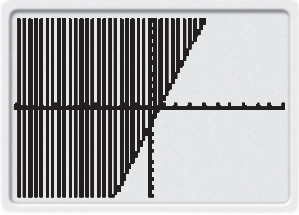


b. Graph $3x - y \leq 2$ on a graphing calculator.

SOLUTION First write the inequality in slope-intercept form.

$$3x - y \leq 2$$
$$-y \leq -3x + 2$$
$$y \geq 3x - 2$$





Example 5 Application: Food Preparation

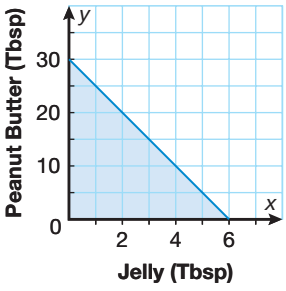
Nellie is making a peanut butter and jelly snack for a school function. Each tablespoon of jelly has 15 grams of carbohydrates and each tablespoon of peanut butter has 3 grams of carbohydrates. She wants the snack to have no more than 90 grams of carbohydrates. Graph the inequality that represents the possible amounts of each ingredient.

SOLUTION Let x represent the tablespoons of jelly and y the tablespoons of peanut butter. The phrase *no more than* indicates *less than or equal to*. Then the inequality is $15x + 3y \leq 90$.

Write $15x + 3y \leq 90$ in slope-intercept form.

$$15x + 3y \leq 90$$
$$3y \leq -15x + 90$$
$$y \leq -5x + 30$$

Graph $y = -5x + 30$ with a solid line. Shade below the line.



The inequality could also be graphed by plotting the points that contain the intercepts (0, 30) and (6, 0). The test point (0, 0) makes the original inequality true and should be in the solution set.

Math Reasoning

Analyze Do all the solutions make sense in context of the application? Explain.

Lesson Practice

Determine if each point is a solution of the inequality.

a. (Ex 1) $y \geq 7 - 4x$, $(-2, 1)$

b. (Ex 1) $-2y + x < 0$, $(0, 0)$

c. (Ex 1) $4y - 6x > 15$, $(4, 13)$

d. (Ex 2) Graph $2y - 6x \leq -12$ by making a table of values.

e. (Ex 3) Graph $4y + 4 > x$ using slope-intercept form.

f. (Ex 3) Graph $4 - 2y > -x$ using slope-intercept form.

g. (Ex 4) Graph $y \geq 0$ on a graphing calculator.

h. (Ex 4) Graph $3y + 9 \geq 3x$ on a graphing calculator.

i. (Ex 5) A school chef is making a vegetable side dish for lunch. Each cup of beans has 35 grams of protein and each cup of peas has 48 grams of protein. He wants the dish to have more than 840 grams of protein. Graph the inequality that represents this.

Practice Distributed and Integrated

Determine if the given points are solutions of the inequalities.

***1.** (39) $y < 6x - 2$, $(3, 18)$

***2.** (39) $y + 2x \geq -4$, $(-1, -2)$

3. (20) Add: $m + \frac{x}{c} + \frac{c}{x^2b}$

4. (29) Solve.
$$\begin{aligned} -8x - 22 + 2y &= 2z \\ -6x + 2y &= 16 + z \\ 6 + 3y + z &= -x \end{aligned}$$

Solve by substitution:

5. (21)
$$\begin{aligned} x &= y + 1 \\ 3x + 2y &= 8 \end{aligned}$$

6. (21)
$$\begin{aligned} 3x - y &= 22 \\ 2x + 3y &= -11 \end{aligned}$$

7. (21)
$$\begin{aligned} -7x + y &= 8 \\ 5x - y &= -6 \end{aligned}$$

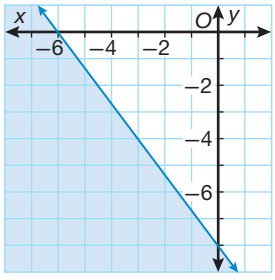
8. (38) Divide $6x^3 + 23x^2 - 20x - 9$ by $3x + 1$.

9. (28) Find the excluded values of $\frac{x^2 - 25}{3x^2 + 15x}$.

10. (23) Factor the polynomial $2x^4 + 4x^3 - 4x$.

11. (28) **Verify** Show why the x's cannot be divided out in $\frac{x + 2}{x + 5}$.

***12.** (39) **Multiple Choice** The graph of which inequality is shown?



A $-3y - 4x \geq 24$

B $-3y - 4x \leq 24$

C $-3y - 4x > 24$

D $-3y - 4x < 24$

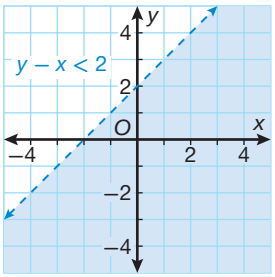
***13.** (37) **Statistics** The fraction of students that are interested in a new brand of sneakers is twice the fraction of those that are not. The fraction of students who are not interested is $\frac{1}{x}$. Those who expressed no opinion in the sneakers were 10% of the students who were interested. What percent of students were interested?

14. (16) **Baseball Equipment** Last month, a baseball coach bought 4 bats and 9 balls for $355. This month, the coach bought 2 bats and a dozen balls for $290. The prices did not change from last month to this month. How much did each bat and ball cost?

a. Set up equations relating the monetary values of the bats and balls last month and this month. Let T represent the cost (in dollars) per bat, and B represent the cost per ball.

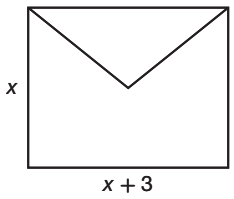
b. Use Cramer's Rule to solve the system of equations.

***15.** (39) **Verify** Using the graph, choose a point in the shaded region and test it. Choose a point in the unshaded region and test it. Is the graph correct? Explain.



***16.** (22) **Graphing Calculator** Graph $y = 4$ on your graphing calculator. Determine if this relation is a function. Determine if the graph is continuous, discontinuous, and/or discrete.

***17.** (35) **U.S. Postal Service** The United States Postal Service uses size, shape, and weight to determine pricing for first class mail. One category of first class mail is large envelope. The maximum length of a large envelope is 3 inches greater than its maximum width, and its maximum area is 180 square inches. Write and solve a quadratic equation to find the maximum length and maximum width of a large envelope.



18. (23) **Multi-Step** Use $x^2 + 8x - 48$.

a. Find the factors of -48.

b. Which of the factors have a sum of 8?

c. What is the factored form of the polynomial?

19. (34) **Multiple Choice** If the graph of $y = x$ is reflected over the x-axis, and then the resulting graph is translated 4 units up, which equation is represented by the final graph?

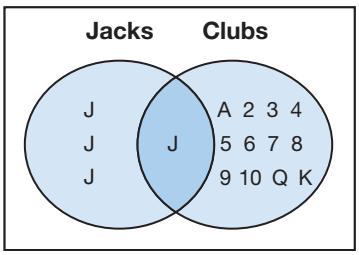
A $y = -\frac{1}{4}x$ **B** $y = -4x$ **C** $y = x - 4$ **D** $y = -x + 4$

20. (21) **Multi-Step** A store sells granola and dried fruit by the pound. Dillon bought 4 pounds of granola and 2 pounds of dried fruit for $22. Carmen bought 2 pounds of granola and 4 pounds of dried fruit for $26.

a. Write a system of equations that represents the price of the granola, g, and the price of the dried fruit, f.

b. How much should one pound of granola and one pound of dried fruit cost individually and together?

***21.** (33) **Model** The figure at the right is a Venn diagram. It shows that jacks and clubs are not mutually exclusive events when choosing a card from a standard 52-card deck. Draw a Venn diagram to show that kings and hearts are not mutually exclusive events when choosing a card from a standard 52-card deck.



22. (32) **Analyze** What system of equations is represented by the matrix equation shown below? Solve the system mentally.

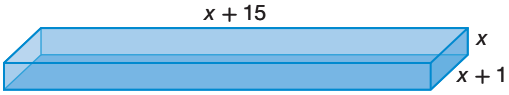
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 50 \end{bmatrix}$$

23. (28) **Error Analysis** A student divided rational expressions as follows: $\frac{20x^2}{y} \div \frac{y}{x^3} = \frac{20x^{\cancel{2}}}{\cancel{y}} \div \frac{\cancel{y}}{x \cdot x^{\cancel{2}}} = \frac{20}{x}$. What error did the student make? Find the correct quotient.

***24.** (38) **Model** Show that the ratio of an odd number to an even number, when each is expressed as a polynomial, has a nonzero remainder.

25. (30) **Verify** Suppose a rocket is fired straight into the air such that the graph of the function representing the height of the rocket after x seconds has its vertex at (3.75, 225) and passes through the point (5, 200). Verify that the value of a in the equation written in vertex form is -16.

*** 26.** (38) **Geometry** What is the ratio of the volume to the surface area for the rectangular prism shown? Does the ratio have a remainder?



27. (29) **Investing** A person invests $50,000 for one year; some is invested at 7%, some at 8%, and the remainder at 12%. The combined interest earned at the end of the year from these investments was $4,770. The amount invested at 8% is $4,000 more then the amount invested at 7% and 12% combined. Find the amount of money invested at each rate.

***28.** (39) **Money** Nelson is building a tower of coins with half-dollars and dimes. He wants the tower to be less than 15 centimeters high. Each half dollar is 2.15 mm thick and each dime is 1.35 mm thick. Write an inequality that represents how many of each coin are needed. Then show how to use the inequality to see if using 30 half-dollars and 70 dimes is a possibility.

29. (27) **Business** Mr. Jing makes and sells carved animals. His monthly profit function can be modeled by $f(x) = -0.15x^2 + 7x$ where x represents the number of carved animals he sells each month.

a. For how many animals does he make the maximum profit? How much profit is it?

b. At what point would Mr. Jing begin to lose money?

***30.** (36) Write the equation of the line in slope intercept form that is parallel to $y = 5x - 3$ and goes through (1, 4).

LESSON
40

Simplifying Radical Expressions

Warm Up

1. (25) **Vocabulary** $\sqrt{16} = 4$. 4 is the _____ of 16.
2. (19) Multiply $(2x - 5)(2x + 5)$.
3. (2) Combine like terms: $3x^2 + 5x^3 - 8x^3 + x^2 - 5x^2$
4. (3) Simplify $7^3 \cdot 7^2$
5. (3) Simplify $(3^2)^3$

New Concepts

Because $4^2 = 16$, 4 is a **square root** of 16. However, $(-4)^2 = 16$, so -4 is also a square root of 16. To indicate that the positive, or **principal,** square root is desired, the **radical symbol** is used. Therefore, $\sqrt{16}$ equals 4 and only 4. The expression under a radical sign is the **radicand.** $\sqrt{16}$ is a **radical,** or radical expression, and 16 is the radicand.

Math Reasoning

Analyze Why is -2 not a cube root of 8?

There are roots other than square roots. For example, because $2^3 = 8$, 2 is a cube root of 8. Because $3^4 = 81$ and $(-3)^4 = 81$, -3 and 3 are fourth roots of 81.

***n* Roots**
Given that a and b are real numbers and n is an integer greater than 1, if $a^n = b$, then a is an nth root of b.
Given that a and b are not negative and n is an even integer greater than 1, or given that a and b are real numbers and n is an odd integer greater than 1, if $a^n = b$, then $\sqrt[n]{b} = a$.

For $\sqrt[n]{b}$, n is the **index.** When n is not shown, it is assumed to be 2.

Example 1 Simplifying Radicals

Simplify each expression.

a. $\sqrt[5]{-32}$

SOLUTION $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

b. $\sqrt[3]{27}$

SOLUTION $\sqrt[3]{27} = 3$ because $3^3 = 27$.

c. $\sqrt[4]{16}$

SOLUTION $\sqrt[4]{16} = 2$ because $2^4 = 16$. Although -2 is a fourth root of 16, the radical sign indicates that only the positive root is to be given.

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The Product Rule for Radicals
Given that a and b are real numbers and n is an integer greater than 1, $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

The product rule states that $\sqrt{10}$ can be written as $\sqrt{2} \cdot \sqrt{5}$. This rule is helpful when one of the radicals in the product can be simplified. For instance, $\sqrt{20} = \sqrt{4} \cdot \sqrt{5}$. Now, $\sqrt{4}$ can be simplified: $\sqrt{20} = 2\sqrt{5}$.

Types of Real Roots
If n is even, $\sqrt[a]{x^n} = x^{\frac{n}{a}}$.
If n is odd, $\sqrt[a]{x^n} = x^{\frac{n-1}{a}} \cdot \sqrt{x}$

Example 2 Using the Product Rule for Radicands

Simplify each expression.

a. $\sqrt{8} \cdot \sqrt{5}$

SOLUTION Either multiply and then simplify, or simplify and then multiply.

Option 1: $\sqrt{8} \cdot \sqrt{5} = \sqrt{40} = \sqrt{4} \cdot \sqrt{10} = 2\sqrt{10}$

Option 2: $\sqrt{8} \cdot \sqrt{5} = \sqrt{4} \cdot \sqrt{2} \cdot \sqrt{5} = 2\sqrt{2} \cdot \sqrt{5} = 2\sqrt{10}$

Caution

Simplifying before multiplying does not always mean that the product will not need to be simplified further.

b. $\sqrt[3]{54}$

SOLUTION

$\sqrt[3]{54} = \sqrt[3]{27} \cdot \sqrt[3]{2}$ Find a factor of 54 that has a cube root.

$= 3\sqrt[3]{2}$ The cube root of 27 is 3.

c. $\sqrt{567}$

SOLUTION

$$\begin{aligned}\sqrt{567} &= \sqrt{9 \cdot 63} \\ &= 3\sqrt{9 \cdot 7} \\ &= 3 \cdot 3\sqrt{7} \\ &= 9\sqrt{7}\end{aligned}$$

When radical expressions have the same radicand and root, they are **like radicals** and can be combined. This is analogous to combining like terms.

Combine like terms: $2x + 3x = 5x$

Combine like radicals: $2\sqrt{x} + 3\sqrt{x} = 5\sqrt{x}$

Example 3 Combining Radical Expressions

Simplify each expression.

a. $12\sqrt[3]{8} - \sqrt[3]{8} + 5\sqrt{8} + 9\sqrt{8}$

SOLUTION

$12\sqrt[3]{8} - \sqrt[3]{8} + 5\sqrt{8} + 9\sqrt{8}$

$11\sqrt[3]{8} + 14\sqrt{8}$ Combine like radicals.

$11 \cdot 2 + 14\sqrt{4} \cdot \sqrt{2}$ $\sqrt[3]{8} = 2$ and $\sqrt{8} = \sqrt{4}\sqrt{2}$.

$22 + 28\sqrt{2}$

b. $\sqrt{242} + \sqrt{72} - \sqrt{48}$

SOLUTION Simplify each term to identify any like terms.

$\sqrt{242} + \sqrt{72} - \sqrt{48}$

$\sqrt{121} \cdot \sqrt{2} + \sqrt{36} \cdot \sqrt{2} - \sqrt{16} \cdot \sqrt{3}$ Use the Product Rule.

$11\sqrt{2} + 6\sqrt{2} - 4\sqrt{3}$ Simplify.

$17\sqrt{2} - 4\sqrt{3}$ Combine like terms.

c. $\sqrt[4]{32} + \sqrt[4]{2} + \sqrt[4]{162}$

SOLUTION

$\sqrt[4]{32} + \sqrt[4]{2} + \sqrt[4]{162}$

$\sqrt[4]{16} \cdot \sqrt[4]{2} + \sqrt[4]{2} + \sqrt[4]{81} \cdot \sqrt[4]{2}$ Use the Product Rule.

$2\sqrt[4]{2} + \sqrt[4]{2} + 3\sqrt[4]{2}$ Take the fourth roots of 16 and 81.

$6\sqrt[4]{2}$ Combine like terms.

Hint

Make a list of square, cube, and fourth roots to look at while simplifying radicals.

All the concepts in this lesson can be extended to variables. Keep in mind, however, that for a positive root to be taken, a variable must represent a positive number. For example, the product rule can be used to expand a variable representing positive values as shown below:

$$\sqrt{x^6} = \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{x^2} = x \cdot x \cdot x = x^3$$

Example 4 Simplifying Radicals with Variables

Simplify each expression. All variables represent non-negative real numbers.

a. $\sqrt{x^4}$

SOLUTION $\sqrt{x^4} = x^2$ because $(x^2)^2 = x^4$.

b. $\sqrt{18h^{17}}$

SOLUTION

$$\sqrt{18h^{17}} = \sqrt{9} \cdot \sqrt{2} \cdot h^{\frac{17-1}{2}} \cdot \sqrt{h}$$
$$= 3 \cdot \sqrt{2} \cdot h^8 \cdot \sqrt{h}$$
$$= 3h^8\sqrt{2h}$$

Hint

With conjugates, the middle two terms of the product will always be opposites.

c. $\sqrt[4]{48x} + \sqrt[4]{3x} + \sqrt[4]{243x}$

SOLUTION Simplify each term to identify any like terms.

$\sqrt[4]{48x} + \sqrt[4]{3x} + \sqrt[4]{243x}$

$\sqrt[4]{16} \cdot \sqrt[4]{3x} + \sqrt[4]{3x} + \sqrt[4]{81} \cdot \sqrt[4]{3x}$ Use the Product Rule.

$2\sqrt[4]{3x} + \sqrt[4]{3x} + 3\sqrt[4]{3x}$ Take the fourth roots of 16 and 81.

$6\sqrt[4]{3x}$ Combine like terms.

d. $\sqrt[4]{16x^2} + \sqrt{4x} + \sqrt[3]{27}$

SOLUTION

$\sqrt[4]{16x^2} + \sqrt{4x} + \sqrt[3]{27}$

$\sqrt[4]{16} \cdot \sqrt[4]{x^2} + \sqrt{4} \cdot \sqrt{x} + 3$ Use the Product Rule.

$2\sqrt[4]{x^2} + 2\sqrt{x} + 3$ Take the indicated roots of 16 and 4.

$2x^{\frac{2}{4}} + 2\sqrt{x} + 3$ Take the 4th root of x^2.

$2\sqrt{x} + 2\sqrt{x} + 3$ Simplify

$4\sqrt{x} + 3$ Combine like terms.

Example 5 Application: Physics

The pendulum in the Pantheon in Paris makes one complete swing in $\sqrt{\frac{110\pi^2}{4}}$ seconds. Simplify this expression.

The length of a pendulum is given by the equation $l = \frac{8s^2}{\pi^2}$, where l is the length in feet and s is the time in seconds the pendulum takes to make a complete swing. Using your simplified expression from the first part of the problem, write an expression to find the length of the pendulum and simplify it to find the length of the pendulum.

SOLUTION Simplify the expression using the product rule for radicals.

$\sqrt{\frac{110\pi^2}{4}} = \sqrt{\pi^2} \cdot \sqrt{\frac{1}{4}} \cdot \sqrt{110}$ Use the Product Rule.

$\pi \cdot \frac{1}{2} \cdot \sqrt{110}$ Take the indicated roots.

$\frac{\pi\sqrt{110}}{2}$ Simplify.

Now, plug this expression into the formula given for the length of a pendulum.

$l = \frac{8s^2}{\pi^2}$

$l = \frac{8\left(\frac{\pi\sqrt{110}}{2}\right)^2}{\pi^2} = \frac{8\left(\frac{\pi^2 110}{4}\right)}{\pi^2} = \frac{2\pi^2 110}{\pi^2} = 2 \cdot 110 = 220$

So the pendulum is 220 feet long.

Lesson Practice

Simplify each expression. Assume all variables represent non-negative real numbers.

a. (Ex 1) $\sqrt[3]{-125}$

b. (Ex 1) $-\sqrt{9}$

c. (Ex 1) $\sqrt[5]{-1}$

d. (Ex 2) $\sqrt{45} \cdot \sqrt{2}$ **e.** (Ex 2) $\sqrt[5]{64}$ **f.** (Ex 3) $\sqrt{50} - \sqrt{32}$

g. (Ex 4) $\sqrt{98w^{10}}$ **h.** (Ex 4) $-\sqrt[4]{y} + 2\sqrt{y} - 7\sqrt[4]{y} - \sqrt{y}$

i. (Ex 5) Refer to the formula given in example 5. If a pendulum makes one complete swing in $\sqrt{2\pi^4} \cdot \sqrt[4]{2}$ seconds, write a simplified expression for the length of the pendulum.

Practice Distributed and Integrated

***1.** (39) Make a table of values to satisfy the inequality $2x - 4y \geq 12$.

Write the equation in standard form for the line that crosses the two points.

2. (26) $(4, 5.5), (-1, 2.5)$

3. (26) $(3, 5), (6, 4)$

Multiply then simplify.

***4.** (40) $\sqrt{30} \cdot \sqrt{6}$

***5.** (40) $\sqrt{180} \cdot \sqrt{10}$

Factor:

6. (23) $5x^2y^2 - 2xy + 10xy^2$

7. (23) $x^2y^3m^5 + 12x^3ym^4 - 3x^2y^2m^2$

8. (20) Given the rational functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{x^2}{x+4}$, find the value of $f(x) \cdot g(x)$ if $x = -2$.

Solve:

9. (35) $-x + x^2 = 12$

10. (35) $-48x = -2x^2 - x^3$

***11.** (40) **Multi-Step** **a.** Expand the expression as much as possible. Do not simplify: $\sqrt{x^8}$

b. Explain how can you rewrite your answer from part a so the expression can be easily simplified.

c. Simplify the expression as much as possible. The variable represents non-negative real numbers.

12. (28) **Error Analysis** Explain and correct the error a student made below.

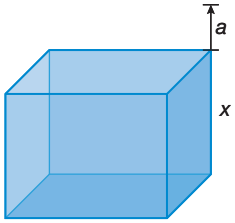
$$\frac{x^2}{x^5} = x^{5-2} = x^3$$

13. *(32)* **Multiple Choice** If $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 5 & 0 \end{bmatrix}$, what is the solution to the matrix equation $AX = B$?

A $X = \begin{bmatrix} 19 & -8 \\ -13 & 6 \end{bmatrix}$ **B** $X = \begin{bmatrix} 11 & -8 \\ -13 & 6 \end{bmatrix}$

C $X = \begin{bmatrix} 19 & 8 \\ -13 & 6 \end{bmatrix}$ **D** $X = \begin{bmatrix} 19 & -8 \\ 13 & 6 \end{bmatrix}$

***14.** *(38)* **Chemistry** Water has a unique property that causes it to increase in volume as it changes from liquid to ice. In the illustration, water just fills a cube-shaped container. The water is frozen, and increased volume of ice changes in one direction. What is the ratio of the new volume to the original volume?



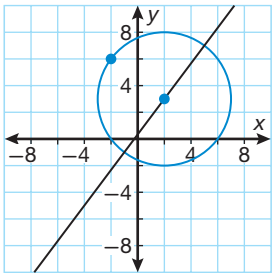
15. *(37)* **Multiple Choice** Subtract. $\dfrac{2}{x^2 - 4x - 45} - \dfrac{1}{x^2 + 9x + 20}$

A $\dfrac{x + 17}{x^2 - 61x - 180}$ **B** $\dfrac{x + 17}{x^3 - 61x - 180}$ **C** $\dfrac{1}{x^3 - 61x - 180}$ **D** $\dfrac{1}{x^2 - 61x - 180}$

16. *(25)* **Error Analysis** A chemist weighs samples obtained from a production run. The weights of the samples are 12 g, 14 g, 65 g, 11 g, 15 g, 12 g, 13 g, 15 g, 14 g, and 11 g. She eliminated the outlier and calculated the mean of the data. What is her error?

$$\overline{x} = \frac{12 + 14 + 11 + 15 + 12 + 13 + 15 + 14 + 11}{10} = \frac{117}{10} = 11.7$$

17. *(36)* **Geometry** For the circle shown, the line $y_1 = \frac{4}{3}x + \frac{1}{3}$ passes through the center of the circle at (2, 3). How do you know that the equation of the tangent at (−2, 6) is parallel to y_1?



18. *(23)* **Predict** Use your knowledge of the factoring formulas to find the product of $(5x^6 + 3)$ and $(5x^6 - 3)$ without multiplying the polynomials. Explain your prediction.

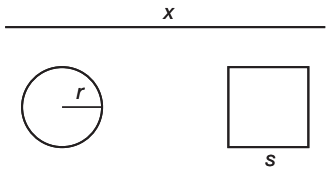
19. *(34)* **Analyze** Does the table of values represent a linear function? Explain your answer without graphing.

x	−4	0	2	8
y	11	5	2	−7

20. *(33)* **Sports** There are 4 teams in the National Football Conference East division. How many ways can the teams finish in first, second, third, and fourth place?

21. *(39)* **Write** Describe how to graph $y - 1 \leq 5x$ on a graphing calculator.

22. *(31)* **Multi-Step** Consider this question: Which is the larger area that a piece of rope of length x will enclose, a circle or a square?



a. Write a simplified expression for the area of the circle in terms of x.

b. Write a simplified expression for the area of the square in terms of x.

c. Which area is larger? Find the ratio of the areas.

d. Express the larger area as a percent of the smaller area. Round to the nearest percent.

23. *(35)* **Write** For the function $f(x) = x^2 + 12x + 36$, describe the relationship among the zero(s) of $f(x)$, the x-intercept(s) of $f(x)$, and the root(s) of $f(x) = 0$.

***24.** *(27)* **Graphing Calculator** Determine the vertex of the function $f(x) = x^2 - 6x + 1$ using a graphing calculator.

***25.** *(40)* **Biology** The metabolism rate of an 18 pound animal is approximately $73\sqrt[4]{625}$ Calories per day. Simplify the metabolism rate.

26. *(24)* **Work** In one month, Angela worked a total of 20 hours, some of it spent babysitting and the rest spent washing cars. She earned a total of $96.25. If she gets paid $5.50 per hour for babysitting and $2.75 per hour for washing cars, how many hours of each type of work did she do?

27. *(39)* **Sports** In American football, a touchdown with a conversion is worth 7 points and a field goal is worth 3 points. Suppose the only scoring in a game was due to touchdowns with conversions and field goals. Write the inequality that represents how many of each could have occurred if the total number of points earned in the game was less than or equal to 105. Give a solution that would be on the boundary line if the inequality were graphed.

28. *(21)* **Coordinate Geometry** You are given equations of three lines on a coordinate grid that intersect to form a triangle. Explain in words how you could find the vertices of the triangle.

***29.** *(40)* **Verify** Supply two examples to show that b and the value of $\sqrt[n]{b}$ can be negative when n is odd.

***30.** *(26)* **Aviation** The Federal Aviation Administration tracks airline delays for airports around the country. From the beginning of the year to the end of March, there were 6,000 delays at Atlanta-Hartsfield Airport. By the end of May, there were 10,000 cumulative delays. Use this information to write an equation in slope-intercept form that could be used to predict the total delays throughout the year. Let x equal the month of the year as a number and y equal the cumulative delays.

INVESTIGATION

4

Understanding Cryptography

Computer Internet security is a top priority for governments and private companies. For information to be secure, it needs to be encrypted when sent out over the Web and decrypted when received. This way only the sender and receiver know the contents of the message. In the area of cryptography, different methods of keeping messages secure have been developed.

One of the simplest ways of encrypting a message is to use a substitution cipher. A number replaces each letter of the alphabet, so that all messages consist of a string of numbers. Without the cipher, it is difficult to decipher the message.

Use a spreadsheet to create a substitution cipher.

1. Use the cipher in the table below and transfer it to a spreadsheet. Input the letters A–Z in cells A1 to A26, and input the numeric cipher in B1 to B26.

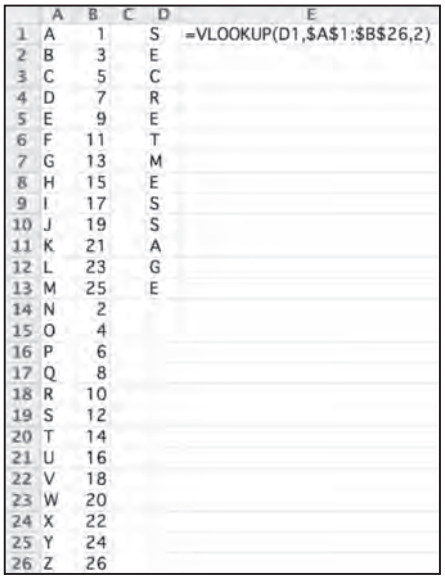
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	3	5	7	9	11	13	15	17	19	21	23	25	2	4	6	8	10	12	14	16	18	20	22	24	26

Math Reasoning

Generalize Suppose you wanted the cipher to include word spaces and special characters. How would you change the cipher?

The cipher is the key to encrypting and decrypting messages. A spreadsheet automates the process of using such a cipher.

2. Input the letters of the words SECRET MESSAGE in cells D1 through D13. The space between the words is not encrypted.
3. The VLOOKUP formula searches for a value in the leftmost column of the table (A) and then returns a value in the same row from a column you specify (B). Input this formula into cell E1: **=VLOOKUP(D1,$A$1:$B$26,2).** Then copy and paste the formula into cells E2 to E13.



	A	B	C	D	E
1	A	1		S	=VLOOKUP(D1,$A$1:$B$26,2)
2	B	3		E	
3	C	5		C	
4	D	7		R	
5	E	9		E	
6	F	11		T	
7	G	13		M	
8	H	15		E	
9	I	17		S	
10	J	19		S	
11	K	21		A	
12	L	23		G	
13	M	25		E	
14	N	2			
15	O	4			
16	P	6			
17	Q	8			
18	R	10			
19	S	12			
20	T	14			
21	U	16			
22	V	18			
23	W	20			
24	X	22			
25	Y	24			
26	Z	26			

4. Write the encrypted message as a string of numbers.

5. Use the spreadsheet to encrypt the following messages. List the message as a string of numbers. If necessary, copy and paste the spreadsheet formula to more cells.

   a. QUADRATIC
   b. ALGEBRA

The spreadsheet you created can be used to encrypt messages. A variation of this spreadsheet can be used to decrypt messages. Suppose your teacher has sent you the following message using the same encryption cipher.

8 16 17 26 14 15 17 12 11 10 17 7 1 24

You need to create a decryption spreadsheet.

6. **Multi-Step** Create a new spreadsheet.

   a. Input the numeric cipher *in ascending order* to cells A1 to A26, and input the alphabet to B1 to B26. The letters will no longer be in alphabetical order.
   b. Input the encrypted message above to cells D1 to D14.
   c. Input this formula into cell E1: **=VLOOKUP(D1,$A$1:$B$26,2).** Then copy and paste the formula into cells E2 to E14.

	A	B	C	D	E
1	1	A		8	=VLOOKUP(D1,$A$1:$B$26,2)
2	2	N		16	
3	3	B		17	
4	4	O		26	
5	5	C		14	
6	6	P		15	
7	7	D		17	
8	8	Q		12	
9	9	E		11	
10	10	R		10	
11	11	F		17	
12	12	S		7	
13	13	G		1	
14	14	T		24	
15	15	H			
16	16	U			
17	17	I			
18	18	V			
19	19	J			
20	20	W			
21	21	K			
22	22	X			
23	23	L			
24	24	Y			
25	25	M			
26	26	Z			

   d. What is your teacher's message?

An additional level of security can be added to the substitution cipher. This involves using matrix multiplication.

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7. Encrypt the following message using the cipher you previously produced.
TOP SECRET MESSAGE

8. Write the string of numbers in two 3 × 3 cipher matrices. The first number goes in the first row first column, the second number goes in the second row first column. The first matrix is started. Complete the two matrices.

$$\begin{bmatrix} 14 & 12 & _ \\ 4 & _ & _ \\ 6 & _ & _ \end{bmatrix} \begin{bmatrix} 25 & _ & _ \\ 9 & _ & _ \\ _ & _ & _ \end{bmatrix}$$

9. **Graphing Calculator** Multiply the cipher matrices by the encoding matrix shown below.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & -5 \\ 2 & -3 & 2 \end{bmatrix}$$

10. Write the newly encrypted message as a string of numbers.

11. **Analyze** Why is this encryption method more secure than a substitution cipher?

Decrypting a message using the cipher-matrix system requires that you know both the encrypted and the encoding matrices. Suppose your teacher has a message encoded in the encrypted matrices:

$$\begin{bmatrix} 38 & -50 & -8 \\ 39 & -51 & -71 \\ 19 & -20 & -9 \end{bmatrix} \begin{bmatrix} 23 & -28 & -24 \\ 41 & -57 & -10 \\ 37 & -47 & 9 \end{bmatrix} \begin{bmatrix} 9 & -9 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

12. **Graphing Calculator** Follow these steps to decrypt the message.

a. Store the encoding matrix in your calculator as matrix [A].
b. Store the encrypted matrices as separate matrices in the calculator.
c. Multiply each encrypted matrix by the inverse of the encoding matrix.
d. Write the partially decrypted message as a string of numbers.
e. Use the decryption spreadsheet to decipher the message. What is the message?

Investigation Practice

Graphing Calculator **Use the cipher-matrix system to decrypt these messages. Use the encoding matrix from problem 9.**

a.
$$\begin{bmatrix} 16 & -17 & -30 \\ 25 & -30 & -45 \\ 25 & -29 & -18 \end{bmatrix} \begin{bmatrix} 31 & -43 & -87 \\ 11 & -12 & -25 \\ 33 & -46 & -20 \end{bmatrix} \begin{bmatrix} 9 & -9 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 38 & -50 & 22 \\ 55 & -75 & 65 \\ 17 & -21 & -34 \end{bmatrix} \begin{bmatrix} 38 & -52 & -12 \\ 27 & -37 & -91 \\ 51 & -68 & 63 \end{bmatrix} \begin{bmatrix} 22 & -22 & 44 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

LESSON
41

Using the Pythagorean Theorem and the Distance Formula

Warm Up

1. (40) **Vocabulary** $\sqrt{75}$ is a _____ expression. (*rational*/*radical*).
2. (40) Simplify $\sqrt{300}$.
3. (19) Expand $(x - 5)^2$.
4. (35) Solve $x^2 + 7x = -10$ by factoring.

New Concepts

A **right triangle** is a triangle with one right angle. The sides that form the right angle are the **legs** of a right triangle and the side opposite the right angle is the **hypotenuse.**

Exploration **Visualizing the Pythagorean Theorem**

Materials needed: grid paper, scissors

Step 1:
On grid paper, draw a right triangle so that the base is 3 squares long and the height is 4 squares high.

Step 2:
Elsewhere on the paper, draw a 3 by 3 square, a 4 by 4 square, and a 5 by 5 square. Cut out these three squares.

Step 3:
Place the squares so that one side of each square aligns with one of the sides of the triangle.

Step 4:
Move the 4 by 4 square onto the 5 by 5 square. Cut the 3 by 3 square into pieces to completely fill in the rest of the 5 by 5 square. The two smaller squares should completely cover the larger square. That is, the sum of the areas of the smaller squares equals the area of the larger square.

Because the area of a square is found by squaring the length of one side of the square, the result of the exploration suggests the Pythagorean Theorem. The converse of the Pythagorean Theorem is also true.

The Pythagorean Theorem
If a triangle is a right triangle, the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse. For legs a and b, and hypotenuse c, $a^2 + b^2 = c^2$.

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The Converse of the Pythagorean Theorem
If the sum of the squares of the lengths of the two shorter sides of a triangle equals the square of the length of the longest side, then the triangle is a right triangle.

Use the Converse of the Pythagorean Theorem when checking if three sides of a triangle are the sides of a *right* triangle.

Example 1 Determining if a Triangle is a Right Triangle

a. The sides of a triangle measure 15 cm, 36 cm, and 39 cm. Is the triangle a right triangle?

SOLUTION Substitute 15 and 36 for a and b, and 39 for c, in $a^2 + b^2 = c^2$.

$$a^2 + b^2 = c^2$$
$$15^2 + 36^2 \stackrel{?}{=} 39^2$$
$$225 + 1296 \stackrel{?}{=} 1521$$
$$1521 = 1521 \quad ✓$$

The triangle is a right triangle.

b. The sides of a triangle measure 21 in., 70 in., and 75 in. Is the triangle a right triangle?

SOLUTION Substitute 21 and 70 for a and b, and 75 for c, in $a^2 + b^2 = c^2$.

$$a^2 + b^2 = c^2$$
$$21^2 + 70^2 \stackrel{?}{=} 75^2$$
$$441 + 4900 \stackrel{?}{=} 5625$$
$$5341 \neq 5625$$

The triangle is not a right triangle.

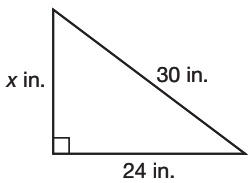
Math Reasoning

Justify Why could 36 be substituted for a and 15 be substituted for b instead?

Use the Pythagorean Theorem when determining the length of any side of a right triangle, given the lengths of the other two sides.

Example 2 Finding the Length of a Missing Side of a Right Triangle

a. Find the value of x.



SOLUTION

$a^2 + b^2 = c^2$

$x^2 + 24^2 = 30^2$ Substitute x and 24 for the legs, a and b.

$x^2 + 576 = 900$ Square 24 and 30.

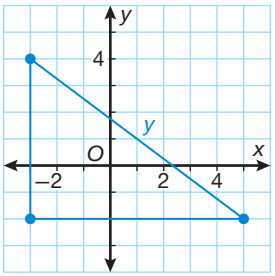
$x^2 = 324$ Subtract 576 from each side.

$x = 18$ Take the square root of both sides.

Hint

Use the fact that the hypotenuse is the longest side of a right triangle to check if your answer is reasonable.

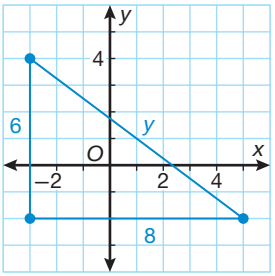
b. Find the value of y.



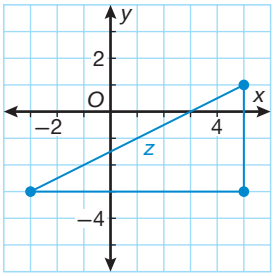
SOLUTION Count the number of units to find the lengths of the legs.

The vertical segment is 6 units long. The horizontal segment is 8 units long.

$$a^2 + b^2 = c^2$$
$$6^2 + 8^2 = y^2$$
$$36 + 64 = y^2$$
$$100 = y^2$$
$$10 = y$$



c. Find the value of z. Give the answer in simplest radical form and to the nearest tenth.

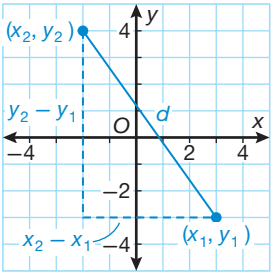


SOLUTION The vertical segment is 4 units long. The horizontal segment is 8 units long.

$$a^2 + b^2 = c^2$$
$$4^2 + 8^2 = z^2$$
$$16 + 64 = z^2$$
$$80 = z^2$$
$$\sqrt{80} = z$$
$$\sqrt{80} = \sqrt{16} \cdot \sqrt{5} = 4\sqrt{5}$$

The value of z is exactly $4\sqrt{5}$ units, or about 8.9 units, long.

The distance between any two points can be found by using the Pythagorean Theorem. As Examples 2b and 2c show, the horizontal distance is the difference between the x-coordinates and the vertical distance is the difference between the y-coordinates. Because the length of the hypotenuse is the square root of the sum of the squares of the leg lengths, the distance between any two points can be represented by the **distance formula** shown below.



Distance Formula
The distance d between any two points with coordinates (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 3 Finding the Distance between Two Points

Find the distance between each pair of points. Give the answer in simplest radical form and to the nearest tenth.

a. $A(-4, 11)$, $B(1, -1)$

SOLUTION Label $(-4, 11)$ as (x_1, y_1) and $(1, -1)$ as (x_2, y_2).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(1 - (-4))^2 + (-1 - 11)^2}$$
$$= \sqrt{(5)^2 + (-12)^2} = \sqrt{25 + 144}$$
$$= \sqrt{169} = 13$$

Math Reasoning

Justify Can $(1, -1)$ be labeled as (x_1, y_1) and $(-4, 11)$ as (x_2, y_2)? Explain.

b. $A(-6, 5)$, $B(-2, -7)$

SOLUTION Label $(-6, 5)$ as (x_1, y_1) and $(-2, -7)$ as (x_2, y_2).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-2 - (-6))^2 + (-7 - 5)^2}$$
$$= \sqrt{(4)^2 + (-12)^2} = \sqrt{16 + 144} = \sqrt{160} = \sqrt{16}\sqrt{10} = 4\sqrt{10} \approx 12.6$$

Example 4 Finding a Missing Coordinate

Find the value of k so that the distance between $(2, k)$ and $(-2, 8)$ is 4.

SOLUTION

Substitute values into $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and simplify.

$$4 = \sqrt{(-2 - 2)^2 + (8 - k)^2}$$
$$= \sqrt{(-4)^2 + 8^2 - 16k + k^2} = \sqrt{16 + 64 - 16k + k^2} = \sqrt{80 - 16k + k^2}$$

Solve $4 = \sqrt{80 - 16k + k^2}$ by factoring.

$16 = 80 - 16k + k^2$	Square both sides of the equation.
$0 = k^2 - 16k + 64$	Make one side 0. Rearrange terms.
$0 = (k - 8)(k - 8)$	Factor the trinomial.

Solve $k - 8 = 0$ to find that $k = 8$.

Example 5 Application: Cartography

A cartographer constructs a map of New Mexico over a grid system such that Silver City is at (63, 142) and Las Cruces is at (192, 98). Find the distance between the cities to the nearest kilometer if each unit represents one kilometer.

SOLUTION

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(192 - 63)^2 + (98 - 142)^2}$$
$$= \sqrt{(129)^2 + (-44)^2} = \sqrt{16{,}641 + 1936} \approx 136$$

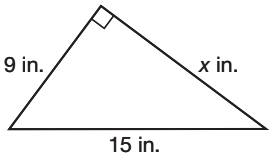
The distance between Silver City and Las Cruces is about 136 kilometers.

Lesson Practice

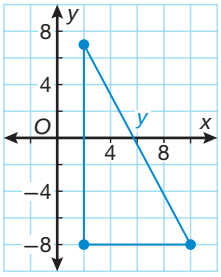
a. (Ex 1) The sides of a triangle measure 14 m, 20 m, and 25 m. Is the triangle a right triangle?

b. (Ex 1) The sides of a triangle measure 12 mm, 16 mm, and 20 mm. Is the triangle a right triangle?

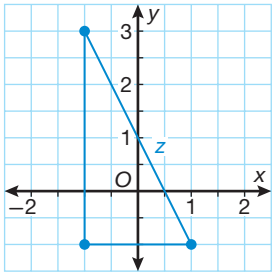
c. (Ex 2) Find the value of x.



d. (Ex 2) Find the value of y.



e. (Ex 2) Find the value of z. Give the answer in simplest radical form and to the nearest tenth.



Find the distance between each pair of points. Give the answer in simplest radical form and to the nearest tenth. (Ex 3)

f. $A(6, 9)$, $B(-18, -1)$

g. $A(5, -5)$, $B(9, -11)$

h. (Ex 4) Find two possible values of k so that the distance between $(1, k)$ and $(6, -3)$ is 13.

i. (Ex 5) A cartographer constructs a map of New Mexico over a grid system such that Silver City is at (200, 205) and Los Alamos is at (225, 287). Find the distance between the cities to the nearest kilometer if each unit represents one kilometer.

Practice Distributed and Integrated

Find the distance between the points.

***1.** (41) $(-3, 8)$ and $(3, 0)$

***2.** (41) $(-1, -1)$ and $(13, 6)$

3. (26) Write the equation of the line that is perpendicular to $y = \frac{1}{3}x - 1$ and passes through the point $(2, -3)$.

Simplify.

***4.** (40) $3\sqrt{3} \cdot 4\sqrt{12} - 5\sqrt{300}$

***5.** (40) $4\sqrt{3}\left(2\sqrt{3} - \sqrt{6}\right)$

Add.

6. (37) $2 + \frac{1}{x}$

7. (37) $\frac{5x^2}{y} + p^2 - \frac{3x}{py}$

8. (36) Find the equation of the line that passes through the point (2, 2) and is parallel to $y = -\frac{3}{7}x + 4$.

Expand.

9. (20) $xy^{-2}\left(\frac{x^0y^2}{x} - \frac{3x^0y^2}{x^2}\right)$

10. (20) $\frac{xy^{-2}}{p^2}\left(\frac{p^2y^2}{x} + \frac{5x^2y^3}{p^{-2}}\right)$

11. (28) **Geography** The state of Wyoming is roughly square in shape. Write the area to perimeter ratio of the state as a simplified rational expression. Then find the ratio given that the length of each side of the state is roughly 300 miles long.

***12.** (41) **Multi-Step** If the origin represents a pier, and one boat travels 4 units due east and 6 units due north, while a second boat travels 4 units due west and 6 units due south, how far apart are the boats in units? Write the answer as both a simplified radical expression and as a decimal rounded to the nearest tenth. Then approximate the distance given that each unit represents 2.5 miles.

13. (30) **Multi-Step** **a.** Identify the vertex and axis of symmetry of $y = (x + 3)^2 - 5$.

b. Find the values of y when $x = -6, -5$, and -4.

c. Use symmetry to find the values of y when $x = -2, -1$, and 0.

***14.** (40) **Geometry** The radius of a right circular cylinder can be represented by $\frac{\sqrt{V}}{\sqrt{\pi h}}$, where V is the volume of the cylinder and h is the height. Simplify the expression.

15. (33) **Multiple Choice** How many three-digit passwords are possible if digits may be repeated?

A 27 **B** 30 **C** 720 **D** 1000

16. (38) **Write** Suppose that $f(x)$ is a polynomial of degree m, and $g(x)$ is a polynomial of degree $n < m$. What can you conclude about $\frac{f(x)}{g(x)}$ if it has a zero remainder?

17. (34) **Error Analysis** A student solved the equation $x^2 - 16 = 0$ as shown below. $x^2 - 16 = 0 \rightarrow (x - 4)^2 = 0 \rightarrow x = 4$. What is the error? Solve the equation correctly.

***18.** (41) **Multiple Choice** The legs of a right triangle are 18 mm and 80 mm. What is the length of the hypotenuse?

A 80 mm **B** 82 mm **C** 89 mm **D** 98 mm

19. (34) **Model** Graph the line represented by the table of values.

x	−2	0	2	4
y	−5	−2	1	4

***20.** (39) **Justify** Why does (0, 0) usually make for a good test point for determining which half-plane to shade? When can it not be used?

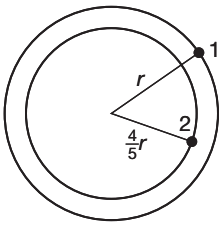
21. (35) **Graphing Calculator** Use a graphing calculator to estimate the root(s) of the equation $x^2 - 78 = 0$ to the nearest integer(s).

22. (32) **Nutrition and Cost** A one-kilogram container of mixed nuts costs \$17.70 and contains 86 grams of protein. The mix contains almonds (\$15 per kilogram, 13% protein), cashews (\$18 per kilogram, 9% protein), and pecans (\$19 per kilogram, 5% protein). How much of each type of nut is in the mix? To find the answer, write a system of equations and use an inverse matrix to solve it.

23. (29) **Ticket Sales** At a college basketball game 350 tickets were sold. Tickets cost \$5 for children and senior citizens, \$7 for students, and \$10 for all others. The total income from ticket sales was \$2,665. The combined number of \$7 and \$10 tickets sold was 6 times more than the number of \$5 tickets sold. How many tickets of each type were sold?

***24.** (40) **Justify** Explain why the expression $\frac{3}{\sqrt{5}}$ is allowed to be written as $\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$.

25. (37) **Physics** A satellite has an orbital acceleration of $\frac{v^2}{r}$, where v is the speed of the satellite and r is the distance from the satellite to the Earth's center. Find the difference in acceleration between satellite 1 and satellite 2.



26. (25) **Rainfall** Annual rainfall amounts in inches for Austin, Texas for 2006 are listed in the table below. What was the mean amount of rainfall in Austin during that year and what was the standard deviation?

Month	J	F	M	A	M	J	J	A	S	O	N	D
Amount (in.)	1.80	0.89	7.54	2.89	5.28	3.18	0.48	0.22	3.00	3.93	1.29	4.20

***27.** (41) **Error Analysis** Find and correct the error a student made in finding the distance between the point (7, 2) and the point (8, −5).

$$\begin{aligned} d &= \sqrt{(8-7)^2-(-5-2)^2} \\ &= \sqrt{1^2-(-7)^2} \\ &= \sqrt{1-49} \\ &= \sqrt{-48} \end{aligned}$$

28. (26) **Justify** Is the equation $2\pi x + 3y = 20$ in standard form? Explain your reasoning.

29. (Inv 1) Create a truth table for $s \wedge \neg t$.

30. (22) **Coordinate Geometry** Use the set of ordered pairs {(2, 3), (2, 1), (2, 0), (2, 4)}. Is the graph of this relation continuous, discontinuous, and/or discrete?

LAB
7

Calculating Permutations and Combinations

Graphing Calculator Lab (*Use with Lesson 42*)

A graphing calculator can be used to calculate permutations and combinations. Use the graphing calculator to calculate ${}_{18}P_5$ and ${}_{14}C_3$.

To Calculate a Permutation: ${}_{18}P_5$

Reading Math

${}_nP_r$ is read "the permutation of *n* things taken *r* at a time."

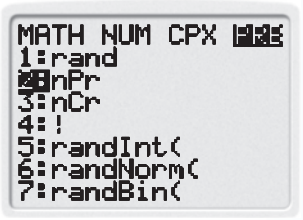
${}_nC_r$ is read "the combination of *n* things taken *r* at a time."

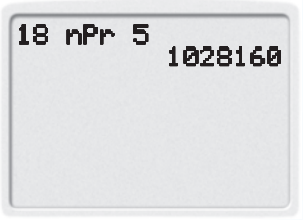
1. Press 1 8 to enter n, the total number of objects to choose from. The subscript n is always located on the left hand side of the general form ${}_nP_r$.

2. Press MATH and then the arrow key ◀, to open the **[PRB]** menu. Use the down arrow ▼ to highlight [2:${}_nP_r$], and press ENTER.

3. Press 5 to enter r, the number of objects chosen. The subscript r is always located on the right hand side of the general form ${}_nP_r$.

4. Press ENTER and the value of ${}_{18}P_5$ is displayed.





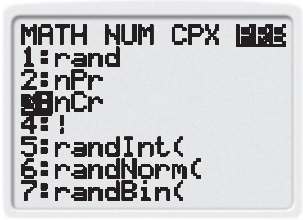
To Calculate a Combination: ${}_{14}C_3$

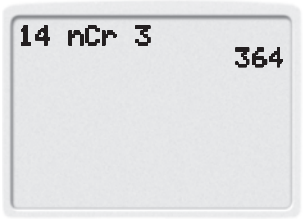
1. Press 1 4 to enter n, the total number of objects to choose from. The subscript n is always located on the left hand side of the general form ${}_nC_r$.

2. Press MATH, and then the arrow key ◀, to open the **[PRB]** menu. Press the down arrow ▼ twice to highlight [3:${}_nC_r$], and press ENTER.

3. Press 3 to enter r, the number of objects chosen. The subscript r is always located on the right hand side of the general form ${}_nC_r$.

4. Press ENTER and the value of ${}_{14}C_3$ is displayed.





Lab Practice

a. Use the graphing calculator to calculate ${}_{12}P_7$.

b. Use the graphing calculator to calculate ${}_{22}P_3$.

c. Use the graphing calculator to calculate ${}_{12}C_7$.

d. Use the graphing calculator to calculate ${}_{22}C_3$.

Online Connection
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LESSON
42

Finding Permutations and Combinations

Warm Up

1. (33) **Vocabulary** An _____ is an outcome of a probability experiment.
2. (33) What is the sample space for tossing a six-sided number cube?
3. (33) Find the sample space of tossing a two-sided coin two times.
4. (33) Determine if the events are dependent or independent: flipping a coin and then tossing a number cube.
5. (33) Use the Fundamental Counting Principal to find how many different outfits consisting of one shirt, one pair of pants, and one pair of shoes can be made from 3 shirts, 2 pairs of pants, and 4 pairs of shoes.

New Concepts

A **factorial** of a positive integer is the product of all positive integers up to and including that integer. Note that 0 factorial (0!) is defined to be 1.

Hint

$4! = 4(4-1)(4-2)(4-3)$
$4! = 4 \times 3 \times 2 \times 1 = 24$

***n* Factorial**
The factorial of n is denoted as $n!$
$$n! = n(n-1)(n-2)(n-3) \ldots 1$$
Zero factorial is defined to be 1. $0! = 1$.

Example 1 Evaluating Expressions Containing Factorials

Evaluate the following expressions that contain factorials:

a. $5!$

SOLUTION

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

A factorial of a positive integer is the multiplication of all positive integers up to and including the integer.

b. $\dfrac{10!}{2!(10-2)!}$

SOLUTION

$$\frac{10!}{2!(10-2)!} = \frac{10!}{2!8!}$$

$$= \frac{10 \times 9 \times \cancel{8!}}{2 \times \cancel{8!}}$$

$$= \frac{90}{2} = 45$$

To help simplify expressions with factorials, expand the numerator just enough to cancel duplicate factors in the denominator.

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Graphing Calculator Tip

For directions on calculating permutations and combinations, see Graphing Calculator Lab #7.

A **permutation** is a selection of items where order is important.

Permutations
The permutation of n objects taken r at a time is: $$P(n, r) = \frac{n!}{(n - r)!}$$

The permutation of 6 objects taken 3 at a time is:

$$P(6, 3) = \frac{6!}{(6 - 3)!} = \frac{6!}{3!} = 120$$

A permutation of n objects arranged into one group of size n, without repetition of objects, can be found by using the formula: $P(n, n) = n!$

For example, the permutations of three letters (**A, B, C**) arranged into groups of three can be calculated as $P(3, 3) = 3! = 3 \times 2 \times 1 = 6$. We can check our calculation by showing all possible arrangements of the three letters, without repetition. The 6 arrangements are **ABC, ACB, BAC, BCA, CAB,** and **CBA.**

When letters are repeated, permutations are not distinguishable from each other. For example, for the letters (**B, B, O**) there are only three distinguishable permutations of the letters: **BOB, BBO,** and **OBB.**

The number of distinguishable permutations of n objects where one object is repeated q_1 times, another object is repeated q_2 times, and so on is: $\frac{n!}{q_1! \bullet q_2! \bullet \ldots q_k!}$. The number of permutations of the letters **BOB** is $\frac{3!}{2!1!} = \frac{6}{2} = 3$.

Example 2 Finding Permutations

a. Find the permutation of 8 objects.

SOLUTION

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ The permutation of n objects is $n!$

There are 40,320 possible permutations of 8 objects.

b. Find the permutation of 6 objects taken 2 at a time.

SOLUTION

Use the formula for the permutation of n objects taken r at a time.

$$P(n, r) = \frac{n!}{(n - r)!} \longrightarrow P(6, 2) = \frac{6!}{(6 - 2)!}$$

$$= \frac{6!}{4!}$$

$$= \frac{6 \times 5 \times \cancel{4!}}{\cancel{4!}}$$

$$= 30$$

There are 30 possible permutations of 6 objects taken 2 at a time.

Hint

You can check your answer by finding the sample space using the digits 1 through 6 to represent the objects.

c. Find the number of distinguishable permutations of the letters in the word **CONNECTION.**

SOLUTION

CONNECTION has 10 letters with **C** repeated 2 times, **O** repeated 2 times, and **N** repeated 3 times. Let n = number of letters in the word **CONNECTION,** q_1 = number of **C**s, q_2 = number of **O**s, and q_3 = number of **N**s.

The number of distinguishable permutations can be found using: $\frac{n!}{q_1!q_2!q_3!}$

$$\frac{10!}{2!2!3!} = \frac{3{,}628{,}800}{2 \times 2 \times 6}$$
$$= 151{,}200$$

There are 151,200 distinguishable permutations of the letters in the word **CONNECTION.**

A **combination** is a selection of items where order does not matter.

Combinations
The combination of n objects taken r at a time is: $$C(n, r) = \frac{n!}{r!(n-r)!}$$

The combination of 5 objects taken 2 at a time is:

$$C(5, 2) = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \times \overset{2}{\cancel{4}} \times \cancel{3 \times 2 \times 1}}{\cancel{2 \times 1} \times \cancel{3 \times 2 \times 1}} = 5 \times 2 = 10$$

To decide if a permutation or combination is needed, ask yourself if order is important.

The arrangement of digits for an alarm system code requires a specific order; the code 352 is different from the code 235, so a permutation is used to determine the number of possible codes.

The arrangement of names of people in a study group does not require a specific order, so a combination is used to calculate the number of possible arrangements.

Example 3 Finding Combinations

a. Find the combination of 7 objects taken 4 at a time.

SOLUTION

Use the formula for the combination of n objects taken r at a time. So,

$$C(n, r) = \frac{n!}{r!(n-r)!} \longrightarrow C(7, 4) = \frac{7!}{4!(7-4)!}$$
$$= \frac{7!}{4!3!}$$
$$= 35$$

There are 35 combinations of 7 objects taken 4 at a time.

Math Language

Multiple **independent events:**

If Event A and Event B can both occur, then multiply their outcomes.

If Event A or Event B can occur, then add their outcomes.

b. How many different pizza varieties can be created if there are two choices of crust, thick or thin, and two out of three possible toppings are selected?

SOLUTION

The combination of the 2 types of crusts is $C(2, 1)$. The combination of 2 out of 3 toppings is $C(3, 2)$. So, the number of pizza varieties that can be created is:

$$C(2, 1) \times C(3, 2) = \frac{2!}{1!(2-1)!} \times \frac{3!}{2!(3-2)!}$$
$$= \frac{2}{1} \times \frac{3 \times \cancel{2}}{\cancel{2} \times 1}$$
$$= 6$$

c. To travel from New York to Boston you could take one of three trains or one of four buses. How many different options for traveling between the two cities do you have?

Hint

When each combination represents a separate outcome, add the combinations.

SOLUTION

The combination of trains is $C(3, 1)$. The combination of buses is $C(4, 1)$. So the number of travel options is $C(3, 1) + C(4, 1)$

$$C(3, 1) + C(4, 1) = \frac{3!}{1!(3-1)!} + \frac{4!}{1!(4-1)!} = \frac{3}{1} + \frac{4}{1} = 7$$

Example 4 Distinguishing Between Permutations and Combinations

a. Explain whether the following is a combination or a permutation and find the solution. How many two-letter arrangements can be formed from the letters **CAT**?

SOLUTION

Order makes a difference in this problem because **CA** and **AC** are not the same arrangement. The solution is found using permutations: $P(3, 2) = \frac{3!}{(3-2)!} = 6$. There are 6 possible arrangements {CA, AC, CT, TC, AT, TA}.

b. Explain whether the following is a combination or a permutation and find the solution. How many two-man crews can be selected from the set of three males: {Charlie, Anton, Tom}?

SOLUTION

For this problem, order does *not* make a difference. The two-man crew of Charlie/Anton is the same as the two-man crew Anton/Charlie. The solution is found using combinations: $C(3, 2) = \frac{3!}{2!(3-2)!} = 3$. There are only 3 different two-man crews {Charlie/Anton, Charlie/Tom, Anton/Tom}.

Hint

Notice that Entry 0 in each row is 1.

Exploration Pascal's Triangle and Combinations

By labeling the top row as Row 0 and the first entry in each row as Entry 0, Pascal's Triangle shows a direct relationship to combinations. For example, to count the number of ways to choose 1 object from 3 objects, look at Entry 1 in Row 3. This entry is 3. $C(3, 1) = 3$

		1				Row 0
	1		1			Row 1
1		2		1		Row 2
	1		3		1	Row 3
1		4		6		4		1	Row 4
1		5		10		10		5		1	Row 5

Explore the different combinations represented in Pascal's Triangle by identifying all of the combinations in the first 5 rows. Justify your results by solving for each combination identified.

SOLUTION

Combinations:

$C(0, 0) = 1$; $C(1, 0) = 1$; $C(1, 1) = 1$; $C(2, 0) = 1$; $C(2, 1) = 2$; $C(2, 2) = 1$; $C(3, 0) = 1$; $C(3, 1) = 3$; $C(3, 2) = 3$; $C(3, 3) = 1$; $C(4, 0) = 1$; $C(4, 1) = 4$; $C(4, 2) = 6$; $C(4, 3) = 4$; $C(4, 4) = 1$; $C(5, 0) = 1$; $C(5, 1) = 5$; $C(5, 2) = 10$; $C(5, 3) = 10$; $C(5, 4) = 5$; $C(5, 5) = 1$.

Example 5 Application: Athletic Tickets

A school has scheduled 5 football games, 6 basketball games, 3 volleyball games, and 4 soccer games. You have a ticket that allows you to attend 5 games. In how many ways can you attend 2 football games, 2 basketball games, and either a volleyball or soccer game?

SOLUTION

1. Understand Since the order of arrangements does not matter, this is a combination problem.

2. Plan To find the number of ways to use your tickets, make a table to help solve the problem.

Five-Game Ticket

Attend 2 games	Attend 2 games	Attend 1 game
Football: 5	Basketball: 6	Volleyball or soccer: 7

Use combinations to solve. Multiply the combination for each event to find the possible ways to use your tickets.

3. Solve $C(5$ football, pick $2) \times C(6$ basketball, pick $2) \times (7$ volleyball or soccer, pick $1)$

$$= C(5, 2) \times C(6, 2) \times C(7, 1) = \frac{5!}{2!3!} \times \frac{6!}{2!4!} \times \frac{7!}{1!6!} = 1{,}050$$

Check Use your calculator to check your calculations for each combination, and then multiply the results. ${}_5C_2 = 10$; ${}_6C_2 = 15$; ${}_7C_1 = 7$; $10 \times 15 \times 7 = 1{,}050$

Lesson Practice

a. (Ex 1) Evaluate 6!

b. (Ex 1) Evaluate $\frac{6!}{4!(6-4)!}$

c. (Ex 2a) Find the number of permutation of 7 objects.

d. (Ex 2b) Find the permutation of 12 objects taken 3 at a time.

e. (Ex 2c) **OHIO** has 4 letters with **O** repeated twice. Find the number of distinguishable permutations of letters in **OHIO.**

f. (Ex 3a) Find the combination of 10 objects taken 7 at a time.

g. (Ex 3b) Find the number of salad varieties using one of four salad dressings and two of four additional vegetables.

h. (Ex 4) How many different 4-card hands are possible if you draw 4 cards from a standard deck of 52 cards? Is this a permutation or combination problem?

i. (Ex 4) How many different ways can 4 cards be picked from a standard deck of 52 cards and be played in a game? Is this a permutation or a combination problem?

j. (Ex 5) In how many ways can 3 blue, 2 red, and 5 green markers be distributed to 10 students?

Practice Distributed and Integrated

Use multipliers to convert.

1. (18) 32 mi^3 to cubic inches

2. (18) 10 square miles to square feet

Use the point-slope formula to write the equation of the line with slope *m* that passes through the given point.

3. (26) $m = -1$, $(4, 5.5)$

4. (26) $m = 1.8$, $(-10, 0)$

Find the distance between the following points.

***5.** (41) $(-2, 2)$ and $(4, 6)$.

***6.** (41) $(-3, 2)$ and $(5, 2)$

Evaluate.

***7.** (42) $P(5, 3)$

***8.** (42) $C(8, 4)$

***9.** (42) 11!

Find the sum.

10. (37) $\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3}$

11. (37) $\frac{1}{1+\frac{1}{x+1}} + \frac{1}{1+\frac{1}{x-1}}$

12. (24) **Verify** Show that $\begin{aligned} 2x + 3y &= 13 \\ 4x - 9y &= -79 \end{aligned}$ can be solved by eliminating either the x-terms or the y-terms.

*13. (42) **Dog Show** In a dog show, how many ways can 3 beagles, 2 poodles, and 5 grey hounds line up if the dogs of the same breed are considered identical?

14. (32) **Error Analysis** Given the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$, a student found the inverse matrix as follows:

$$A^{-1} = \frac{1}{-5}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0.2 & -0.2 \\ -0.4 & -0.6 \end{bmatrix}.$$

What is the error? Find the correct inverse matrix.

15. (11) **Multi-Step** The graph of a polynomial function is continuous. That is, there are no breaks in the graph. The x-intercepts of a function are the x-coordinates of the points at which the graph of the function crosses the x-axis.

a. Complete the table of values for the polynomial function $f(x) = x^3 - 4x - 2$.

x	−2	−1	0	1	2	3
$f(x)$						

b. The function $f(x) = x^3 - 4x - 2$ has three x-intercepts. Between what pair of consecutive integer values of x does each x-intercept lie?

*16. (42) **Write** Describe the difference between a permutation and a combination if you select two-letter sets from the three-letter set {A, B, C}.

*17. (30) **Graphing Calculator** Use a graphing calculator to graph $f(x) = -x^2 - 8x - 11$. Explain how the graph differs from the parent function $y = x^2$.

18. (35) **Multiple Choice** What are the x-intercept(s) of the function $f(x) = x^2 + 2x + 1$?

A −1 only **B** 1 only **C** −1 and 1 **D** −1 and −2

*19. (41) **Probability** A teacher writes the numbers shown below on 6 sheets of paper and a student randomly chooses two papers from the hat, without replacement. What is the probability that the two papers chosen show measures that could be the sides of a right triangle?

30, 72, 78	24, 45, 51	12, 18, 20	25, 60, 65	9, 40, 41	14, 50, 54

20. (39) **Analyze** Suppose you test a point that is on the boundary line of the graph of an inequality. What information does the result give about making the graph? What information does it not give about making the graph?

21. (28) **Multiple Choice** Which of the following shows $\frac{3x^2 + 6x + 3}{x^2 - 3x - 4}$ completely simplified?

A $-\frac{1}{4}$ **B** $\frac{1}{x - 4}$ **C** $\frac{3(x + 1)}{x - 4}$ **D** $\frac{(x + 1)^2}{x - 4}$

*22. (41) **Cartography** A cartographer constructs a map of Montana over a grid system such that Hamilton is at (132, 4) and Missoula is at (146, 72). Find the distance between the cities to the nearest kilometer if each unit represents one kilometer.

23. **Multi-Step** If an object is thrown straight down with an initial velocity of 32 feet per second, the number of feet it travels in t seconds is given by the expression $16t^2 + 32t$ (neglecting air resistance). (31)

a. Write a rational expression for the average velocity of the object before it hits the ground over the time interval from 0 seconds to t seconds. Then simplify the expression. (Average velocity equals distance traveled divided by elapsed time.)

b. Evaluate the expression to find the average velocity of the object over the time interval from 0 seconds to 2 seconds.

24. **Error Analysis** Two students are dividing $2x^5 - 7x^4 - 4x^3 + 27x^2 - 18x$ by $x + 1$ but get different results. Which student made the mistake? (38)

Student A

$$
\begin{array}{r}
2x^4 - 9x^3 + 5x^2 + 32x \\
(x+1)\overline{)\,2x^5 - 7x^4 - 4x^3 + 27x^2 - 18x} \\
\underline{-(2x^5 + 2x^4)} \\
0 - 9x^4 - 4x^3 \\
\underline{-(-9x^4 - 9x^3)} \\
0 + 5x^3 + 27x^2 \\
\underline{-(5x^3 + 5x^2)} \\
0 + 32x^2 - 18x \\
\underline{-(32x^2 + 32x)} \\
0 - 50x
\end{array}
$$

Student B

$$
\begin{array}{r}
2x^4 - 9x^3 + 5x^2 + 22x \\
(x+1)\overline{)\,2x^5 - 7x^4 - 4x^3 + 27x^2 - 18x} \\
\underline{-(2x^5 + 2x^4)} \\
0 - 9x^4 - 4x^3 \\
\underline{-(-9x^4 - 9x^3)} \\
0 + 5x^3 + 27x^2 \\
\underline{-(5x^3 + 5x^2)} \\
0 + 22x^2 - 18x \\
\underline{-(22x^2 + 22x)} \\
0 - 40x
\end{array}
$$

25. **Model** A lunch special includes soup (tomato or chicken noodle), a sandwich (tuna, roast beef, or ham), and a beverage (juice or milk). Draw a tree diagram to show all the different possible lunches. How many different lunches are possible? (33)

26. **Multi-Step** Use $\begin{aligned} y &= -5x + 2z - 18 \\ -6x &= 2y - 3z + 23 \\ z &= -10 - 4x + y \end{aligned}$ (29)

a. Write the equations with the variables aligned.

b. Estimate the number of solutions. Explain your answer.

c. Solve the system of equations to verify your estimate.

27. **Physics** To convert from Fahrenheit to Celsius, use the formula $F = \frac{9}{5}C + 32$. Find the formula to convert Celsius to Fahrenheit and determine if the graphs of the equations are parallel, perpendicular, or neither. (36)

***28.** **Multi-Step** **a.** Identify the like radicals in the expression $4\sqrt[3]{5k} - 4\sqrt{5} + \sqrt[3]{4k} + \sqrt{5}$ (40)

b. Simplify the expression by combining the like radicals.

29. **Geometry** Give all the names that apply to the polygon that is formed by the graphs of $y = \frac{3}{4}x + 2$, $y = 6$, $y = \frac{3}{4}x - 1$, and $y = 2$. (34)

30. **Model** Make sketches of parabolas to show that there can be zero, one, or two x-intercepts in the graph of a quadratic function. (27)

LESSON 43

Solving Systems of Linear Inequalities

Warm Up

(39) **1.** **Vocabulary** An ordered pair that makes a linear inequality true is a _____ of that inequality.

(39) **2.** Is (3, 5) a solution of $2x + y > 10$?

(39) **3.** Is $(-1, 4)$ a solution of $-3x + 2y \leq 11$?

New Concepts

A **system of linear inequalities** is formed by two or more linear inequalities. An ordered pair is a solution of a system if it is a solution of every inequality in the system.

Math Language

To graph an inequality or system means to graph its **solution set.** The solution set is the set of all solutions.

To graph a linear inequality, first graph its boundary line. Then shade the entire half-plane that contains its solutions. To decide which half-plane to shade, pick a point in either half-plane. If the point is a solution, then shade that half-plane; if the point is not a solution, then shade the other half-plane. The graph of the system is the intersection of the graphs of the inequalities in the system.

Example 1 Identifying Solutions and Graphing a System

a. Determine whether (2, 1) and (2, 2) are solutions of the system of linear inequalities.

$$y < 3$$
$$x + 2y \leq 4$$

SOLUTION Substitute the ordered pairs for x and y.

Substitute (2, 1).

$y < 3 \rightarrow 1 < 3$ **true**

$x + 2y \leq 4 \rightarrow 2 + 2(1) \leq 4$ **true**

(2, 1) is a solution.

Substitute (2, 2).

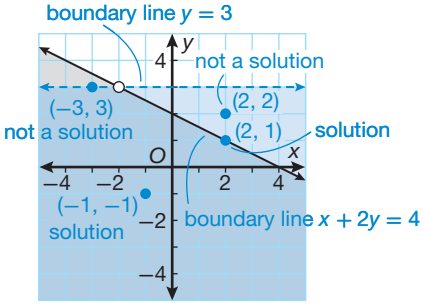
$y < 3 \rightarrow 2 < 3$ **true**

$x + 2y \leq 4 \rightarrow 2 + 2(2) \leq 4$ **false**

(2, 2) is not a solution.

b. Graph the system.

SOLUTION



Graph the dashed boundary line $y = 3$. The solution set for $y < 3$ consists of all points below the line.

Graph the solid boundary line $x + 2y = 4$. The solution set for $x + 2y \leq 4$ consists of all points on the line and all points below the line. The solution set for the system consists of all points that are solutions of both inequalities. It is represented by the dark shaded region and the thick portion of the solid boundary line, not including $(-2, 3)$.

Hint

For $<$ and $>$, the boundary line is dashed, indicating that points on the line are not solutions. For $\leq$ and $\geq$, the boundary line is solid, indicating that points on the line are solutions.

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Example 2 Graphing Systems with Parallel Boundary Lines

a. Graph the system of linear inequalities. Describe the solution set.

$$3x - 4y > 8$$

$$y \geq \frac{3}{4}x + 1$$

SOLUTION

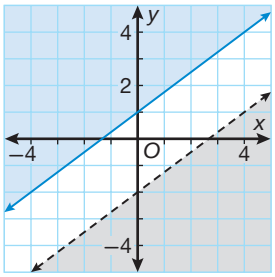
Solve $3x - 4y > 8$ for y.

$$3x - 4y > 8$$

$$-4y > -3x + 8$$

$$\frac{-4y}{-4} < \frac{-3x + 8}{-4}$$

$$y < \frac{3}{4}x - 2$$



The graph of $3x - 4y > 8$ consists of all points below the boundary line. The graph of $y \geq \frac{3}{4}x + 1$ consists of all points on the boundary line and all points above the line. The boundary lines are parallel, so the graphs do not intersect. The solution set of the system is the empty set.

Hint

Remember to reverse the inequality symbol when you multiply or divide both sides of an inequality by a negative number.

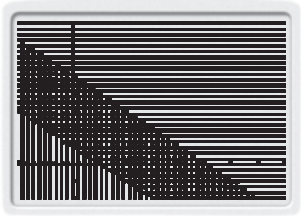
b. Graph the system on a graphing calculator. Describe the solution set.

$$y \geq -x + 1$$

$$x + y \leq 5$$

SOLUTION

The equations of the boundary lines must be entered in the form $y = mx + b$, so solve $x + y \leq 5$ for y. Enter $y = -x + 1$ and $y = -x + 5$.



Then use the SHADE feature to shade the appropriate half-plane for each inequality. The boundary lines are parallel. The solution set consists of all points on both boundary lines and all points between the boundary lines.

Graphing Calculator Tip

To shade in the Y= editor, move the cursor to the graph style icon in the first column, just left of Y_1. Then press ENTER repeatedly until you get the appropriate icon.

Example 3 Graphing a System of Three Linear Inequalities

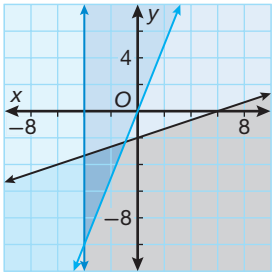
Graph the system of linear inequalities. Describe the solution set.

$$x \geq -4$$

$$5x - 2y \leq 0$$

$$x - 3y \geq 6$$

SOLUTION



The solution set of the system consists of all points on the triangle formed by the three boundary lines and all points inside the triangle.

Example 4 Application: Tickets to an Entertainment Event

Ticket prices for a major league baseball game are \$63 for a main box seat and \$20 for a tier reserved seat. Suppose you are a CEO of a company, and you want to buy tickets for some of your employees. You decide to buy up to 30 tickets and to spend up to \$1000. Write and graph a system of linear inequalities to represent all the possible combinations of main box seats and tier reserved seats you can buy.

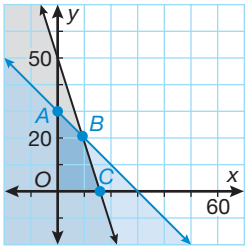
SOLUTION Write the system. Let x equal the number of main box seats and y equal the number of tier reserve seats.

$x + y \leq 30$ The maximum sum of tickets is 30.

$63x + 20y \leq 1000$ The maximum amount to spend is \$1000.

Graph the system.

The solution set of the system consists of all points in the region where the shading overlaps and all points on the boundary of that region. However, only points with nonnegative integer coordinates make sense for this application because x and y represent numbers of tickets. Therefore, all the possible combinations of main box seats and tier reserved seats you can buy are represented by all points (x, y) on and inside polygon $OABC$, where x and y are integers.



Math Reasoning

Write Is the number of points on and inside polygon $OABC$ finite or infinite? Is the number of possible combinations of tickets finite or infinite? Explain.

Lesson Practice

a. *(Ex 1)* Determine whether $(-5, 0)$, $(-6, 0)$, and $(1, -2)$ are solutions of the system of linear inequalities.

$$x \geq -5$$
$$2x + 3y < -3$$

b. *(Ex 1)* Graph the system of linear inequalities.

$$x \geq -5$$
$$2x + 3y < -3$$

c. *(Ex 2)* Graph the system of linear inequalities. Describe the solution set.

$$y > -x - 2$$
$$x + y < 3$$

d. *(Ex 2)* Graph the system of linear inequalities on a graphing calculator. Describe the solution set.

$$y \geq -\frac{3}{4}x + 4$$
$$3x + 4y \leq 4$$

e. *(Ex 3)* Graph the system of linear inequalities. Describe the solution set.

$$y < 4$$
$$2x + y \geq 4$$
$$x - 2y \leq -2$$

f. (Ex 4) Two of the ticket prices for a major league baseball game are \$65 for a tier box MVP seat and \$12 for a bleachers seat. Suppose you want to buy tickets for some of your friends. You decide to buy up to 20 tickets and to spend up to \$500. Let x represent a number of tier box MVP seats and y represent a number of bleachers seats. Write and graph a system of linear inequalities to represent all the possible combinations of tier box MVP seats and bleachers seats you can buy.

Practice Distributed and Integrated

Factor.

1. (23) $63x^3 + 108x^2 + 81x$

2. (23) $x^{10} - 400$

Determine if there are any excluded values. If so, list them.

3. (28) $\dfrac{10}{5x^2 - 20x - 60}$

4. (28) $\dfrac{4 - x}{3}$

***5.** (43) **Justify** Explain why all solutions of the system below lie in quadrant III.

$$y > \frac{1}{2}x$$

$$y < -2$$

6. (30) **Small Business** The profits, in dollars, for x weeks in one year for a small business can be modeled by the function $y = -0.55(x - 12)^2 + 630$. Does the graph of the function have a maximum or a minimum? What is that value and during what week did it occur?

7. (16) **Analyze** Explain the difficulty you encounter when trying to use Cramer's Rule to solve the system $\begin{aligned} 4x - 12y &= -4 \\ -6x + 18y &= -6 \end{aligned}$

***8.** (43) **Ticket Prices** Two of the ticket prices for a baseball game are \$45 for a right field box seat and \$90 for a pavilion box seat. Suppose it is the year 2007 and you are a business owner. You decide to buy tickets to give away as prizes to customers. You decide to buy up to 15 tickets and to spend up to \$900. Let x represent a number of right field box seats and y represent a number of pavilion box seats.

a. Write and graph a system of linear inequalities that represents the situation.

b. Describe the points in the graph of the system that represent all the possible combinations of right field box seats and pavilion box seats you can buy.

c. Determine the point of intersection of the boundary lines in the graph and explain its meaning.

9. (20) **Tennis** The length of a regulation tennis court is given by the function $f(x) = x + 42$ feet. The width of a regulation tennis court is given by the function $g(x) = x$ feet. Write a function $h(x)$ that represents how many feet longer the court is than it is wide.

*10. (29) **Graphing Calculator** Use a graphing calculator to solve.

$$\begin{aligned} 9z - 7 &= 10x + 7y \\ -x - 3 &= -8y + 11z \\ z &= -2 - x + y \end{aligned}$$

11. (36) **Generalize** Suppose that $y_1 = m_1x + b_1$ is perpendicular to $y_2 = m_2x + b_2$. What must be true for b_1 to equal b_2?

*12. (43) **Multiple Choice** Which system has no solution?

A $\begin{aligned} y &\le 2x + 2 \\ y &\ge 2x + 3 \end{aligned}$ **B** $\begin{aligned} y &\le 2x + 2 \\ y &\le 2x + 3 \end{aligned}$ **C** $\begin{aligned} y &\ge 2x + 2 \\ y &\le 2x + 3 \end{aligned}$ **D** $\begin{aligned} y &\ge 2x + 2 \\ y &\ge 2x + 3 \end{aligned}$

13. (23) **Write** In the difference of 2 squares, explain why the product of the two binomials $(a + b)$ and $(a - b)$ is $a^2 - b^2$.

Determine whether the given points are solutions of the systems of linear inequalities.

*14. (43) $\begin{aligned} y + x &\le 6 \\ 2x - 3y &> 12 \end{aligned}$ $(6, 0)$

*15. (43) $\begin{aligned} 10x + 5y &\ge 16 \\ 4x - 6y &< 22 \end{aligned}$ $(1, 3)$

16. (28) **Multiple Choice** What are the excluded value(s) for $\frac{x + 4}{2x^2 - x - 21}$?

A -4 **B** -3 **C** -3 and 3.5 **D** -4, -3, and 3.5

17. (40) **a.** Show the steps to multiply $\sqrt{12} \cdot \sqrt{24}$ by simplifying each radical first.

b. Show the steps to multiply $\sqrt{12} \cdot \sqrt{24}$ by multiplying the radicands first.

18. (38) **Justify** Suppose that the graph of $y = ax^2 + bx + c$, for real numbers a, b, and c, does not cross the x-axis. Is there a linear polynomial $x + d$, for real number d, such that $\frac{ax^2 + bx + c}{x + d}$ has a zero remainder? Explain.

19. (32) **Data Analysis** Students at a high school were asked how they traveled to and from school. The table shows data about the results of the survey. How many students in each grade were surveyed? To find the answer, write a system of equations and use an inverse matrix to solve it.

	Tenth Graders	Eleventh Graders	Twelfth Graders	Total
Walk	30%	20%	16%	114
School Bus	50%	50%	40%	245
Other	20%	30%	44%	171

*20. (41) **Error Analysis** Find and correct the error a student made in finding the distance between the point at $(-1, 3)$ and the point at $(0, 2)$.

$$\begin{aligned} d &= \sqrt{(0 - (-1))^2 + (2 - 3)^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

21. (21) **Business School** This table shows the change in rank according to Business Week of two undergraduate business schools from 2006 to 2007. If the rankings continue to change at these constant rates, at what rank will University of Florida and University of Wisconsin be equal?

School	2006	2007	Change
University of Florida	47	43	–4
University of Wisconsin	27	28	+1

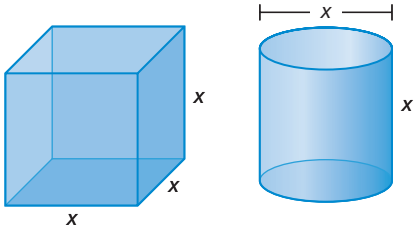
State the expanded form.

***22.** (42) 8!

***23.** (42) $\frac{10!}{3!}$

24. (39) **Geometry** Suppose 5 inch by 5 inch squares and 5 inch by 10 inch rectangles are placed end to end to make a long strip that is 5 inches high. Write and graph the inequality that shows how many of each shape can be used if the area of the strip must remain less than 1500 square inches.

25. (31) **Multi-Step** A container manufacturer is comparing a cubic container and a cylindrical container with the dimensions shown.



a. Write a rational expression in simplest form for the ratio of surface area to volume for the cubic container.

b. Write a rational expression in simplest form the ratio of surface area to volume for the cylindrical container.

c. Explain the meaning of the results in parts *a* and *b*.

26. (27) **Error Analysis** A student was asked to convert a function written in vertex form to its standard form. The student's work is shown below. Find and correct the error the student made.

$$\begin{aligned} f(x) &= -2(x-4)^2 - 1 \\ &= -2(x^2 - 8x + 16) - 1 \\ &= -2x^2 - 8x + 16 - 1 \\ &= -2x^2 - 8x + 15 \end{aligned}$$

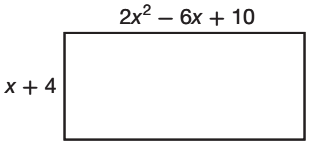
Divide and identify the roots.

27. (38) $15x^3 - 17x^2 - 66x + 56$ by $x + 2$

28. (38) $-8x^3 + 50x^2 - 93x + 45$ by $x - 3$

***29.** (42) **Student Government** There are nine seniors and six juniors running for the class officer positions (President, Secretary, and Treasurer). Each class must elect 3 officers. How many possible outcomes are there for this election? Explain.

30. (19) **Multi-Step** The diagram at the right shows the garden in Lucy's backyard.



a. Write an expression for the area of the garden.

b. Simplify the expression using multiplication.

c. Find the area of the garden if $x = 4$ feet.

LESSON 44

Rationalizing Denominators

Warm Up

1. (40) **Vocabulary** In the radical expression $3\sqrt{2}$, 2 is called the _____.
2. (40) Multiply and write the product in simplest form: $\frac{-2x+10}{x^2-25} \cdot \frac{x+5}{6}$
3. (40) Simplify: $2\sqrt{3} \cdot 4\sqrt{2}$
4. (40) Simplify: $4\sqrt{5} \cdot \sqrt{5}$

New Concepts

The expressions $\frac{3}{2\sqrt{5}}$ and $\frac{4+\sqrt{3}}{2-3\sqrt{3}}$ have irrational denominators because $\sqrt{5}$ and $\sqrt{3}$ are irrational. To rationalize the denominator of an expression, write an equivalent expression so that there are no radicals in any denominator and no denominators in any radical. An expression containing square root radicals is in simplest form when no radicand has a perfect square root factor and there are no irrational radical expressions in any denominator.

Hint

$\frac{3\sqrt{5}}{2\sqrt{25}}$ has a rational denominator, but is not in simplest form because the radicand 25 is a perfect square.

Example 1 Monomials in Numerator and Denominator

a. Simplify $\frac{3}{2\sqrt{5}}$.

SOLUTION

$$\frac{3}{2\sqrt{5}} = \frac{3}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$

Multiply the numerator and denominator by $\sqrt{5}$ to eliminate radical from the denominator.

$$= \frac{3\sqrt{5}}{2\sqrt{5}\sqrt{5}}$$

$$= \frac{3\sqrt{5}}{2 \cdot 5}$$

$\sqrt{5} \cdot \sqrt{5} = \sqrt{25} = 5$

$$= \frac{3\sqrt{5}}{10}$$

b. Simplify $\sqrt{\frac{1}{8}}$.

SOLUTION

$$\sqrt{\frac{1}{8}} = \sqrt{\frac{1}{8} \cdot \frac{2}{2}}$$

The radicand is a fraction. Multiply the numerator and denominator of the radicand by 2 to get a perfect square in the denominator.

$$= \sqrt{\frac{2}{16}}$$

$$= \frac{\sqrt{2}}{\sqrt{16}}$$

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$$= \frac{\sqrt{2}}{4}$$

Hint

Here is another method:

$$\sqrt{\tfrac{1}{8} \cdot \tfrac{8}{8}} = \sqrt{\tfrac{8}{64}}$$
$$= \tfrac{\sqrt{8}}{8}$$
$$= \tfrac{\sqrt{4}\sqrt{2}}{8}$$
$$= \tfrac{2\sqrt{2}}{8}$$
$$= \tfrac{\sqrt{2}}{4}$$

To rationalize some binomial denominators, you need to multiply by a conjugate.

Radical Conjugates

If a, b, c, and d are rational numbers, then $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$ are **conjugates** of each other, and their product is a rational number.

If denominator has the form	Multiply numerator and denominator by
$a\sqrt{b} + c\sqrt{d}$	$a\sqrt{b} - c\sqrt{d}$
$a\sqrt{b} - c\sqrt{d}$	$a\sqrt{b} + c\sqrt{d}$

Example 2 Monomial in Numerator, Binomial in Denominator

a. Simplify $\dfrac{2}{-4+\sqrt{3}}$.

SOLUTION

$$\frac{2}{-4+\sqrt{3}} = \frac{2}{-4+\sqrt{3}} \cdot \frac{-4-\sqrt{3}}{-4-\sqrt{3}}$$

$-4-\sqrt{3}$ is the conjugate of $-4+\sqrt{3}$.

$$= \frac{2(-4-\sqrt{3})}{(-4+\sqrt{3})(-4-\sqrt{3})}$$

Multiply.

$$= \frac{-8-2\sqrt{3}}{(-4)^2-(\sqrt{3})^2}$$

Use $(a+b)(a-b) = a^2 - b^2$ in the denominator.

$$= \frac{-8-2\sqrt{3}}{16-3}$$

Simplify the powers.

$$= \frac{-8-2\sqrt{3}}{13}$$

Simplify the denominator.

Hint

$-\frac{8}{13} - \frac{2\sqrt{3}}{13}$ is an alternate simplified form for Example 2a.

b. Simplify $\dfrac{3}{\sqrt{5}+\sqrt{2}}$.

SOLUTION

$$\frac{3}{\sqrt{5}+\sqrt{2}} = \frac{3}{\sqrt{5}+\sqrt{2}} \cdot \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$\sqrt{5}-\sqrt{2}$ is the conjugate of $\sqrt{5}+\sqrt{2}$.

$$= \frac{3(\sqrt{5}-\sqrt{2})}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})}$$

Multiply.

$$= \frac{3(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2}$$

Use $(a+b)(a-b) = a^2 - b^2$ in the denominator.

$$= \frac{3(\sqrt{5}-\sqrt{2})}{5-2}$$

Simplify the powers.

$$= \frac{3(\sqrt{5}-\sqrt{2})}{3}$$

Simplify the denominator.

$$= \sqrt{5}-\sqrt{2}$$

Simplify.

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Example 3 Binomials in Numerator and Denominator

Simplify $\dfrac{4+\sqrt{3}}{2-3\sqrt{3}}$.

SOLUTION

$$\frac{4+\sqrt{3}}{2-3\sqrt{3}}$$

$$=\frac{4+\sqrt{3}}{2-3\sqrt{3}}\cdot\frac{2+3\sqrt{3}}{2+3\sqrt{3}}$$ $2+3\sqrt{3}$ is the conjugate of $2-3\sqrt{3}$.

$$=\frac{(4+\sqrt{3})(2+3\sqrt{3})}{(2-3\sqrt{3})(2+3\sqrt{3})}$$ Multiply.

$$=\frac{8+12\sqrt{3}+2\sqrt{3}+3\sqrt{3}\,\sqrt{3}}{(2)^2-(3\sqrt{3}\,)^2}$$ Use FOIL method in the numerator. Use $(a+b)(a-b)=a^2-b^2$ in the denominator.

$$=\frac{8+14\sqrt{3}+9}{4-27}$$ Simplify.

$$=\frac{17+14\sqrt{3}}{-23}$$

Lesson Practice

Simplify.

a. (Ex 1) $\dfrac{2}{5\sqrt{7}}$

b. (Ex 1) $\sqrt{\dfrac{3}{20}}$

c. (Ex 2) $\dfrac{2}{2-\sqrt{7}}$

d. (Ex 2) $\dfrac{1}{4\sqrt{6}+\sqrt{3}}$

e. (Ex 3) $\dfrac{3+\sqrt{5}}{1-\sqrt{5}}$

Practice Distributed and Integrated

Simplify:

***1.** (44) $\dfrac{5}{3\sqrt{3}}$

***2.** (44) $\dfrac{4}{2+\sqrt{7}}$

***3.** (44) $\sqrt{\dfrac{9}{10}}$

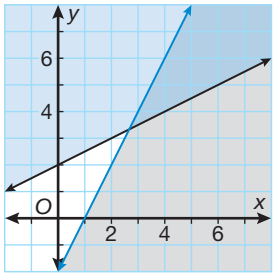
***4.** (41) Hiking If the origin represents a campsite, and one hiker hikes 3 units due east and 4 units due south, while a second hiker hikes 5 units due west and 7 units due south, how much further from the campsite is the second hiker hike than the first? Write the answer as a decimal to the nearest tenth of a mile, given that each unit represents 1.5 miles.

5. (24) Solve $\begin{aligned}4x&=12-5y\\24&=8x+10y\end{aligned}$.

*6. (43) **Error Analysis** The graph at the right is a student's incorrect graph of the system below. What is the error? Graph the system correctly.

$$y \geq \frac{1}{2}x + 2$$

$$2x - y \leq 2$$



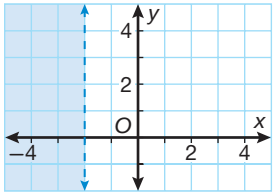
*7. (Inv 4) Describe a substitution cipher.

*8. (43) **Justify** Are there more arrangements of 10 items taken 4 at a time with order not important, or 10 items taken 6 at a time with order not important?

*9. (40) **Physics** The time in seconds it takes for an object to fall from a certain height can be given by $\frac{\sqrt{h}}{\sqrt{16}}$ where h is the initial height of the object in feet.

Find how long it takes for a rock to fall from the top of the Hoover Dam which has a height of 726 feet.

*10. (39) **Multiple Choice** The graph of which inequality is shown?



A $x \leq -2$ **B** $x < -2$

C $y \leq -2$ **D** $y < -2$

*11. (38) **Generalize** Suppose the polynomial $ax^2 + bx + c$ has no real roots. Is there a linear factor $x + d$ such that the remainder is zero? Explain.

12. (33) **Geometry** Are the events mutually exclusive? Explain your answer.

a. Draw an obtuse triangle or draw an isosceles triangle.

b. Draw an acute triangle or draw a right triangle.

13. (36) **Analyze** Suppose that $y_1 \parallel y_2$, $y_2 \perp y_3$, and $y_3 \perp y_4$. What must be true for y_1 and y_4 to share the same y-intercept?

14. (23) Factor $x^3y^2z^3 + x^2yz^2 - 3x^3yz$.

15. (34) Given $f(x) = \frac{x}{x+1}$ and $g(x) = \frac{3}{2x}$, find $g(f(x))$ in simplest form.

16. **Graphing Calculator** Use a graphing calculator to find the inverse of $A = \begin{bmatrix} 6 & -2 \\ 4 & 12 \end{bmatrix}$.

17. (30) How does the graph of $y = 5x^2$ differ from the graph of $y = x^2$?

18. (29) **Multi-Step** Use $\begin{aligned} 2x + 3y &= 2z \\ 4x &= -5y + 4z \\ 3x + 5z &= 2y \end{aligned}$.

a. Write the equations with the variables aligned.

b. Estimate the number of solutions. Explain your answer.

c. Solve the system of equations to verify your estimate.

19. (27) **Utilities** A utility cable is hung between two poles that are 150 feet apart. The height of the cable between the two poles can be modeled by $f(x) = 0.0005x^2 + 35$ where x is the distance from the halfway point to either pole. How far up each pole is the cable attached? Round to the nearest tenth of a foot.

20. (27) Written in standard form, what are the values of a, b, and c in $f(x) = 3x^2$?

21. (35) **Write** Consider a quadratic function $f(x) = ax^2 + bx + c$ with $a < 0$ and whose vertex is in Quadrant III. How many real zeros does the function have? Explain.

22. (26) **Error Analysis** Two students used the point-slope form to write the equation of a line with slope 3 that passes through the point (5, 2). One student wrote the equation as $y = 3x - 13$, while the other wrote the equation as $y = 3x - 1$. Which student is correct? What error did the other student make?

Student A	Student B
$y - 5 = 3(x - 2)$ $y = 3x - 6 + 5$ $y = 3x - 1$	$y - 2 = 3(x - 5)$ $y = 3x - 15 + 2$ $y = 3x - 13$

23. (23) **Measurement** Convert $36x^2 + 72x + 96$ inches to feet.

24. (23) Factor $16m^2p^3y - 8y^4mp^3 + 4m^2p^2y^2$

25. (23) The United States Postal Service will mail a letter that is no greater than $2x + 1.5$ inches long and no more than $x + 1$ inches high. The total area of an envelope with the maximum dimensions is approximately 69 in^2. Calculate the approximate maximum dimensions of an envelope that can be mailed through the USPS.

26. (38) Divide $x^4 - 2$ by $x + 1$ and check.

27. (43) **Write** Assume you have graphed the boundary line for $x + 2y > 6$. Describe two ways to determine which half-plane to shade.

28. (41) **Multi-Step** If the origin represents a parking lot, and one mountain biker is located at $(6, k)$ and another is located at $(-2, -7)$:

a. Find two possible values of k if the bikers are 10 units apart.

b. Determine the first biker's coordinates given that his distance from the parking lot is about 6.1 units.

29. (42) **Error Analysis** At an ice cream store, there are five different flavors of frozen yogurt. A customer orders three scoops. One customer says that there could be 60 different variations, and another customer says there could be only 10. Who is correct? Explain your reasoning.

30. (38) Divide $3x^3 - 3$ by $-x - 2$ and check.

LAB
8

Applying Linear and Median Regression

Graphing Calculator Lab (*Use with Lesson 45*)

The graphing calculator can be used to calculate and plot a line of best fit using linear and median regression. Use the following data:

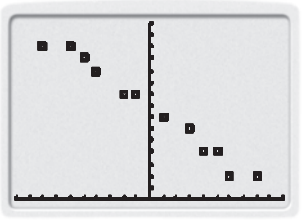
x	−8	−6	−5	−4	−2	−1	1	3	4	5	6	8
y	13	13	12	11	9	9	7	6	4	4	2	2

Graphing Calculator Tip

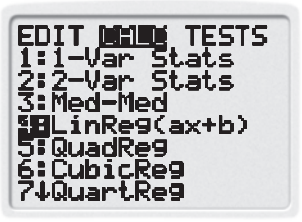
For directions on how to store and plot data using lists, see Graphing Calculator Lab 5.

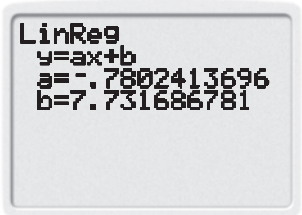
Calculating and Plotting Linear Regression Lines

1. Store and plot the data points from the table above.
   - **a.** Store the x-values in L1 and the y-values in L2.
   - **b.** Plot the data points on a coordinate graph.
   - **c.** Change the viewing window to $Y_{min} = 0$ and $Y_{max} = 15$ to display all the points. (For help in changing the window of a graph, see Graphing Calculator Lab 4.)

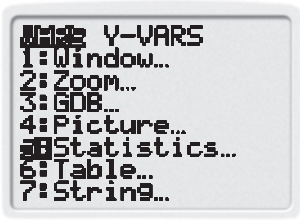


2. Once the data is stored into the lists, the linear regression line can be calculated.
   - **a.** Press STAT to open the [**CALC**] menu.
   - **b.** Press ▼ to select [**4: LinReg(ax + b)**] and press ENTER.
   - **c.** Press 2nd 1 to enter L1 as the x-values. Then, press , 2nd 2 to enter L2 as the y-values.
   - **d.** Press ENTER, and the values for a and b will be displayed. These values, substituted in the equation $y = ax + b$, form the Linear Regression line.





3. The Linear Regression line can now be graphed and compared to the plotted data points.
   - **a.** Press Y=, to open the [**Y=**] editor.
   - **b.** Press VARS to open the Variables menu.
   - **c.** Press the ▼ key to highlight **[5: Statistics…]** and press ENTER. Press the ▶ key twice to open the **[EQ]** menu. Select **[1: RegEQ]** and press ENTER. The equation for the Linear Regression line is stored in the RegEQ (regression equation) tool and will now be displayed next to Y_1=.



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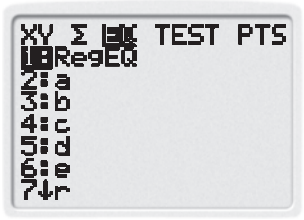
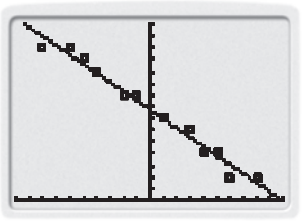
d. Press GRAPH and the Linear Regression line will be graphed next to the plotted points.

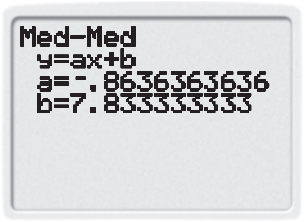


4. The RegEQ tool can also be used to graph the Median Fit line.



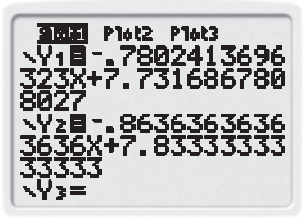
a. Press 2nd MODE to return to the home screen. Press STAT ▶ to open the **[CALC]** menu again, but use the ▼ key to highlight **[3: Med-Med]** and press ENTER.

b. Press 2nd 1 , 2nd 2 to enter L1 and L2 as the x- and y-values.

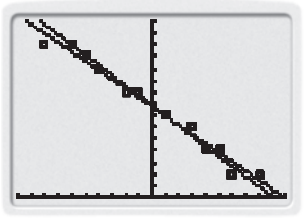


c. Press ENTER and the values of a and b for the Median Fit line will be displayed. This equation will now be saved in the RegEQ tool.

d. Press Y= to open the **[Y=]** editor and move the cursor to $Y_2=$. Repeat Step 3, to insert the equation stored in the RegEQ tool.



e. Press GRAPH and now the Median Regression line is graphed along with the Linear Regression line, and can be compared.



Lab Practice

Use the following data:

x	23	19	11	8	6	2	0	−1	−7	−9	−19	−22
y	0	2	5	7	6	9	11	13	12	18	19	21

a. Find the equation of the Linear Regression line.

b. Find the equation of the Median Regression line.

c. Graph the data and both regression lines on the graphing calculator.

LESSON
45

Finding the Line of Best Fit

Warm Up

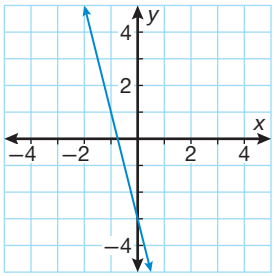
(13) **1.** **Vocabulary** The y-coordinate of the point where a graph crosses the y-axis is the _____.

(13) **2.** Identify the slope and y-intercept in $2x - 4y = 8$.

(2) **3.** Evaluate $15.5x + 70$ when $x = 90$.

(26) **4.** Write the equation of the line graphed at right.

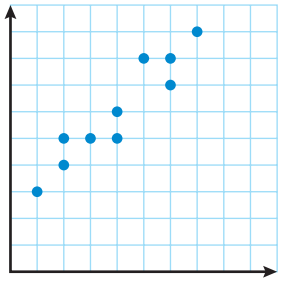
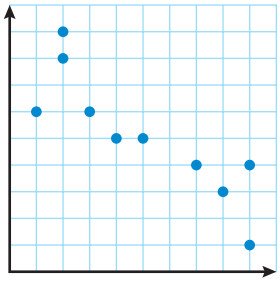
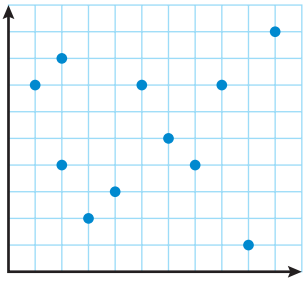
(13) **5.** Tell whether the slope of line graphed at the right is positive, negative, zero, or undefined.



New Concepts

Regression is the process of identifying a relationship between variables. A measure of the strength and direction of the relationship between two variables or data sets is called **correlation.**

When the points tend to gather about a line, there is a linear relationship, or linear correlation. A scatter plot of the points will indicate whether there is a positive correlation or a negative correlation. When the points are widely scattered, there will be little to no correlation.

Positive Correlation	Negative Correlation	No Correlation
		
Points **rise** from left to right	Points **fall** from left to right	Points are widely scattered

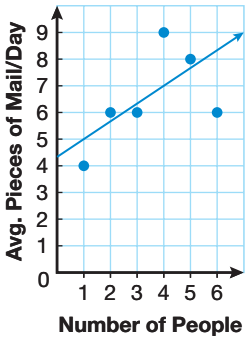
A line that best fits the points in a scatter plot is a **line of best fit.** It is also called a regression line. A line of best fit will have about the same number of points above and below it. Also, it may or may not go through one or more of the points on the graph.

Example 1 Describing Correlation and Estimating a Line of Best Fit

Create a scatter plot of the data and describe the correlation. Sketch a line of best fit, write the equation of the line, and describe the slope in terms of the data.

Number of People in Household	1	2	3	4	5	6
Average Pieces of Mail per Day	4	6	6	9	8	6

SOLUTION Plot the points (people, mail). The points tend to rise from left to right. There is a positive correlation. Sketch a line of best fit and use two points on the line to write the equation.



$$(1, 5), (4, 7): m = \frac{7-5}{4-1} = \frac{2}{3}$$

$$y - y_1 = m(x - x_1)$$

$$y - 5 = \frac{2}{3}(x - 1)$$

$$y = \frac{2}{3}x + 4\frac{1}{3}$$

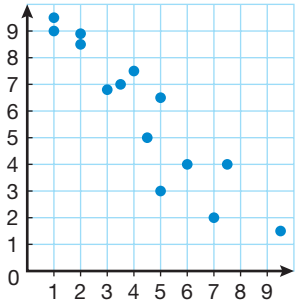
The slope indicates that the number of pieces of mail increases by 2 for every increase of 3 people in the household.

Math Reasoning

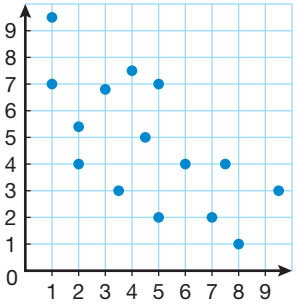
Generalize Describe a data set whose *r*-value is –0.8.

The strength and direction of a linear correlation is measured by the **correlation coefficient,** r. The values of r can range from –1 to 1, where negative values indicate a negative correlation and positive values indicate a positive correlation. The further r is from 0, in either direction, the closer the points are to a straight line. When there is no linear correlation, r equals zero.

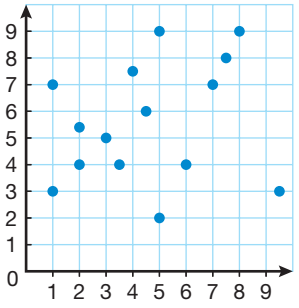
$r \approx -0.93$



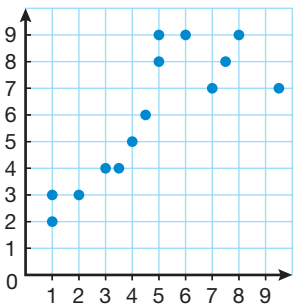
$r \approx -0.67$



$r \approx 0.21$



$r \approx 0.81$



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Hint

Predicting within a given range of data is more accurate than predicting outside the range of data.

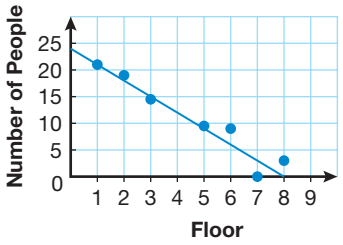
Example 2 Estimating the Value of *r*

Create a scatter plot of the data and estimate the correlation coefficient. Then predict the number of people who took the stairs to the fourth floor.

Building Floor Number	1	2	3	5	6	7	8
Number of People who Took Stairs	22	18	14	9	8	0	3

SOLUTION Plot the points (floor, people).

The points fall, so the *r*-value is negative. They are very close to forming a line, so the value is near −1, but not −1 because the points do not form a line exactly. A good estimate of the *r*-value would be −0.97.



Sketching a line of best fit shows that a good prediction for the number of people who take the stairs to the fourth floor is 12.

Exploration Collecting and Analyzing Data

Materials needed: tennis ball, stop watch

Step 1:

Have one student bounce a tennis ball and one student time how long it takes for the ball to stop bouncing. Meanwhile, the other students should be counting the number of times the ball bounces during that time. It will help for students to agree on the number of bounces if the timer yells "stop" when he or she stops the stop watch.

Step 2:

Record the number of seconds the ball bounced as well as the number of bounces in a table similar to the one below.

Time (sec)										
Number of Bounces										

Step 3:

Repeat the activity nine more times for a total of 10 data points.

Step 4:

Have each student plot the points, sketch a line of best fit, determine the equation of the line, and estimate the correlation coefficient. Have students share and discuss their results.

A graphing calculator can be used to determine the correlation coefficient and line of best fit for a set of data.

Example 3 Using a Graphing Calculator

Enrollment at a high school for the years 2000 through 2007 are shown. Make a scatter plot of the data. Then use a graphing calculator to find the correlation coefficient. Predict the enrollment for 2008.

Year	2000	2001	2002	2003	2004	2005	2006	2007
Number of Students	149	175	198	240	262	332	418	451

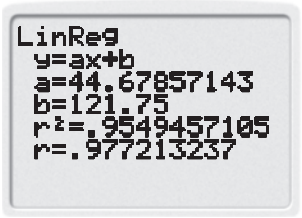
Graphing Calculator Tip

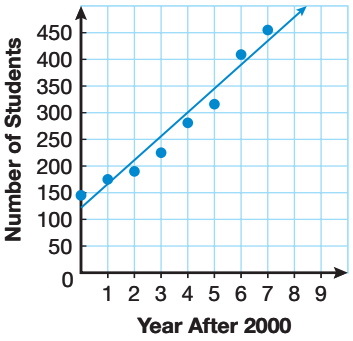
For help using your calculator for regression, see the Lab on page 323. For the correlation coefficient, turn on the diagnostics feature. Press 2nd 0 to open the **CATALOG** menu. Scroll down to **DiagnosticsOn** and press ENTER twice.

SOLUTION

To avoid a large gap on the x-axis, write the years as the number of years after 2000.

Enter the data into a graphing calculator. Choose **LinReg($ax + b$)** from the CALC menu. Then type the list names.





The r-value is about 0.98. The line of best fit is also shown. It is $y \approx 44.7x + 121.75$. Sketch this on the graph to predict that enrollment in 2008 is 479 students.

Lesson Practice

(Ex 1) **a.** Create a scatter plot of the data and describe the correlation. Sketch a line of best fit, write the equation of the line, and describe the slope in terms of the data.

Minutes Studied for Quiz	5	9	20	15	12	18
Number of Incorrect Answers	4	1	0	1	3	1

(Ex 2) **b.** Create a scatter plot of the data and estimate the correlation coefficient. Then predict the amount of the heating bill when the average temperature is 40°F.

Average Temperature, °F	15	22	24	30	49	56	59
Monthly Heating Bill, $	130	98	75	81	82	65	51

(Ex 3) **c.** Enrollment at a fitness center for the years 2000 through 2007 are shown. Make a scatter plot of the data. Then use a graphing calculator to find the correlation coefficient. Predict the enrollment for 2008.

Year	2000	2001	2002	2003	2004	2005	2006	2007
Number of Members	310	295	412	488	322	592	514	522

Practice Distributed and Integrated

*1. (45) **Multi-Step**

x	1	5	2	7	4	8	4	7
y	−75	−20	−32	−31	−44	−16	−64	−39

a. Make a scatter plot of the data set. **b.** Sketch a line of best fit.

c. Give the equation of the line.

*2. (45) **Graphing Calculator** Use a graphing calculator to find the value of r for the data set below. Tell what this value means.

Weight of Bicycle (lb)	21.1	20.4	38.4	41.3	29.9	29.5	14.2	15.6	26.9
Price of Bicycle ($)	650	710	490	330	650	535	4500	5610	2200

*3. (43) **Measurement** Find the area of the graph of the system of linear inequalities.

$$y \le 10 \text{ and } y \ge \frac{3}{4}x + 2 \text{ and } y \ge -\frac{5}{2}x + 15$$

4. (42) **Window Display** A video store wants to display 4 identical DVD movies, 3 identical CDs, and 2 identical hand-held devices. How many distinguishable displays are possible?

5. (41) **Verify** Show that 16 cm, 40 cm, and 43 cm *cannot* be the lengths of the sides of a right triangle.

Simplify.

6. (40) $2\sqrt{27} - 3\sqrt{75}$

7. (40) $3\sqrt{2}\left(2\sqrt{2} - \sqrt{6}\right) \cdot 4\sqrt{3} + 2$

8. (44) $\dfrac{3}{5\sqrt{12}}$

9. (44) $\dfrac{14}{3\sqrt{75}}$

10. (44) $\dfrac{2}{3\sqrt{6}}$

11. (40) **Multiple Choice** Simplify $\sqrt[3]{250}$.

A 10 **B** $\sqrt[3]{10}$ **C** $2\sqrt[3]{5}$ **D** $5\sqrt[3]{2}$

12. (39) **Sightseeing** In 2007, tickets to the Space Needle's Observation Deck in Seattle Washington cost $16 per adult and $8 per child. A group of tourists has up to $200 to spend on tickets. Write and graph an inequality that shows how many of each ticket they could buy. Use the graph to explain whether they could buy 6 adult tickets and 15 children's tickets.

13. (45) **Error Analysis** A student claimed that the correlation coefficient for a set of data was 1.25. Why is the student incorrect?

14. (38) **Physics** One of the properties of metals is that they expand in size when heated. The coefficient of thermal expansion is an indication of how a particular metal increases in size. A metal cube is heated. The cube has side lengths, x. After heating, the linear dimensions of the cube increase by a value of a. What is the ratio of the new volume to the original volume?

15. (40) **Generalize** When do roots of negative numbers exist and when are they undefined?

16. (28) **Geography** On a map, a square is drawn around Washington DC. Write the perimeter to area ratio of the square as a simplified rational expression. Then find the ratio given that the length of each side of the square is 10 miles long.

17. (37) **Write** Can a rational number be a rational expression? Can a rational expression be a rational number? Explain.

Expand.

18. (19) $(x - 1)^3$

19. (19) $(x + 5)^3$

20. (36) Which is the equation of the line parallel to $y = -5x + 7$ that crosses the origin?

A $y = -5x$ **B** $y = -\frac{1}{5}x + 7$ **C** $y = 5x$ **D** $y = \frac{1}{5}x + 7$

21. (35) **Error Analysis** A student solved the equation $4x^2 - 20x + 25 = 0$ as shown below.

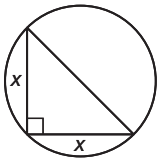
$$4x^2 - 20x + 25 = 0$$
$$(2x + 5)(2x - 5) = 0$$
$$2x + 5 = 0 \quad \text{or} \quad 2x - 5 = 0$$
$$2x = -5 \quad \text{or} \quad 2x = 5$$
$$x = -\frac{5}{2} \quad \text{or} \quad x = \frac{5}{2}$$

What is the error? Solve the equation correctly.

22. (34) **Temperature** The function $C = \frac{5}{9}(F - 32)$ describes the relationship between Celsius temperature and Fahrenheit temperature. Write the function in point-slope form. Graph the function, state the rate of change, and describe what it means.

23. (33) **Analyze** These characters are written on 10 index cards, a different character on each card: A, B, C, D, E, 1, 2, 3, 4, and 5. A letter card is chosen and then a number card is chosen. How many outcomes are in the event *choose a vowel and then choose an odd number*? List all those outcomes.

24. (31) **Geometry** An isosceles right triangle is inscribed in a circle as shown. Its hypotenuse is the diameter of the circle. Write the ratio of the area of the circle to the area of the right triangle in simplest form.



25. (32) **Write** Describe how to determine whether two matrices are inverses.

Solve.

26. (35) $-x + x^2 = 12$

27. (35) $-48x = -2x^2 - x^3$

28. (27) Write a quadratic function in vertex form where $a = -1$, $h = -2$, and $k = 3$.

Add.

29. (37) $\frac{4x}{x + 4} + \frac{6}{x + 2}$

30. (37) $\frac{3m}{m^2 + 3m + 2} - \frac{5m}{m + 1}$

LESSON
46

Finding Trigonometric Functions and their Reciprocals

Warm Up

1. (41) **Vocabulary** The longest side of a right triangle is the _____.

2. (7) Solve $4.2 = \frac{x}{35}$.

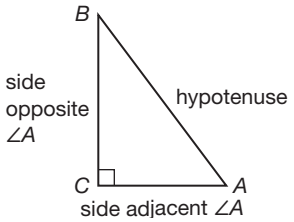
3. (7) Solve $0.06x = 42$.

4. (31) What is the reciprocal of $\frac{3}{4}$?

5. (41) Can the sides of a right triangle measure 20 inches, 25 inches, and 30 inches?

New Concepts

This lesson introduces **trigonometry,** which is the study of right triangles. A **trigonometric ratio** is a ratio of the lengths of any two sides of a right triangle. A function whose rule is a trigonometric ratio is a **trigonometric function.** Three common trigonometric functions are sine, cosine, and tangent, abbreviated sin, cos, and tan, respectively.

Trigonometric Functions		
The **sine** of angle A is the ratio of the length of the **opposite** side to the length of the **hypotenuse.**	$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB}$	
The **cosine** of angle A is the ratio of the length of the **adjacent** side to the length of the **hypotenuse.**	$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB}$	
The **tangent** of angle A is the ratio of the length of the **opposite** side to the length of the **adjacent** side.	$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC}$	

Math Language

The side **adjacent** to an angle is one of the sides that forms the angle but is not the hypotenuse.

To determine the sine, cosine, and tangent functions for $\angle B$ in the diagram above, be sure to use AC for the length of the opposite side and BC for the length of the adjacent side.

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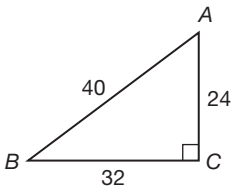
The trigonometric functions are only related to the acute angles of a right triangle. It does not make sense to find the sine, cosine, or tangent of $\angle C$.

Example 1 Finding the Values of Sine, Cosine, and Tangent

Find the values of the sine, cosine, and tangent of $\angle A$.



SOLUTION Identify the hypotenuse: $\overline{AB}$.

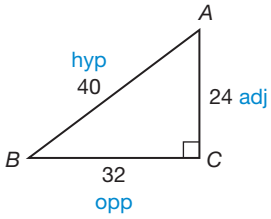
Identify the side opposite $\angle A$: $\overline{BC}$.

Identify the side adjacent $\angle A$: $\overline{AC}$.

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{32}{40} = \frac{4}{5} = 0.8$$

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{24}{40} = \frac{3}{5} = 0.6$$

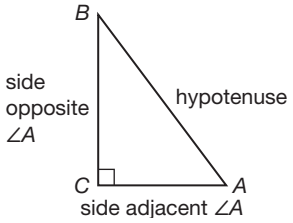
$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{32}{24} = \frac{4}{3} = 1.\overline{3}$$



Three other trigonometric functions are **cosecant, secant,** and **cotangent,** abbreviated csc, sec, and cot, respectively. They are the reciprocals of the sine, cosine, and tangent functions, respectively.

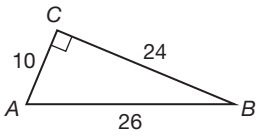
Math Reasoning

Verify Show how $\frac{1}{\sin A} = \frac{\text{hyp}}{\text{opp}}$.

Reciprocal Trigonometric Functions		
The **cosecant** of angle A is the reciprocal of the **sine** function.	$\csc A = \frac{1}{\sin A} = \frac{\text{hyp}}{\text{opp}} = \frac{AB}{BC}$	
The **secant** of angle A is the reciprocal of the **cosine** function.	$\sec A = \frac{1}{\cos A} = \frac{\text{hyp}}{\text{adj}} = \frac{AB}{CA}$	
The **cotangent** of angle A is the reciprocal of the **tangent** function.	$\cot A = \frac{1}{\tan A} = \frac{\text{adj}}{\text{opp}} = \frac{AC}{BC}$	

Example 2 Finding the Values of Cosecant, Secant, and Cotangent

Find the values of the cosecant, secant, and cotangent of $\angle B$.



SOLUTION Identify the hypotenuse: $\overline{AB}$.

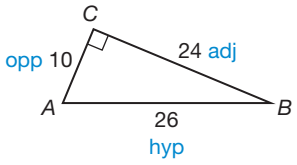
Identify the side opposite $\angle B$: $\overline{AC}$.

Identify the side adjacent $\angle B$: $\overline{BC}$.

$$\csc B = \frac{\text{hyp}}{\text{opp}} = \frac{26}{10} = \frac{13}{5} = 2.6$$

$$\sec B = \frac{\text{hyp}}{\text{adj}} = \frac{26}{24} = \frac{13}{12} = 1.08\overline{3}$$

$$\cot B = \frac{\text{adj}}{\text{opp}} = \frac{24}{10} = \frac{12}{5} = 2.4$$



The trigonometric ratios are related to the angle measures, rather than to the lengths of the sides. For instance, the ratios for a right triangle whose sides measure 3, 4, and 5 will be the same as for those whose sides measure twice as much, 6, 8, and 10, or three times as much, 9, 12, and 15. This is because the angle measures of similar triangles are the same.

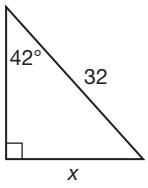
The values of the trigonometric ratios for given angles are listed in trigonometric tables, and are also stored in scientific and graphing calculators.

Graphing Calculator Tips

To put the calculator in degree mode, press MODE and use the arrow keys to switch from Radian to Degree.

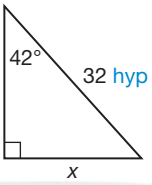
Example 3 Finding the Side of a Triangle

Find the value of x. Round to the nearest tenth.



SOLUTION

Identify the known and unknown lengths as the hypotenuse and the opposite side to the given angle.



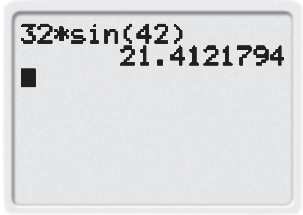
Use a trigonometric function that uses the hypotenuse and opposite side and substitute the given information.

$$\sin 42° = \frac{\text{opp}}{\text{hyp}}$$

$$\sin 42° = \frac{x}{32}$$ Solve for x.

$$x = 32(\sin 42°)$$ Use a calculator to find x.

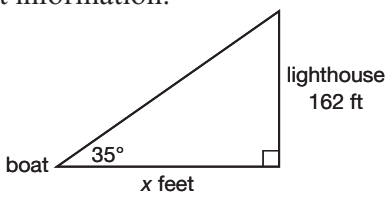
$$x \approx 21.4$$



Example 4 Application: Navigation

The top of the Yaquina Bay Lighthouse on the coast of Oregon is 162 feet above sea level. The angle of elevation from a boat to the top of the lighthouse is 35°. Find the distance from the boat to the base of the lighthouse to the nearest foot.

SOLUTION Draw a diagram with the relevant information.



$$\tan 35° = \frac{\text{opp}}{\text{adj}}$$

$$\tan 35° = \frac{162}{x}$$ Solve for x.

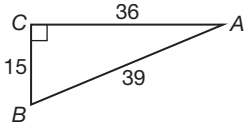
$$x = \frac{162}{\tan 35°}$$ Use a calculator to find x.

$$x \approx 231$$

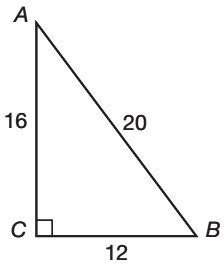
The boat is about 231 feet away from the base of the lighthouse.

Lesson Practice

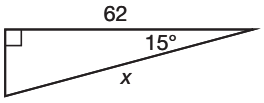
a. *(Ex 1)* Find the values of the sine, cosine, and tangent of $\angle A$.



b. (Ex 2) Find the values of the cosecant, secant, and cotangent of $\angle A$.



c. (Ex 3) Find the value of x. Round to the nearest tenth.



d. (Ex 4) A surveyor is located 380.7 feet from the base of the Chrysler Building in New York, NY. The angle of elevation from the surveyor to the top of the building is 70°. Find the height of the Chrysler Building to the nearest foot.

Practice Distributed and Integrated

For each equation and point, find the equation of the lines that are parallel and perpendicular to the line, and that pass through the point.

1. (36) $y = 3x - 6$; $(3, 4)$

2. (36) $y = -2x + 8$; $(-7, 0)$

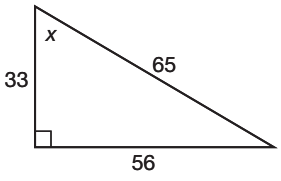
Convert:

3. (20) 1800 seconds to hours.

4. (20) 16 yd^2 to in^2.

5. (30) Write the vertex form of a quadratic equation if the parent function is shifted 5 units right.

Use the right triangle to find the value of the following trigonometric functions:



***6.** (46) $\tan(x)$

***7.** (46) $\csc(x)$

Factor the expressions:

8. (23) $-12x^4 - 18x^2 - 6$

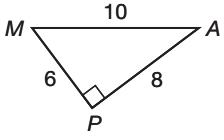
9. (23) $x^2 - 28x + 196$

Write the equation of the line with slope *m* that passes through the point, in slope-intercept form.

10. (26) $m = -8$, $(-2, 9)$

11. (26) $m = \frac{4}{9}$, $(6, 3)$

***12.** (46) **Multiple Choice** What is the value of the cosecant of $\angle M$?



A 0.6 **B** 0.8

C 1.25 **D** $1.\overline{7}$

***13.** (16) **Coordinate Geometry** Use Cramer's Rule to find the intersection of the lines with equations $9x + 3y = 21$ and $2x - 2y = 2$.

14. *(35)* **Gravity** The height of a free-falling object (neglecting air resistance) is given by the function $h(t) = -16t^2 + v_0t + h_0$, where h_0 is the initial height in feet, v_0 is the initial velocity in feet per second, and $h(t)$ is the height in feet at t seconds. An object is dropped from a height of 56 feet. How long does it take for the object to reach a height of 20 feet? (Hint: The initial velocity of a dropped object is zero.)

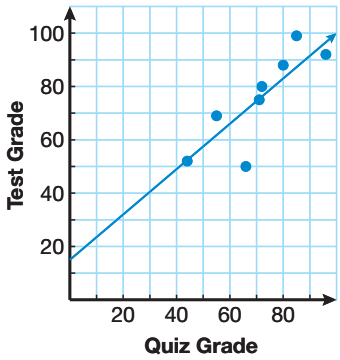
15. *(43)* **Analyze** What ordered pair of integers in the solution set of the system below has the least sum? Explain your answer.

$$y < \frac{3}{4}x - 3$$
$$y > \frac{1}{3}x + 4$$

***16.** *(45)* **Multi-Step** The graph compares students' quiz and test grades.

a. Write an equation for the line of best fit.

b. Describe the slope in terms of the data.



17. *(39)* **Error Analysis** Find and correct the error made below where a student was determining if $(4, -8)$ was a solution to $2y + 3 \leq -4x$.

$2(4) + 3 \stackrel{?}{\leq} -4(-8)$

$8 + 3 \stackrel{?}{\leq} 32$

$11 \leq 32$ True, $(4, -8)$ is a solution.

18. *(23)* **Multiple Choice** Circle the letter that represents the complete factorization of $x^2 - 9x - 36$.

A $(x + 6)(x - 6)$ **B** $(x + 4)(x - 9)$

C $(x + 3)(x - 12)$ **D** $(x + 2)(x - 18)$

19. *(40)* **Density** The side length of a metal cube, in centimeters, is given by $\frac{\sqrt[3]{m}}{\sqrt[3]{p}}$, where m is its mass in grams and p is its density.

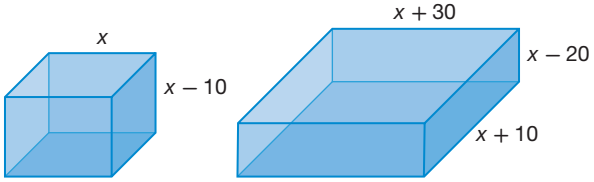
a. Write the expression in simplified form.

b. Approximate the side length of a 1000 gram cube of nickel given that the density of nickel is 88 g/cm^3.

***20.** *(45)* **Justify** Why does a table showing number of feet and the corresponding number of inches have a correlation of 1?

21. *(33)* **Baseball** There are 5 teams in the American League East division of professional baseball. How many ways can teams finish in first and second place?

22. (38) **Physics** Through the process of annealing, metals are made more flexible, allowing a piece of metal to be flattened without breaking. Suppose a rectangular piece of metal with square ends goes through the annealing process and is flattened into a thin sheet. What is the ratio of the new surface area to the original? Does the quotient have a zero remainder? Explain.



23. (41) **Cartography** A cartographer constructs a map of North Dakota over a grid system such that Bismarck is at (225, 87) and Minot is at (192, 232). Find the distance between the cities to the nearest kilometer if each unit represents one kilometer.

***24.** (32) **Graphing Calculator** If $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix}$, use a graphing calculator to find $AX = B$.

***25.** (46) **Write** How are the sine and cosine functions alike? How are they different?

26. (27) **Error Analysis** A student was asked to convert a function written in vertex form to its standard form. The student's work is shown below. Find and correct the error the student made.

$$\begin{aligned} f(x) &= -(x+3)^2 + 14 \\ &= -(x^2 + 9) + 14 \\ &= -x^2 - 9 + 14 \\ &= -x^2 + 5 \end{aligned}$$

27. (42) **Chess Tournament** Ten people enter a chess tournament. How many ways can matches be determined for the first round?

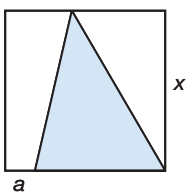
28. (36) **Formulate** A staircase needs to be built that crosses a horizontal distance of 10 feet and a vertical distance of 15 feet. Write an equation of a line that could model the handrail that is 3 feet above the stairs.

***29.** (46) **Sports** A baseball diamond is shaped like a square that is 90 feet on each side. A line drawn from second base to home plate forms a right triangle with acute angles measuring 45°.

a. Show how to use a trigonometric function to find the distance from second base to home base to the nearest tenth of a foot.

b. Check your answer to part a by using the Pythagorean Theorem.

30. (37) **Geometry** What is the ratio of the area of the shaded region to that of the unshaded region in the square to the right?



LESSON 47

Graphing Exponential Functions

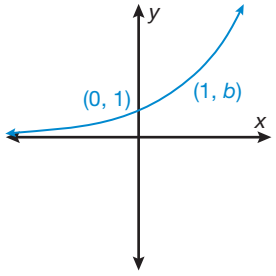
Warm Up

1. (3) **Vocabulary** In the expression 2^3, the exponent is 3 and the _____ is 2.
2. (3) What is the value of $\left(\frac{2}{5}\right)^3$?
3. (27) What is the minimum value of the function $y = x^2 - 5$?

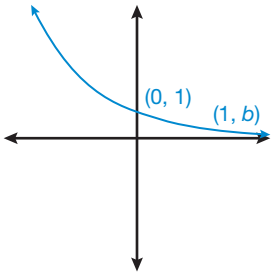
New Concepts

A function of the form $y = ab^x$ is an **exponential function** if x is a real number, $a \neq 0$, $b > 0$, and $b \neq 1$. The parent function of all exponential functions with base b is $y = b^x$.

Parent function $y = b^x$, $b > 1$



Parent function $y = b^x$, $0 < b < 1$



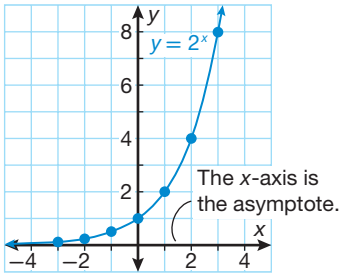
For each graph above, the line $y = 0$, or the x-axis, is an asymptote. An **asymptote** is a line that a graph approaches as the value of a variable becomes extremely large or small. The line $y = k$ is a **horizontal asymptote** of a graph if y approaches k as x increases or decreases without bound. The base of an exponential function must be positive, but the exponent can be negative. To graph exponential functions, use the property $b^{-m} = \left(\frac{1}{b}\right)^m$. For example, $2^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ and $\left(\frac{3}{2}\right)^{-2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$.

Example 1 Graphing $y = b^x$

Graph $y = 2^x$. Identify the domain, the asymptote, and the range.

SOLUTION Make a table, plot the ordered pairs, and draw a smooth curve through them.

x	-3	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



The domain is the set of all real numbers; the graph extends without end to the left and to the right. The asymptote is the line $y = 0$; as x decreases without bound, y approaches 0. The range is the set of all positive real numbers; the graph gets closer and closer to the x-axis from above, and it extends up without end.

Math Reasoning

Verify The domain of $y = 2^x$ includes numbers such as $x = \frac{1}{2}$ and $x = \sqrt{5}$.

Verify these approximate function values on the graph:

$2^{\frac{1}{2}} = \sqrt{2} \approx 1.4$

$2^{\sqrt{5}} \approx 4.7$

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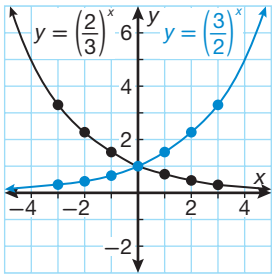
Example 2 Graphing $y = b^x$ and $y = \left(\frac{1}{b}\right)^x$

Graph $y = \left(\frac{3}{2}\right)^x$ and $y = \left(\frac{2}{3}\right)^x$.

SOLUTION

Make a table of values and graph each function.

x	-3	-2	-1	0	1	2	3
$y = \left(\frac{3}{2}\right)^x$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$
$y = \left(\frac{2}{3}\right)^x$	$\frac{27}{8}$	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$



Transformations of the Parent Function $y = b^x$	
$y = ab^x$	vertical stretch if $\lvert a\rvert > 1$
	vertical compression if $0 < \lvert a\rvert < 1$
	reflection over x-axis if $a < 0$
$y = b^{x-h} + k$	shift horizontally h units, vertically k units

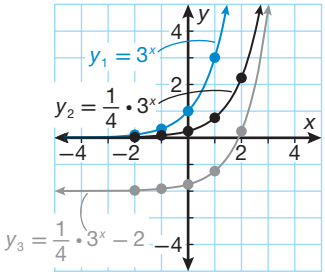
Example 3 Graphing Transformations of $y = b^x$

Graph the functions $y_1 = 3^x$, $y_2 = \frac{1}{4} \cdot 3^x$, and $y_3 = \frac{1}{4} \cdot 3^x - 2$. Describe the graphs, including domains and ranges.

SOLUTION

Make a table of values and graph each function.

x	-2	-1	0	1	2
$y_1 = 3^x$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$y_2 = \frac{1}{4} \cdot 3^x$	$\frac{1}{36}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{9}{4}$
$y_3 = \frac{1}{4} \cdot 3^x - 2$	$-\frac{71}{36}$	$-\frac{23}{12}$	$-\frac{7}{4}$	$-\frac{5}{4}$	$\frac{1}{4}$



y_1 is the parent function of all exponential functions with base 3. The domain of all the functions is the set of all real numbers. The range of y_1 and y_2 is the set of all positive real numbers. The range of y_3 is the set of all real numbers greater than -2.

The graph of y_2 is a vertical compression of the graph of y_1 by a factor of $\frac{1}{4}$.

The graph of y_3 is a vertical shift 2 units down of the graph of y_2.

Math Reasoning

Analyze What is the horizontal asymptote for the graph of y_3?

A function of the form $y = -b^x$ has the form $y = ab^x$, with $a = -1$.

The graph of $y = -b^x$ is a reflection over the x-axis of the graph of $y = b^x$.

Caution

$y = (-1.4)^x$ has a negative base and is not an exponential function.

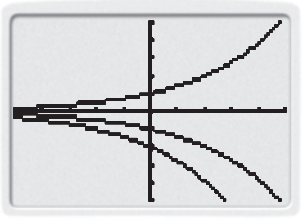
$-(1.4)^x = -1 \cdot (1.4)^x$

$-(1.4)^x \neq (-1.4)^x$

Example 4 Graphing Transformations of $y = b^x$ with a Calculator

Use a graphing calculator to graph the functions $y_1 = (1.4)^x$, $y_2 = -(1.4)^x$, and $y_3 = -2\,(1.4)^x$. Describe the graphs.

SOLUTION



y_1 is the parent function of all exponential functions with base 1.4.

The graph of y_2 is a reflection of the graph of y_1 over the x-axis.

The graph of y_3 is a vertical stretch of the graph of y_2 by a factor of 2.

Example 5 Application: Compound Interest

The amount A in an account that earns interest is given by the exponential function $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where P is the initial deposit, r is the annual interest rate, n is the number of times per year the interest is compounded, and t is the number of years. Suppose \$20,000 is deposited in a college savings account when a person is born. How much will be in the account on that person's 18th birthday if the account earns 9% compounded quarterly? If it is compounded daily?

SOLUTION

Math Reasoning

Write Is the function $A = P\left(1 + \frac{r}{n}\right)^{nt}$ in the form $y = ab^x$? Explain.

Understand Quarterly means 4 times per year. Daily means 365 times per year.

Plan Evaluate the function, letting $n = 4$ and $n = 365$.

Solve

Compounded Quarterly:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 20{,}000\left(1 + \frac{0.09}{4}\right)^{4 \cdot 18}$$

$$A = 20{,}000(1 + 0.0225)^{72}$$

$$A = 99{,}263.32$$

Compounded Daily:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 20{,}000\left(1 + \frac{0.09}{365}\right)^{365 \cdot 18}$$

$$A = 20{,}000\left(1 + \frac{0.09}{365}\right)^{6570}$$

$$A = 101{,}041.63$$

There will be \$99,263.32 in the account if compounded quarterly and \$101,041.63 if compounded daily.

Check The amounts from quarterly compounding and daily compounding are reasonable because the quarterly compounding should be less than the amount from the daily compounding.

Lesson Practice

a. (Ex 1) Graph $y = \left(\frac{1}{2}\right)^x$. Identify the domain, the asymptote, and the range.

b. (Ex 2) Graph $y = \left(\frac{3}{4}\right)^x$ and $y = \left(\frac{4}{3}\right)^x$. Describe the graphs.

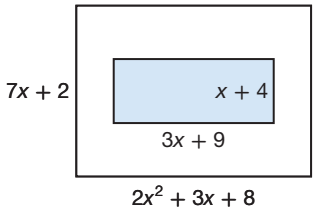
c. (Ex 3) Graph the functions $y_1 = 2^x$, $y_2 = \frac{1}{2} \cdot 2^x$ and $y_3 = \frac{1}{2} \cdot 2^x + 3$. Describe the graphs, including domains and ranges.

d. (Ex 4) Use a graphing calculator to graph the functions $y_1 = (1.6)^x$, $y_2 = -(1.6)^x$, and $y_3 = -\frac{1}{2}(1.6)^x$. Describe the graphs.

e. (Ex 5) Suppose \$1,000 is deposited in a savings account. How much will be in the account in 2 years if the account earns 6% compounded quarterly? If it is compounded monthly?

Practice Distributed and Integrated

1. (19) **Probability** The local amusement park has a dart game where you win a prize if you can throw a dart into the shaded area of a rectangle. The probability of getting a dart to land in the shaded area is $\frac{\text{area of the shaded area}}{\text{total area of the rectangle}}$. Write an expression that represents the probability of winning a prize for the dart game.



2. (38) **Multiple Choice** Which of the following is the remainder when dividing $3x^3 - 16x^2 + 12x + 16$ by $(x + 1)$?

A -16 **B** -15 **C** 0 **D** 15

3. (27) **Generalize** If the x-intercepts of a quadratic function are x_1 and x_2, give a general equation for the x-coordinate of the vertex.

***4.** (47) **Graphing Calculator** Graph the functions $y_1 = 2.3^x$, $y_2 = \frac{1}{6}(2.3)^x$ and $y_3 = -\frac{1}{6}(2.3)^x$ on a calculator.

Add:

5. (37) $\frac{5x^2}{pm} - 4 + \frac{c}{p^2m}$

6. (37) $3xy^2m + \frac{4}{x}$

7. (23) **Running** The expression representing a runner's speed is $120x^2 - 60x - 240$ feet per minute. Find the expression for the runner's speed in feet per second.

***8.** (43) **Multi-Step** A college student earns \$10 per hour working in the library and \$15 per hour on a landscaping crew. He decides to work not more than 20 hours in a week, and his goal is to earn at least \$240 in a week. Let x represent a number of hours working in the library and y represent a number of hours working on the landscaping crew.

a. Write and graph a system of linear inequalities that represents the situation.

b. Describe the portion of the solution set that contains meaningful solutions for this situation.

c. Determine the vertices of the polygon you described in part b. Describe what each vertex represents.

*9. (39) **Formulate** Write a list of rules describing which half-plane to shade and what type of line to use (dashed or solid) based on the inequality symbol in an inequality.

10. (35) **Gravity** The height of a free-falling object (neglecting air resistance) is given by the function $h(t) = -16t^2 + v_0t + h_0$, where h_0 is the initial height in feet, v_0 is the initial velocity in feet per second, and $h(t)$ is the height in feet at time t seconds. How long does it take for an object to hit the ground after it is thrown straight up at a speed of 36 feet per second from a height of 52 feet?

11. (15) Graph the system below. Describe the shape that the lines form.

$$0.9x - 1.5y = 0.21$$
$$-1.2x + 2.0y = -0.28$$
$$0.6x - y = 0.14$$

*12. (47) **Savings** London deposited $3000 into a savings account when she got her first job after graduating from college. How much will be in her account when she finishes graduate school in 2 years if the account earns 8.3% compounded daily?

Determine whether the ordered triples are solutions of the systems.

13. (29) Is (2, −5, 3) a solution of the system $\begin{aligned} 4x + 3y + 5z &= 8 \\ -6x + 3y + 7z &= -6 \\ 8x + y - 2z &= 5 \end{aligned}$?

14. (29) Is (0, 1, 2) a solution of the system $\begin{aligned} 10x + 9y + 6z &= 21 \\ -x + 4y - 7z &= -10 \\ -9x + 5y + 2z &= 9 \end{aligned}$?

*15. (44) **Write** Explain why multiplying a radical denominator by its radical conjugate eliminates the radical from the denominator.

16. (16) **Analyze** Explain the difficulty you encounter when trying to use Cramer's Rule to solve the system $\begin{aligned} 4x - 12y &= -4 \\ -6x + 18y &= -6 \end{aligned}$

17. (41) **Verify** Show that 30 cm, 72 cm, and 78 cm can be the lengths of the sides of a right triangle.

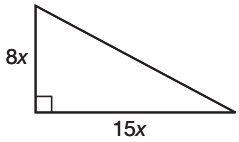
*18. (45) **Depreciation** The table shows the trade-in value for seven cars in good condition with typical mileage and standard equipment. Make a scatter plot of the data, using the number of years before and after 2000 on the x-axis. Sketch a line of best fit.

	A	B	C	D	E	F	G
Year	1987	1992	1996	1999	2002	2005	2007
Value ($)	450	650	1950	2050	3435	4265	8610

19. *(40)* **Error Analysis** A student submitted the work below and reported that $\sqrt{8} \cdot \sqrt{6}$ simplifies to $2\sqrt{12}$. Explain her error.

$$\sqrt{8} \cdot \sqrt{6} = \sqrt{4} \cdot \sqrt{2} \cdot \sqrt{6} = 2\sqrt{2}\sqrt{6} = 2\sqrt{12}$$

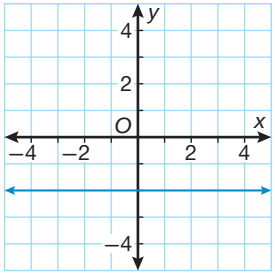
20. *(31)* **Geometry** Write a rational expression in simplest form for the ratio of the triangle's perimeter to its area.



21. *(36)* **Economics** The exchange rate from dollars to different currencies is shown. A company charges a fee of \$2.50 for amounts up to \$50 and \$7.50 for amounts above \$50. Write the equations for exchanging British pounds to dollars. What do the graphs of the equations have in common?

Currency	US \$
1 Euro =	1.377
1 Canadian dollar =	0.9472
1 British pound =	2.0284

22. *(13)* **Error Analysis** A student incorrectly graphed the equation $x = -2$ as shown. What equation does the graph represent? Explain how you know.



23. *(28)* **Multi-Step** Consider the expression $\frac{3x - 6}{2}$.

a. Is the expression simplified? Tell why or why not.

b. Are there any excluded values? Explain.

***24.** *(46)* **Tourism** The angle of elevation from a tourist on the beach in Cape May, New Jersey to the top of the lighthouse is 38°. The lighthouse has a height of 158 feet. To the nearest foot, how far away is the tourist from the base of the lighthouse?

25. *(15)* Graph the system below. Describe the shape that the lines form.

$$4x - 2y = 2$$
$$-6x - 3y = 3$$
$$x - 2y = 6$$

26. *(42)* **Multiple Choice** How many different committees of three people can be formed from a group of 15?

A 455 **B** 910 **C** 1365 **D** 2370

Write the distance from each point to the origin as an absolute value expression. Then simplify the expression.

27. *(17)* (5, 0) **28.** *(17)* (−7, 0) **29.** *(17)* (0, 9)

***30.** *(44)* Which of the following shows the simplified radical expression for $\frac{4\sqrt{7} - 2}{5\sqrt{7} + 3}$?

A $\frac{73 - 11\sqrt{7}}{83}$ **B** $\frac{54}{83}$ **C** $\frac{134 + 22\sqrt{7}}{184}$ **D** $\frac{108}{146 + 19\sqrt{7}}$

LESSON
48

Understanding Complex Fractions

Warm Up

1. (31) **Vocabulary** $\frac{2}{x}$ is the _____ of $\frac{x}{2}$.

2. (31) Divide and write the quotient in simplest form: $\frac{3}{10} \div \frac{6}{5}$.

3. (37) What is the Least Common Denominator of $\frac{1}{4x}$, $\frac{5}{6x}$, and $\frac{x}{x+5}$?

New Concepts

A **complex fraction** is a fraction that contains one or more fractions in its numerator or denominator.

Some examples of complex fractions are shown below.

$$\frac{(x-2)}{\frac{5}{x}} \qquad \frac{2+\frac{1}{3}}{2x+5} \qquad \frac{x+\frac{2}{x}}{x-\frac{1}{3x}}$$

Simplifying Complex Fractions

Method 1: Simplify as needed to write the numerator and the denominator each as a single term. Then divide.

Method 2: Multiply the numerator and the denominator by the Least Common Denominator (LCD) of all the fractions in the complex fraction. Then simplify.

Example 1 Simplifying a Numerical Complex Fraction (Method 1)

Simplify $\dfrac{\frac{1}{6}+\frac{3}{4}}{2+\frac{1}{5}}$.

SOLUTION

$$\frac{\frac{1}{6}+\frac{3}{4}}{2+\frac{1}{5}} = \frac{\frac{2}{12}+\frac{9}{12}}{\frac{10}{5}+\frac{1}{5}}$$

Add the terms in the numerator. Add the terms in the denominator.

$$= \frac{\frac{11}{12}}{\frac{11}{5}}$$

Simplify.

$$= \frac{11}{12} \cdot \frac{5}{11}$$

To divide by a fraction, multiply by its reciprocal.

$$= \frac{5}{12}$$

Hint

You need to use an LCD in Method 1, but not necessarily the LCD of *all* fractions in the complex fraction. The LCD of $\frac{1}{6}$ and $\frac{3}{4}$ is 12.

Online Connection
www.SaxonMathResources.com

Example 2 Simplifying Algebraic Complex Fractions (Method 2)

a. Simplify $\dfrac{3+\frac{2}{a}}{5-\frac{1}{b}}$.

SOLUTION

$$\frac{3+\frac{2}{a}}{5-\frac{1}{b}} = \frac{\left(3+\frac{2}{a}\right)\cdot ab}{\left(5-\frac{1}{b}\right)\cdot ab}$$

The LCD of $\frac{2}{a}$ and $\frac{1}{b}$ is ab. Multiply the numerator and denominator by ab.

$$= \frac{3ab+2b}{5ab-a}$$

Distribute ab to simplify.

b. Simplify $\dfrac{\frac{3}{x}+\frac{1}{x-5}}{\frac{4x}{x-5}}$.

SOLUTION

$$\frac{\frac{3}{x}+\frac{1}{x-5}}{\frac{4x}{x-5}} = \frac{\left(\frac{3}{x}+\frac{1}{x-5}\right)\cdot x(x-5)}{\left(\frac{4x}{x-5}\right)\cdot x(x-5)}$$

The LCD of $\frac{3}{x}$, $\frac{1}{x-5}$, and $\frac{4x}{x-5}$ is $x(x-5)$. Multiply the numerator and denominator by $x(x-5)$.

$$= \frac{3(x-5)+x}{4x\cdot x}$$

Simplify.

$$= \frac{4x-15}{4x^2}$$

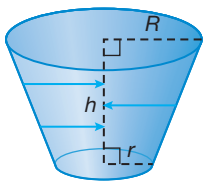
Caution

Do not divide out factors of individual terms to attempt to simplify in a rational expression.

$\frac{3ab+2b}{5ab-a} \neq \frac{3+2b}{5-a}$.

Example 3 Application: Solar Heating

A solar collector in the shape of a frustum of a cone heats collected water by reflecting the sun's rays from its lateral surface toward its vertical axis. Express the volume formula in simplest form in terms of h and R if the radius r of the smaller base is half the radius R of the larger base.



Volume: Frustum of a Cone

$$V = \frac{\pi h(r^2+rR+R^2)}{3}$$

SOLUTION

Substitute $\frac{R}{2}$ for r, and then simplify.

$$V = \frac{\pi h(r^2+rR+R^2)}{3} = \frac{\pi h\left(\left(\frac{R}{2}\right)^2+\left(\frac{R}{2}\right)R+R^2\right)}{3} = \frac{\pi h\left(\frac{R^2}{4}+\frac{R^2}{2}+R^2\right)}{3}$$

$$= \frac{\pi h\left(\frac{R^2}{4}+\frac{R^2}{2}+R^2\right)\cdot 4}{3\cdot 4} = \frac{\pi h(R^2+2R^2+4R^2)}{12} = \frac{7\pi hR^2}{12}$$

Check The expression is in simplest form in terms of h and R.

Math Language

A **frustum** of a cone is formed by a plane cutting through a cone, perpendicular to its vertical axis. It has two parallel circular bases of different sizes.

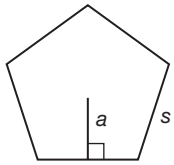
Lesson Practice

a. (Ex 1) Simplify $\dfrac{\frac{1}{10}-1}{\frac{5}{8}+\frac{3}{20}}$.

b. (Ex 2) Simplify $\dfrac{\frac{5}{x}-1}{\frac{3}{y}+1}$.

c. (Ex 2) Simplify $\dfrac{\frac{a}{b}-\frac{c}{d}}{\frac{a}{b}+\frac{c}{d}}$.

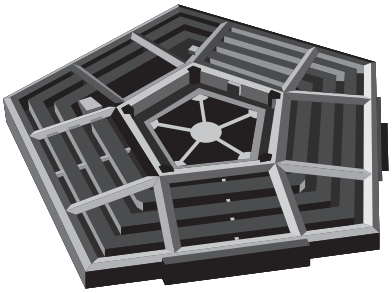
Hint



Area: Regular n-sided Polygon

$A = \frac{1}{2}nas$

d. (Ex 3) The Pentagon building in Washington D.C. has the approximate shape of a regular pentagon. In a regular pentagon, the apothem a and the side length s are related by $a = \frac{\frac{s}{2}}{0.727}$ (where 0.727 is correct to the nearest thousandth). Express the formula for the area of a regular pentagon in simplest form in terms of s.



Practice Distributed and Integrated

Simplify:

***1.** (48) $\dfrac{\frac{x}{m+p}}{\frac{y}{m+p}}$

***2.** (48) $\dfrac{x+\frac{4xy}{x}}{\frac{1}{x}-y}$

***3.** (48) $\dfrac{\frac{m}{p}+\frac{3}{xp}}{\frac{y}{p}}$

4. (20) **Basketball** The length of a regulation NCAA basketball court is given by the function $f(x) = x + 24$ feet. The width of a regulation NCAA basketball court is given by the function $g(x) = 2x - 90$ feet. Write a function $h(x)$ that represents how many feet longer the court is than it is wide.

5. (Inv 2) **Airplane Altitude** A plane's altitude increases at a rate of 40 ft/s as it ascends after takeoff. Its ground distance from the airport is increasing at a rate of 240 ft/s. What is the location of the plane 20 seconds after takeoff?

6. (36) **Justify** Is the triangle defined by vertices $(-3, 6)$, $(4, 4)$, and $(1, -8)$ a right triangle? Explain your answer.

7. (44) **Geometry** The area of a rectangular garden is 30 ft^2. The width of the garden is $(\sqrt{3}+1)$ ft. Find a simplified expression for the length of the garden.

Use the description to write the vertex form of a quadratic equation.

8. (30) The parent function is shifted 5 units right.

9. (30) The parent function is shifted 2 units to the left and 7 units up.

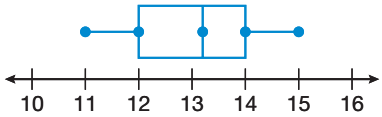
10. (40) **Write** Explain why absolute value symbols are used in the following sentence. For any real number x, $\sqrt{x^2} = |x|$.

11. (22) **Data Analysis** A store records the number of video games sold each month. Examine the table. Does this table represent a continuous, discontinuous, and/or discrete function? Justify your answer.

Month	Games Sold
1	10
2	8
3	9
4	20

***12.** (42) **Verify** If you have 5 digits that can be arranged in any order, then specify that one digit must be in the second position; verify that the number of arrangements of the other four digits must be 4!

13. (25) **Error Analysis** A chemist weighs samples obtained from a production run. The weights of the samples are 12 g, 14 g, 65 g, 11 g, 15 g, 14 g, 14 g, 12 g, 13 g, 15 g, 14 g, and 11 g. He eliminated the outlier and drew a box-and-whisker plot to represent the data. What is his error?



***14.** (48) Simplify $\dfrac{\frac{5}{2x} - \frac{2}{x+5}}{\frac{1}{2(x+5)} + \frac{1}{x}}$.

15. (27) **Formulate** **a.** Use a table to graph $f(x) = (x-2)^2 + 6$.

b. Compare the graph to its function. Formulate a conjecture about how you can find the vertex of a parabola based on its related function given in vertex form.

Simplify:

16. (40) $5\sqrt{18} - 10\sqrt{50} + 3\sqrt{72}$

17. (40) $3\sqrt{18}\left(4\sqrt{2} - 2\sqrt{3}\right)$

18. (40) $2\sqrt{\frac{2}{9}} - 3\sqrt{\frac{9}{2}}$

19. (40) $-3\sqrt{\frac{2}{3}} + 2\sqrt{\frac{3}{2}}$

20. (40) $3\sqrt{24}\left(2\sqrt{6} - 3\sqrt{12}\right)$

21. (30) **Multi-Step** **a.** Graph the function $y = (x-5)^2 + 1$.

b. On the same graph, sketch the reflection of the parabola across the x-axis.

c. Give the equation of the function represented by the reflected parabola.

22. *(24)* **Multiple Choice** To solve by elimination, which linear system will require that both equations first be multiplied by a constant before the equations are added?

A $\begin{aligned} 4x - 4y &= 0 \\ -4x + 6y &= 2 \end{aligned}$ **B** $\begin{aligned} 4x - 4y &= 0 \\ 4x + 5y &= 9 \end{aligned}$ **C** $\begin{aligned} 4x - 4y &= 0 \\ -\frac{1}{4}x + \frac{5}{4}y &= 1 \end{aligned}$ **D** $\begin{aligned} 4x - 4y &= 0 \\ 3x + 5y &= 8 \end{aligned}$

23. *(37)* **Error Analysis** Two students are adding the rational expressions $\frac{1}{x+3}$ and $\frac{1}{x-3}$ but get different results. Which student made the mistake?

Student A	Student B
$\frac{1}{x+3} + \frac{1}{x-3} = \frac{x-3}{(x+3)(x-3)} + \frac{x-3}{(x-3)(x+3)} = \frac{2x-6}{x^2-9}$	$\frac{1}{x+3} + \frac{1}{x-3} = \frac{x-3}{(x+3)(x-3)} + \frac{x+3}{(x-3)(x+3)} = \frac{2x}{x^2-9}$

*__24.__ *(47)* **Write** Describe the transformations you need to make to the graph of $y = \left(\frac{1}{5}\right)^x + 3$ to get the graph of $y = 5^x + 1$.

*__25.__ *(15)* **Graphing Calculator** Solve the system of equations using a graphing calculator.

$$\begin{aligned} 5x + y &= 15 \\ 2y &= 5 + 10x \end{aligned}$$

*__26.__ *(48)* **Verify** Simplify the expression on the left side of the equation; verify that it is equal to the right side.

$$\frac{\frac{4}{x+2} - \frac{2}{x+3}}{\frac{2}{x+3}} = \frac{x+4}{x+2}$$

27. *(39)* **Money** Dejah is collecting pennies and nickels for a charity event. She's going to continue until her collection of coins weighs at least 2 kilograms. Every penny weighs 2.5 grams and every nickel weighs 5 grams. Write and graph an inequality that represents how many of each coin are needed.

28. *(18)* **Swimming** In order to qualify for the 2008 Olympic Swim Team, a male swimmer must swim the 100-meter freestyle in under 51.59 seconds.

a. What is this time in feet per second? (Hint:1 meter = 3.2808399 feet)

b. If a male swimmer swims the 100-meter freestyle in 48.62 seconds, what is his time in feet per second?

*__29.__ *(47)* **Compound Interest** Ann deposits $2,800 in a savings account. How much will be in her account in 3 years if the account earns 4% compounded annually? If it is compounded semi-annually?

30. *(41)* **Write** When using the Pythagorean Theorem to find a missing side of a right triangle, how is finding the length of a leg different from finding the length of the hypotenuse?

LESSON
49

Using the Binomial Theorem

Warm Up

1. (42) **Vocabulary** A _____ is a selection of items where order doesn't matter.
2. (42) Expand $(x + 2)^2$
3. (42) Evaluate 5!
4. (42) Evaluate ${}_6C_2$
5. (42) Evaluate $\frac{2!}{6!}$

New Concepts

Expand a few powers of the binomial $(a + b)$.

$$(a + b)^0 = 1a^0b^0$$
$$(a + b)^1 = 1a^1b^0 + 1a^0b^1$$
$$(a + b)^2 = 1a^2b^0 + 2a^1b^1 + 1a^0b^2$$
$$(a + b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$

Math Reasoning

Generalize What pattern do you observe in the exponents for each term?

The coefficients form a pattern known as Pascal's Triangle.

$(a + b)^0$ 1

$(a + b)^1$ 1 1

$(a + b)^2$ 1 2 1

$(a + b)^3$ 1 3 3 1

$(a + b)^4$ 1 4 6 4 1

Example 1 Expanding a Binomial Using Pascal's Triangle

Expand $(a + b)^4$.

SOLUTION

Use the fourth row of Pascal's Triangle to write the coefficients.

$$(a + b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Another way to show the coefficients is by writing them as combinations.

Hint

The top row of Pascal's Triangle is called the zero row.

$(a + b)^0$ ${}_0C_0$

$(a + b)^1$ ${}_1C_0$ ${}_1C_1$

$(a + b)^2$ ${}_2C_0$ ${}_2C_1$ ${}_2C_2$

$(a + b)^3$ ${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$

$(a + b)^4$ ${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$

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The pattern can be summarized in the **Binomial Theorem.**

Math Language

The Greek capital letter *sigma*, Σ, is used to notate the sum of a series.

Binomial Theorem

If n is a nonnegative integer, then

$$(a+b)^n = ({}_nC_0)a^nb^0 + ({}_nC_1)a^{n-1}b^1 + ({}_nC_2)a^{n-2}b^2 + \ldots + ({}_nC_{n-1})a^1b^{n-1} + ({}_nC_n)a^0b^n$$

$$= \sum_{r=0}^{n}({}_nC_r)a^{n-r}b^r \text{ where } {}_nC_r = \frac{n!}{r!(n-r)!}$$

Example 2 Expanding a Binomial Using the Binomial Theorem

Use the Binomial Theorem to expand each power.

a. $(x+y)^5$

SOLUTION

Use the binomial theorem to expand $(a+b)^n$, where $a = x$, $b = y$, and $n = 5$.

$$(x+y)^5 = ({}_5C_0)x^5y^0 + ({}_5C_1)x^4y^1 + ({}_5C_2)x^3y^2 + ({}_5C_3)x^2y^3 + ({}_5C_4)x^1y^4 + ({}_5C_5)x^0y^5$$

Evaluate the coefficients and simplify.

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

b. $(2r-s)^3$

SOLUTION

Use the binomial theorem to expand $(a+b)^3$. Substitute $2r$ for a and $-s$ for b.

$$(a+b)^3 = ({}_3C_0)a^3b^0 + ({}_3C_1)a^2b^1 + ({}_3C_2)a^1b^2 + ({}_3C_3)a^0b^3$$

$$= (2r)^3 + 3(2r)^2(-s) + 3(2r)(-s)^2 + (-s)^3$$

$$= (2)^3r^3 + 3(2)^2r^2(-s) + 3(2r)(-s)^2 + (-s)^3$$

$$= 8r^3 - 12r^2s + 6rs^2 - s^3$$

Hint

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$${}_5C_2 = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3!}{2!3!} = \frac{5 \cdot 4}{2!}$$

Caution

$(2r)^3 \neq 2r^3$

$(2r)^3 = 2^3r^3$

Example 3 Finding the *n*th Term of a Binomial

Find the fifth term of the expansion of $(p+q)^{10}$.

SOLUTION

First, use the Binomial Theorem to write the expansion in summation notation.

$$(p+q)^{10} = \sum_{r=0}^{10}({}_{10}C_r)p^{10-r}q^r$$

In the fifth term, $r = 4$.

$$({}_{10}C_r)p^{10-r}q^r = ({}_{10}C_4)p^{10-4}q^4$$

$$= \frac{10!}{(10-4)!4!}p^{10-4}q^4$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}p^6q^4$$

$$= 210p^6q^4$$

The Binomial Theorem can also be used to find probabilities for a **binomial experiment.** A binomial experiment has exactly two possible outcomes for each trial, where one outcome is a success (p) and one outcome is a failure (q). The probability of success is the same for each trial. Because there are only two outcomes, $p + q = 1$ or $q = 1 - p$.

These are some examples of binomial experiments.

Experiment	Success	Failure	P(success)	P(failure)
100 flips of a coin	Heads	Tails	$p = 0.5$	$p = 0.5$
20 rolls of a number cube	Roll a 2	Roll 1, or 3–6	$p = \frac{1}{6}$	$p = \frac{5}{6}$

Binomial Probability

If p is the probability of success and q is the probability of failure in one trial of a binomial experiment, then the **binomial probability** of exactly n successes in m trials is given by

$$ {}_mC_n p^n q^{m-n} $$

Example 4 Finding a Binomial Probability

Suppose you toss a coin 4 times. Use the Binomial Theorem to find each probability.

a. P(exactly 2 tails)

SOLUTION There are ${}_4C_2$ ways of getting 2 tails.

$$P(\text{exactly 2 tails}) = ({}_4C_2)p^2q^2$$
$$= ({}_4C_2)\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$$
$$= \frac{6}{16} = \frac{3}{8}$$

The probability of getting exactly 2 tails in 4 tosses is $\frac{3}{8}$.

b. P(at least 2 tails)

SOLUTION Instead of adding the probabilities of getting exactly 2, 3, and 4 tails, it may be easier to subtract the probabilities of getting exactly 0 or 1 tail from 1.

$$P(\text{at least 2 tails}) = 1 - P(0 \text{ tails}) - P(1 \text{ tail})$$
$$= 1 - ({}_4C_0)p^0q^4 - ({}_4C_1)p^1q^3$$
$$= 1 - \left(\frac{1}{2}\right)^4 - 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3$$
$$= \frac{11}{16}$$

The probability of getting at least 2 tails in 4 tosses is $\frac{11}{16}$.

Graphing Calculator Tip

For help calculating combinations see the lab on page 303.

Example 5 Using a Calculator to Find Binomial Probability

Use a calculator to find the probability of getting exactly 50 heads if you toss a coin 100 times.

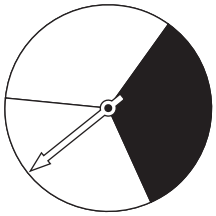
SOLUTION

There are ${}_{100}C_{50}$ ways of getting exactly 50 heads.

$$P(50 \text{ heads}) = ({}_{100}C_{50})h^{50}t^{50}$$
$$= ({}_{100}C_{50})\left(\frac{1}{2}\right)^{50}\left(\frac{1}{2}\right)^{50}$$
$$\approx 0.08$$

Example 6 Application: Spinner Probabilities

Suppose you spin the following spinner 3 times. What is the probability that you will spin white exactly 2 times?



SOLUTION

Understand The probability of spinning white is $\frac{2}{3}$ and the probability of spinning black is $\frac{1}{3}$.

Plan Find the probability of spinning white 2 times.

Solve $P(\text{spinning white 2 times}) = ({}_3C_2)p^2q^1$

$$= ({}_3C_2)\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1$$
$$= 3\left(\frac{4}{27}\right)$$
$$= \frac{12}{27} = \frac{4}{9}$$

The probability of spinning white exactly 2 times is $\frac{4}{9}$.

Lesson Practice

(Ex 1) **a.** Use Pascal's Triangle to expand $(2x + 4)^3$.

Use the Binomial Theorem to expand each expression.

(Ex 2) **b.** $(f + 2)^5$

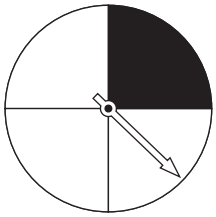
c. $(1 - 7x)^3$

(Ex 3) **d.** Find the fourth term of the expansion of $(x + 3)^{10}$.

(Ex 4) **e.** Find the probability of getting exactly 5 tails if a coin is tossed 6 times.

(Ex 4) **f.** Find the probability of getting at least 1 tail if a coin is tossed 6 times.

(Ex 6) **g.** Suppose you spin the following spinner 4 times. What is the probability that you will spin white exactly 1 time?



Practice Distributed and Integrated

Simplify.

1. (23) $\dfrac{4xy + 4x^2y^2}{4xy}$

2. (23) $p^5x^3 + p^4x^2 - p^3x$

***3.** (48) $\dfrac{ax^2 - \frac{4}{a}}{\frac{x^2}{a} + 6}$

***4.** (48) $\dfrac{\frac{m^2p}{x} - 6}{m^2p - \frac{4}{x}}$

5. (40) $6\sqrt{18} + 5\sqrt{8} - 3\sqrt{50}$

6. (40) $2\sqrt{5}(3\sqrt{15} - 2\sqrt{5})$

7. (24) Solve by using either substitution or elimination.

$$3x - 2y = 10$$
$$y = -\frac{1}{2}$$

Expand.

***8.** (49) $(b - 2)^4$

***9.** (49) $(2x + y)^6$

10. (36) Find the equation of the line that passes through (2, 4) and is perpendicular to the line that passes through the points (2, 4) and (−3, −2).

11. (31) **Analyze** Given the functions $f(x) = \frac{2x - 2}{x}$ and $g(x) = \frac{x^2 + 3x}{x - 1}$, describe the graph of the product function $f(x) \cdot g(x)$. Include points of discontinuity in the description. (Note: If a function is undefined at some value of x, then there is a point of discontinuity in the function graph at that value of x.)

***12.** (49) **Baseball** A baseball player has a batting average of 0.32 (32 hits in 100 times at bat). Find each probability for the next 4 times at bat.

a. P(exactly 2 hits)

b. P(at least 1 hit)

13. (45) **Multi-Step**

x	0	1	3	3	4	5	5	6	7	7	7
y	8	11	5	8	6	3	7	5	7	3	1

a. Make a scatter plot of the data

b. Sketch the line of best fit and write its equation.

c. Estimate the value of r.

14. (22) **Data Analysis** A customer visits the post office to weigh a package. The cost of mailing a package depends on its weight. Examine the table below. Does this table represent a continuous, discontinuous, or discrete function? Justify your answer.

Weight	Cost
$0 \le x < 5$	\$2.00
$5 \le x < 10$	\$4.00
$10 \le x < 15$	\$6.00
$15 \le x < 20$	\$8.00
$20 \le x < 25$	\$10.00

15. (43) **Ticket Prices** Two of the ticket prices for a 2007 baseball game were: \$27 for an outfield grandstand seat and \$45 for a right field box seat. Suppose you are an automobile dealer. You decide to buy tickets to give away during a sale. You decide to buy up to 24 tickets and to spend up to \$720. Let x represent a number of outfield grandstand seats and y represent a number of right field box seats.

a. Write and graph a system of linear inequalities that represents the situation.

b. Describe the points in the graph of the system that represent all the possible combinations of outfield grandstand seats and right field box seats you can buy.

c. Determine the point of intersection of the boundary lines in the graph and explain its meaning.

16. (41) **Multiple Choice** Which could be the sides of a right triangle?

A 21, 72, and 75 **B** 30, 35, and 45 **C** 35, 58, and 48 **D** 41, 82, and 90

17. (17) **Health** An adult is considered to be in a healthy weight range if his or her BMI (Body Mass Index) is within 3.2 of 21.7. Write an absolute value equation to determine the minimum and maximum BMI that is considered healthy. And then solve the equation

***18.** (49) **Write** Describe how to expand the binomial $(y + z)^5$ using Pascal's Triangle.

19. (25) **Geometry** The table below shows the measurements in inches of various shipping boxes available from the United States Postal Service.

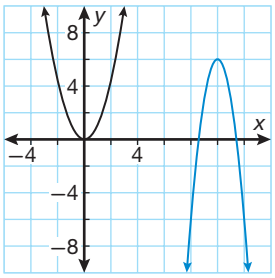
Box	Measurement 1	Measurement 2	Measurement 3
1	11.875	3.375	13.625
2	11	8.5	5.5
3	7.5	5.125	14.375

a. What is the range of volumes?

b. If you had to choose from these boxes to ship something, how could the range of volumes be useful?

20. (42) **Lottery** A lottery game contains 24 numbers. In order to win, four numbers must be chosen that match the winning numbers. If the winning numbers are selected one at a time and not replaced, how many possible ways are there for the winning numbers to be determined?

21. (30) **Estimate** The graphs of the parent quadratic function and a second function are shown. Use the graph to estimate the value of a for the equation of the second function.



22. (44) **Error Analysis** A student worked out the problem below using radical conjugates. Is the procedure correct? If not, identify and correct the error.

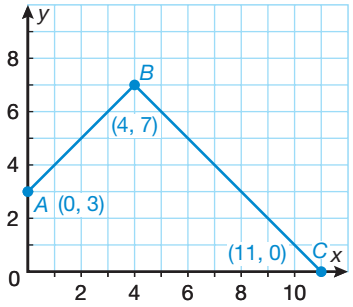
Simplify: $\frac{2\sqrt{2}+6}{3\sqrt{2}+\sqrt{5}}$.

Student work: $\frac{2\sqrt{2}+6}{3\sqrt{2}+\sqrt{5}} \cdot \frac{2\sqrt{2}-6}{3\sqrt{2}-\sqrt{5}} = \frac{8-36}{18-5} = -\frac{28}{13}$

23. (36) **Multi-Step** The speed of an object at time t is based on its initial speed (v_o) and its acceleration (a): $v = v_o + at$.

a. Find the equations of the lines from A to B and B to C and describe the motion of the object.

b. What is the relationship between the equations of motion?



24. (46) **Architecture** The Petronas Towers in Malaysia are two twin towers joined by a sky bridge that is 192 feet long and 558 feet above the ground. The angle of elevation from either end of the sky bridge to the top of the opposite tower is about 78.27°. Find the height of the towers to the nearest foot.

25. (28) **Verify** Prove that $2x - 7 = -(7 - 2x)$. Supply a reason for every step.

***26.** (40) **Graphing Calculator** Graph $y = \sqrt[3]{x}$ and $y = \sqrt[4]{x}$ on a graphing calculator. (Use the equivalent equations of $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{4}}$.) How do the graphs confirm the differences in rules for $\sqrt[n]{b}$ as far as n being odd or even?

***27.** (49) **Multiple Choice** What is the fifth term of the expansion of $(y - 3)^8$?

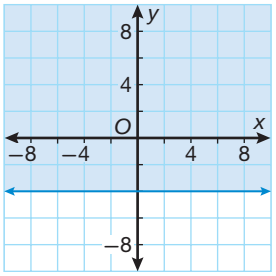
A $-13{,}608y^3$ **B** $5670y^4$ **C** $1890y^5$ **D** $1458y^6$

***28.** (48) **Multi-Step** Use the formula $r = \frac{d}{t}$ relating rate, distance, and time. A car traveled a distance of $3x + 60$ miles in 2 hours. A second car traveled $2x + 40$ miles in 3 hours.

a. Write a rational expression to represent the ratio of the rate of the first car to the rate of the second car.

b. Simplify the rational expression. What does this expression tell you about the rates of the two cars?

29. (39) **Error Analysis** Explain the error(s) the student made when graphing $-4 > y$.



***30.** (47) **Multiple Choice** Which of the following equations does **not** represent an exponential function?

A $y = -3^x$ **B** $y = x^3$ **C** $y = \left(\frac{1}{3}\right)^x$ **D** $y = 3^{-x}$

LESSON
50

Finding Inverses of Relations and Functions

Warm Up

1. (4) **Vocabulary** If (x, y) is an ordered pair in a relation, then x is an element of the domain and y is an element of the _____.

2. (7) Solve $2y - 8 = 4x$ for y.

3. (30) What is the reflection image of $(2, 6)$ over the y-axis?

New Concepts

A relation is a set of ordered pairs. If r represents a relation, then the **inverse relation** is the set of ordered pairs obtained by reversing the coordinates in each ordered pair of r. So if (a, b) is in relation r, then (b, a) is in the inverse relation. The inverse of a relation may or may not be a function. If a relation and its inverse are both functions, they are **inverse functions.**

Math Language

A **function** is a relation in which each element in the domain is paired with exactly one element in the range.

Example 1 Describing a Relation and its Inverse

Find the inverse of relation r. Determine whether each relation is a function. Describe the graphs.

Relation r

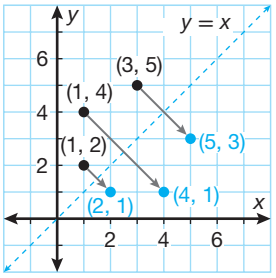
Domain	x	1	1	3
Range	y	2	4	5

Inverse of r

Domain	x	2	4	5
Range	y	1	1	3

SOLUTION Reverse the coordinates in each ordered pair.

Relation r is not a function because the domain element 1 is paired with two different range elements: 2 and 4. The inverse of r is a function because each element in the domain is paired with exactly one element in the range.



Each point in the graph of the inverse of r is the reflection image of the corresponding point in relation r. The line of reflection is $y = x$.

The line $y = x$ is important in the graph of any pair of inverse relations.

Graphing Calculator Tip

To get undistorted graphs on a calculator, use ZOOM 5:ZSquare after setting the $[-2, 4]$ window.

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Exploration Graphing a Function and its Inverse

a. Use a graphing calculator to graph the following functions. Use a window of $[-2, 4]$ for both x and y.

Y1: $f(x) = x^2$ Y2: $g(x) = \sqrt{x}$ Y3: $h(x) = -\sqrt{x}$ Y4: $I(x) = x$

b. Describe the inverse of f.

c. Is the inverse of f a function? Explain.

d. Describe the role of the identity function I.

If a relation is given by an equation, interchange the variables to find the equation for the inverse.

Example 2 Finding the Inverse of a Linear Function

Find an equation for the inverse of $y = 2x + 6$.

SOLUTION

$y = 2x + 6$ Original equation.

$x = 2y + 6$ Interchange the variables.

$x - 6 = 2y$ Solve for y.

$\frac{1}{2}x - 3 = y$

The inverse of $y = 2x + 6$ is $y = \frac{1}{2}x - 3$.

Math Reasoning

Generalize In what form are both equations $y = 2x + 6$ and $y = \frac{1}{2}x - 3$? Is the inverse of *any* equation of the form $y = mx + b$ where $m \neq 0$, a function?

The vertical line test to determine whether a relation is a function leads to a horizontal line test to determine whether the inverse of a relation is a function.

Horizontal and Vertical Line Tests

Vertical Line Test A relation is a function if and only if no vertical line intersects the relation's graph in more than one point.

Horizontal Line Test A relation's inverse is a function if and only if no horizontal line intersects the relation's graph in more than one point.

Example 3 Finding an Inverse that is a Radical Expression

Find an equation for the inverse of $y = \frac{1}{4}x^3 - 2$. Use a graphing calculator and the vertical and horizontal line tests to determine whether each relation is a function.

SOLUTION

$y = \frac{1}{4}x^3 - 2$ Original equation

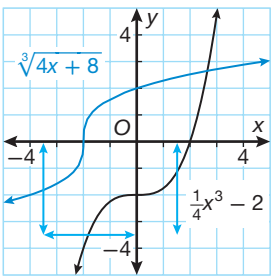
$x = \frac{1}{4}y^3 - 2$ Interchange the variables.

$x + 2 = \frac{1}{4}y^3$ Add 2 to each side.

$4x + 8 = y^3$ Multiply each side by 4.

$\sqrt[3]{4x + 8} = y$ Write the cube root of each side to solve for y.

The inverse of $y = \frac{1}{4}x^3 - 2$ is $y = \sqrt[3]{4x + 8}$. $y = \frac{1}{4}x^3 - 2$ is a function because no vertical line intersects its graph in more than one point. The inverse, $y = \sqrt[3]{4x + 8}$, is also a function because no horizontal line intersects the graph of $y = \frac{1}{4}x^3 - 2$ in more than one point.



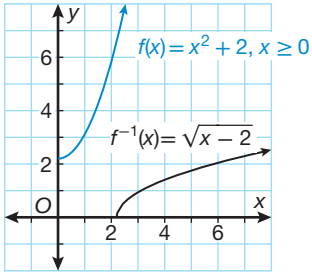
Hint

The graph of $y = \frac{1}{4}x^3 - 2$ appears to be almost horizontal near the point $(0, -2)$. Use coordinates of some nearby points to verify that it is not really horizontal.

The inverse relations in Example 3 are inverse functions, and the domain and range of both functions is the set of all real numbers. In some cases, an inverse is not a function, but a domain can be restricted to make it a function. The inverse of a function f is denoted f^{-1}.

Example 4 Restricting a Domain to Create Inverse Functions

Restrict the domain of $y = x^2 + 2$ so that its inverse is a function. Identify the domain and range of each function.



SOLUTION

Require that $x \geq 0$. Then when x and y are interchanged, $y \geq 0$, and the inverse consists only of $y = \sqrt{x - 2}$.

So if $f(x) = x^2 + 2$ with domain $x \geq 0$ and range $y \geq 2$, then its inverse function is $f^{-1}(x) = \sqrt{x - 2}$ with domain $x \geq 2$ and range $y \geq 0$.

Hint

An alternate approach using a different domain for $f(x)$ yields these inverse functions: $f(x) = x^2 + 2, x \leq 0$ and $f^{-1}(x) = -\sqrt{x - 2}$.

Formulas used in applications have variables that represent specific quantities, such as d for distance and t for time. When finding the inverse of a formula, it is better to *not* interchange variables. For example, if d and t were interchanged, then d would represent time and t would represent distance; this would be confusing.

Example 5 Application: Distance, Time, and Gravity

The Sears Tower in Chicago is 1451 feet tall. The distance traveled by a free-falling object is given by $d = 16t^2$, where d is distance in feet and t is time in seconds. How long would it take an object to reach the ground if dropped from the top of the Sears Tower?

SOLUTION

Write t as a function of d. That will be the inverse function. Substitute 1451 for d and evaluate t.

$d = 16t^2, t \geq 0$	The domain is restricted naturally. Time cannot be negative.
$\frac{d}{16} = t^2$	Do not interchange variables. Solve for t.
$\sqrt{\frac{d}{16}} = t$	The inverse function is $t = \sqrt{\frac{d}{16}}$.
$\sqrt{\frac{1451}{16}} = t$	Substitute 1451 for d.
$9.5 \approx t$	Use a calculator to approximate.

It will take about 9.5 seconds for the object to reach the ground.

Check Substitute 9.5 for t in the original formula.

$d = 16t^2 = 16(9.5)^2 = 1444$, which is close to 1451. The answer is reasonable.

Lesson Practice

a. (Ex 1) Find the inverse of relation r. Graph both relations. Determine whether each relation is a function. Describe the graphs.

Relation *r*

Domain	x	0	3	−3
Range	y	2	2	1

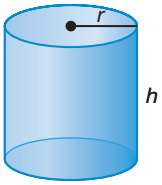
b. (Ex 2) Find an equation for the inverse of $y = -\frac{1}{3}x - 1$.

c. (Ex 3) Find an equation for the inverse of $y = -\frac{1}{2}x^3 + 1$. Determine whether each relation is a function. (Hint: Use a graphing calculator if necessary.)

d. (Ex 4) Find an equation for the inverse of the function $y = \frac{1}{2}x^2 - 4$. Identify the domain and range of each relation.

e. (Ex 4) Restrict the domain of $y = \frac{1}{2}x^2 - 4$ if needed so that its inverse is a function. Identify the domain and range of the inverse functions.

f. (Ex 5) The volume of a cylinder is given by $V = \pi r^2 h$. A manufacturer wants a cylindrical container to have a volume of one cubic foot, and the height to be twice the radius. Write an inverse function that gives r as a function of V, and find the required radius to the nearest tenth of an inch.



Practice Distributed and Integrated

Multiply:

1. (19) $(3x^3 - 2x)(2x^2 - x - 4)$

2. (19) $(4x + 2)(x^3 - 2x + 4)$

Write the equation in standard form for the line that crosses the two points.

3. (26) $(-3, 0), (-5, 1)$

4. (26) $(-7, 9), (-10, 11)$

Expand:

***5.** (49) $(2 - m)^7$

***6.** (49) $(3g + 2h)^3$

Find the inverse of each linear function.

***7.** (50) $y = 3x^2 - 1$

***8.** (50) $y = \sqrt{x}$

***9.** (50) $y = \frac{1}{2}x + 6$

***10.** (45) **Graphing Calculator** The table shows the category and amount of damage to the United States from seven hurricanes that hit the US after the year 2000. Use a graphing calculator to find the value of r for the data set. Tell what this value means.

	Wilma	**Charley**	**Ivan**	**Rita**	**Isabel**	**Lili**	**Francis**
Category	3	4	3	3	2	1	2
Damage to US ($, billions)	20.6	15	14.2	11.3	3.37	0.86	8.9

11. *(31)* **Velocity** The velocity of a certain automobile during the first 40 seconds starting from rest is approximated by the rational function $f(t) = \frac{90t}{t+6}$, where t is the number of elapsed seconds and $f(t)$ is velocity in feet per second.

Write a division expression for the average acceleration of the automobile over the time interval from 0 seconds to t seconds. Then simplify the expression. (Average acceleration equals change in velocity divided by elapsed time.)

***12.** *(49)* **Probability** You have a fifty-fifty chance of correctly guessing each answer on a test. Suppose you guess at 10 true/false questions. Find the probability that you will get exactly 4 questions correct.

13. *(17)* **Multiple Choice** Solve $|2x + 1| < 4$.

A $-\frac{5}{2} < x < \frac{3}{2}$ **B** $\frac{3}{2} < x < -\frac{5}{2}$

C $x < -\frac{5}{2}$ or $x > \frac{3}{2}$ **D** $x < \frac{3}{2}$

14. *(42)* **Error Analysis** In selecting five items out of ten, a student says there would be ten times more permutations than combinations. Is the student correct? If not provide the correct answer.

15. *(25)* **Model** The table below shows the measurements in inches of various shipping boxes available from the United States Postal Service. Draw a box-and-whisker plot to represent the approximate areas of all sides of the boxes. Consider that a box has 6 faces.

Box	Measurement 1	Measurement 2	Measurement 3
1	11.875	3.375	13.625
2	11	8.5	5.5
3	7.5	5.125	14.375

16. *(30)* **Multi-Step** The graph of a quadratic function has a vertex $(2, -3)$ and passes through the point $(5, -12)$. Give the equation of the function. Tell if the function has a maximum or minimum.

17. *(17)* **Grizzly Bears** The table shows amounts of three foods that a 200-lb grizzly bear can eat in a day.

Type	Amount (lbs)
Deer	24
Salmon	30
Vegetation	67

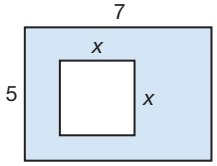
a. Write an inequality to show the amounts of these foods that a 200-lb grizzly bear can eat in a day.

b. Verify that your answer is equivalent to the absolute value inequality $|f - 45.5| \leq 21.5$.

18. *(40)* **Multiple Choice** Which number has two fourth roots?

A -27 **B** 4 **C** 16 **D** 125

19. *(27)* **Geometry** **a.** Write a quadratic equation in standard form to represent the area of the shaded region.



b. Identify the values of a, b, and c in the equation in part a.

20. (19) **Error Analysis** Two students completed the problem $(x - 8)(x + 8)$. Who is correct? Explain your answer.

Student A	Student B
$= x^2 + 8x + 8x - 64$ $= x^2 + 16x - 64$	$= x^2 + 8x - 8x - 64$ $= x^2 - 64$

21. (43) **Justify** Explain why the system $\begin{array}{l} y > 3x + 1 \\ y < 3x - 1 \end{array}$ has no solution.

22. (20) **Births** From 1990 to 1999, the number of births $b(x)$ in the U.S. can be modeled by the function $b(x) = -27x + 4103$, and the number of deaths $d(x)$ can be modeled by the function $d(x) = 23x + 2164$. Write a function for the net increase in population, $p(x) = b(x) - d(x)$ from 1990 to 1999.

23. (28) **Justify** Explain why $4x^2 + 3x$ is a rational expression.

24. (41) **Cartography** A student constructs a map of the eastern United States over a grid system such that Indianapolis, IN is the origin and Raleigh, NC is at $(6, -3)$. Find the distance between the cities to the nearest kilometer if each unit represents 120 kilometers.

25. (31) If $g(x) = \frac{1}{x + 2}$, what is $g(x)^2$ in simplest form?

26. (46) **Analyze** Why are the sine and cosine values of an acute angle in a right triangle always less than 1?

27. (34) **Engine Size** In past years, engine size for American automobiles was usually measured in cubic inches. Today, most engine sizes are measured in liters. The table shows three typical engine sizes for modern automobiles in both liters and cubic inches.

Liters (x)	2.3	2.5	3.0
Cubic Inches (y)	140.3	152.5	183

Graph the linear function that relates liters and cubic inches, state the rate of change, and describe what it means.

28. (29) Use $\begin{array}{l} 9x = y - 3z - 21 \\ 4y - 6z = 8x + 10 \\ 56x - 28y = 2 - 42z \end{array}$.

a. Rewrite the equations lining up the variables.

b. Estimate the number of solutions. Explain your answer.

c. Solve the system of equations to verify your estimate.

29. (41) **Analyze** In the Pythagorean Theorem, why are a, b, and c all greater than 0?

30. (11) **Write** Explain in words what you are doing when you "factor out the GCF" to factor a polynomial.

INVESTIGATION

5

Finding the Binomial Distribution

Suppose you are flipping a coin two times. However simple the task, this coin-flipping activity is an example of a binomial experiment. There are four conditions that need to be met for a probability experiment to qualify as a binomial experiment. The four conditions are listed below and next to each item is an explanation of how the coin-flipping experiment complies.

Binomial Experiment	Does the Test Scenario Comply?
There are n trials in the experiment	Yes. There are 2 trials.
Only two possible outcomes per trial	Yes. The coin lands either heads or tails.
Each trial is independent	Yes. The result of one trial does not influence the result of any other.
The probability of success is the same from trial to trial	Yes. Each toss has a 1 in 2 (or 50%) chance of heads (or tails).

1. Suppose you were flipping a coin 100 times. Is this also an example of a binomial experiment? Why?
2. Suppose you were flipping two coins twice. Is this an example of a binomial experiment? Why?
3. Suppose you were rolling a six-sided number cube twice. Is this an example of a binomial experiment? Why?

Math Reasoning

Verify Two six-sided number cubes are rolled, and the values are added. Show that there are 36 possible outcomes, but not all outcomes are unique.

Some events that are usually not binomial experiments can be structured in such a way that they are. Consider two six-sided number cubes. There are 36 possible outcomes when you roll the number cubes, so this is not a binomial experiment.

4. Now suppose that you want to keep track of how often the sum of the number cubes is greater than 3 after rolling the two cubes 15 times. Determine if this is a binomial experiment by completing the table.

Binomial Experiment	Does the Test Scenario Comply?
There are n trials in the experiment	
Only two possible outcomes per trial	
Each trial is independent	
The probability of success is the same from trial to trial	

5. Consider a probability experiment where you keep track of how many results are even and how many are odd that involves rolling a six-sided number cube five times.

   a. **Verify** Show that this is a binomial experiment.
   b. **Predict** If you perform the experiment, how many even number outcomes will there be?

6. Perform the binomial experiment from question 5.

   a. Create a frequency diagram of your results.
   b. Create a bar graph.
   c. Compare your results to the prediction you previously made.

You can use a spreadsheet to simulate a probability experiment. In fact, using a spreadsheet, you can simulate the binomial experiment similar to the one from question 5. In this case you will simulate rolling a number cube 25 times to see how many times the results are an even number and how many times the results are an odd number.

Math Reasoning

Analyze The MOD function can be used to find the remainder when one number is divided by another. Why is Mod(#, 2) a test for even or odd?

7. **Multi-Step** Use a spreadsheet for the following activity.

   a. Input the following formula to cell A1: **=RANDBETWEEN(1, 6).** This formula generates a random integer from 1 to 6, each number representing one of the faces of the number cube.
   b. Copy and paste this formula from cell A2 to A5.
   c. Press the F9 key, and you'll see that the numbers change randomly. The goal is to keep track of when the results are even and when they are odd.
   d. Input the following formula to cell B1: **=IF(MOD(A1, 2)=0, "Even", "Odd").** This tests if the outcome is an even or odd number.
   e. Copy and paste the formula to cells B2 to B5.
   f. Input the following formula to cell B7: **=COUNTIF(B1:B5, "Even").** Input the following formula to cell B8: **=COUNTIF(B1:B6, "Odd").** These formulas will count the number of even and odd numbers for each trial.
   g. Run 25 trials (by repeatedly using the F9 key). Keep track of the number of even and odd results.
   h. Create a bar graph of your results.
   i. How do your results compare to your prediction in question 5?

8. Run an additional 75 trials to have a total of 100 trials.

   a. Create a bar graph of your results.
   b. How do your results compare to your previous predictions?

Test Prep Does it help to guess if you don't know the answer to a question? What are your chances of getting a good score on a test if all you do is guess?

Suppose you are taking a 20-item, multiple-choice test. There are five possible answer choices, A through E, for each item. Your plan is to guess the answer to each of the 20 questions. What is the probability that you will get 0, 1, … up to 20 questions correct?

9. **Verify** A binomial experiment needs to satisfy the four conditions stated below. Complete the table to verify that guessing on the 20-item test is a binomial experiment.

Binomial Experiment	Does the Test Scenario Comply?
a. There are n trials in the experiment	
b. Only two possible outcomes per trial	
c. Each trial is independent	
d. The probability of success is the same from trial to trial	

10. **Predict** If you randomly guess on 20 questions, how many are you likely to get right?

11. **Multi-Step** Use a spreadsheet for the following activity.

    a. Input the following formula to cell A1: **=RANDBETWEEN(1, 5).** Copy and paste this formula to cells A2 to A20. These 20 numbers represent the correct answers to the items in the test.

    b. Repeat part **a** for cells B1 to B20. These 20 numbers represent the 20 random guesses.

    c. Input the following formula to cell C1: **=IF(A1-B1=0, "Correct", "N").** This tests if the guess for a particular row is correct or not. (A guess is correct when both numbers in a row are equal.)

    d. Run 30 trials (by using the F9 key) and record the cumulative number of correct guesses in a table with one column to record trial numbers and one column to record number of correct answers.

    e. Which number of correct guesses has the highest probability?

    f. How does your answer to part e compare to your original prediction?

Investigation Practice

Determine if each of the probability experiments is a binomial experiment or not. If not, explain why.

a. Rolling two six-sided number cubes 500 times to see the probability of rolling a 12 a total of 100 times.

b. Taking a 20-question multiple choice test and guessing on half the items.

c. Conduct a binomial experiment that involves simulating rolling two six-sided number cubes 50 times to determine the frequency of getting a 12. Create a bar graph of your results.

LESSON
51

Using Synthetic Division

Warm Up

1. (38) **Vocabulary** If $x^2 - 9$ is divided by $x - 3$, then $x^2 - 9$ is the dividend and $x - 3$ is the _____.

2. (19) Multiply. $(2x + 1)(x^2 + 2x - 3)$

3. (11) Find $f(2)$ for $f(x) = x^4 + x^3 - 3x - 1$.

4. (2) Divide $\frac{6x + 15y}{3}$

5. (SB) $12.18 \div 2.1$

New Concepts

To divide a polynomial by another polynomial, you can use long division. The process is similar to long division with numbers.

Example 1 Using Long Division to Divide Polynomials

Divide $5x^4 - 10x^2 + 3x - 1$ by $x^2 + 2x - 3$.

SOLUTION Determine the first term in the quotient, and then multiply it by the divisor. Next subtract the product from the dividend. Then bring down the next term in the dividend.

$$
\begin{array}{r}
5x^2 \\
x^2 + 2x - 3 \overline{)\ \boxed{5x^4} + 0x^3 - 10x^2 + 3x - 1} \\
\underline{-(5x^4 + 10x^3 - 15x^2)\ \downarrow} \\
\boxed{-10x^3} + 5x^2 + 3x
\end{array}
$$

Quotient ← $5x^2$

Dividend ← $5x^4 + 0x^3 - 10x^2 + 3x - 1$

Divisor ↑ $x^2 + 2x - 3$

Determine the second term in the quotient. Continue the process above until all of the terms in the divisor have been exhausted.

$$
\begin{array}{r}
5x^2 - 10x + 25 \\
x^2 + 2x - 3 \overline{)\ 5x^4 + 0x^3 - 10x^2 + 3x - 1} \\
\underline{-5x^4 - 10x^3 + 15x^2\ \ \ \ \ \ \downarrow} \\
\boxed{-10x^3} + 5x^2 + 3x \\
\underline{-(-10x^3 - 20x^2 + 30x)} \\
\boxed{25x^2} - 27x - 1 \\
\underline{-(25x^2 + 50x - 75)} \\
-77x + 74
\end{array}
$$

The quotient is $5x^2 - 10x + 25$ and the remainder is $-77x + 74$.

Caution

Be sure to write the dividend and the divisor with the terms in descending order of degree. Use placeholders for any missing terms.

Hint

To determine the first term in the quotient, divide the first term in the dividend by the first term in the divisor.

$\frac{5x^4}{x^2} = 5x^2$ ← 1st term

Caution

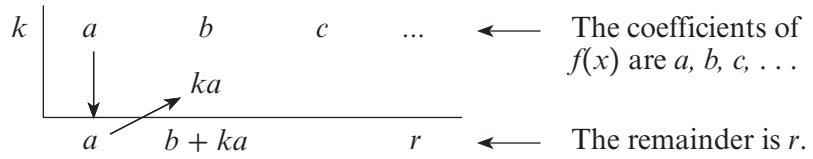
Remember to distribute the negative sign when subtracting the product from the dividend .

Online Connection
www.SaxonMathResources.com

When the divisor is a linear binomial of the form $x - k$, you can use synthetic division. **Synthetic division** is an abbreviated form of long division in which only coefficients are written.

Synthetic Division

To divide polynomial $f(x)$ by $x - k$, begin by writing k and the coefficients of $f(x)$ on the first line.



Bring down a; this is the first coefficient of the quotient.

Multiply a by k. Write the result, ka, under b.

Add b and ka. The result, $b + ka$, is the second quotient coefficient.

Continue the pattern. Multiply each new quotient coefficient by k and add the result to the next coefficient of $f(x)$. The result of the last addition is the remainder r.

Example 2 Using Synthetic Division to Divide by a Linear Binomial

Divide $3x^3 - x^2 - 10x + 11$ by $x + 2$.

SOLUTION $x + 2 = x - (-2)$, so $k = -2$.

$$\begin{array}{r|cccc} -2 & 3 & -1 & -10 & 11 \\ & & -6 & 14 & -8 \\ \hline & 3 & -7 & 4 & 3 \end{array}$$

Dividend coefficients ← 11

Remainder ← 3

Quotient coefficients: $3 \quad -7 \quad 4$

The quotient is $3x^2 - 7x + 4$ and the remainder is 3.

$$\frac{3x^3 - x^2 - 10x + 11}{x + 2} = 3x^2 - 7x + 4 + \frac{3}{x + 2}$$

Hint

Remember to write the remainder over the divisor. The remainder of 3 is written as $\frac{3}{x+2}$.

If both sides of the equation above are multiplied by $x + 2$, the result can be written $3x^3 - x^2 - 10x + 11 = (x + 2)(3x^2 - 7x + 4) + 3$.

If $f(x) = 3x^3 - x^2 - 10x + 11$, then $f(-2)$ can be evaluated as follows:

$$f(x) = (x + 2)(3x^2 - 7x + 4) + 3$$

$$f(-2) = (-2 + 2)(3(-2)^2 - 7(-2) + 4) + 3$$

$$= 0 + 3 = 3$$

Notice that $k = -2$, the remainder r is 3, and $f(k) = f(-2) = 3 = r$.

This demonstrates the Remainder Theorem.

Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, the remainder is $r = f(k)$.

By the Remainder Theorem, you can divide $f(x)$ by $x - k$ to find $f(k)$. If synthetic division is used to divide, the process is called **synthetic substitution.**

Example 3 Using Synthetic Substitution to Evaluate a Polynomial

Find $f(5)$ for $f(x) = -2x^4 + 6x^3 + 15x - 1$.

SOLUTION

Divide $f(x)$ by $x - 5$ to find $f(5)$.

5	−2	6	0	15	−1
		−10	−20	−100	−425
	−2	−4	−20	−85	−426

The remainder is -426, so $f(5) = -426$.

Hint

To verify the solution in Example 3, evaluate $f(5)$ directly by substituting 5 for x in the polynomial.

Example 4 Application: Predicting Population

Based on the U.S. census results for the years 1950 through 2000, the population of Nevada can be modeled by the function $P(x) = 12x^3 - 16x^2 + 150x + 154$, where x is the number of decades since 1950 and $P(x)$ is the population in thousands. Use $P(x)$ to predict the population of Nevada in the year 2010.

SOLUTION

Understand Make a table to help understand the problem.

Year	1950	1960	1970	1980	1990	2000	2010
Number of decades since 1950	0	1	2	3	4	5	6

Plan Use synthetic substitution to evaluate $P(6)$.

Solve

6	12	−16	150	154
		72	336	2916
	12	56	486	3070

Using $P(x)$ to predict, the population of Nevada in the year 2010 will be about 3070 thousand, or 3,070,000.

Check Evaluate $P(6)$ directly.

$P(6) = 12(6)^3 - 16(6)^2 + 150(6) + 154$

$= 3070$, so the answer is verified.

Math Reasoning

Predict Use $P(x)$ to predict population in the year 2050. Do you think $P(x)$ is a good model for predictions that far in the future?

Lesson Practice

a. (Ex 1) Use long division to divide $2x^4 + x^3 - 5x + 2$ by $x^2 - x + 4$.

b. (Ex 2) Use synthetic division to divide $-2x^3 + x^2 + 5x - 1$ by $x - 3$.

c. (Ex 3) Use synthetic substitution to find $f(-2)$ for $f(x) = x^5 - 2x^3 - x^2 + 10$.

d. (Ex 4) The yearly enrollment at a high school for its first six years can be modeled by the function $f(x) = -x^3 + 12x^2 - 8x + 800$, where x is the year and $f(x)$ is the enrollment. Use $f(x)$ to predict the enrollment for the seventh year.

Practice Distributed and Integrated

Use synthetic division to divide.

***1.** (51) $2x^4 - x$ by $(x - 2)$

***2.** (51) $2 + 4x - 2x^2 + 4x^3$ by $(-3 + x)$

Find the distance between the given points.

3. (41) $(-3, 5)$ and $(-1, 2)$

4. (41) $(2, 4)$ and $(6, -2)$

5. (41) $(-3, 5)$ and $(4, -2)$

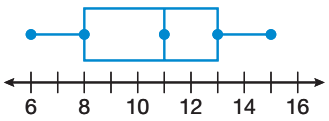
***6.** (14) **Computer Graphics** A computer programmer wants a shape to rotate and expand on the screen. The programmer multiplies the coordinates of each vertex by the matrix $\begin{bmatrix} 1.5 & 0.71 \\ -0.71 & 1.5 \end{bmatrix}$. The area of the shape expands by an amount that equals the determinant of this matrix. By what amount does that area expand?

Add.

7. (37) $\frac{m^2}{x^2a} + \frac{5}{ax} - \frac{m}{a}$

8. (37) $\frac{a}{x^2} - a - \frac{3x}{2a^4}$

9. (25) **Multiple Choice** Which data set is represented by the box-and-whisker plot shown?



A {15, 8, 10, 13, 11, 12, 6, 13, 7, 15}

B {6, 9, 10, 12, 7, 8, 12, 15, 12, 14}

C {8, 10, 12, 6, 15, 12, 6, 13, 14, 8}

D {6, 11, 12, 7, 12, 8, 14, 15, 9, 12}

Simplify.

10. (44) $\frac{3}{4\sqrt{15}}$

11. (44) $\frac{5\sqrt{3} - \sqrt{2}}{4\sqrt{3} + 3\sqrt{2}}$

12. (44) $\frac{5}{2\sqrt{7} - \sqrt{5}}$

Use synthetic substitution to evaluate the polynomials.

***13.** (51) Find $f(3)$ for $f(x) = 3x^3 - 7x + 15$

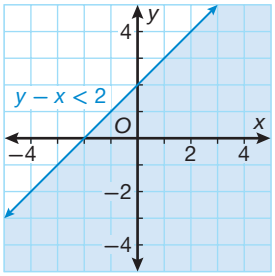
***14.** (51) Find $f(6)$ for $f(x) = -3x^5 - 12x^4 + 30x^3 - 16x^2 - x$

*15. (48) **Electronics** If two resistors in an electrical circuit have resistances R_1 and R_2 (both in ohms), then the total resistance R_t (in ohms) is given by $R_t = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$.

a. Simplify the complex fraction.

b. If there are two resistors in an electrical circuit with resistances 3 ohms and 6 ohms, what is the total resistance of the circuit?

16. (39) **Verify** Determine if the graph is correct by choosing points in the shaded and unshaded regions and testing them.



17. (37) **Generalize** Suppose that $f(x)$ is a polynomial of degree m and $g(x)$ is a polynomial of degree n. What must be true for rational expression $\frac{f(x)}{g(x)}$ to be of degree $m - 2$ in the numerator and $n - 2$ in the denominator?

18. (11) **Error Analysis** Explain the error in the student's work.

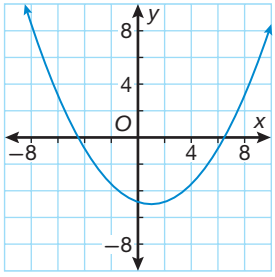
$$(3x^4 - 2x^3 + 1) - (x^4 - x^3 + 5)$$
$$= 3x^4 - 2x^3 + 1 - x^4 - x^3 + 5$$
$$= 2x^4 - 3x^3 + 6$$

*19. (45) **Graphing Calculator** The table shows the elevation of where each record cold temperature reading took place. Use a graphing calculator to find the value of r and the equation for the line of best fit. Use it to predict the record cold temperature for an elevation of 5000 feet.

	Arizona	Idaho	Wyoming	Rhode Island	Alabama	Tennessee	Virginia	Mississippi
Elev. (ft)	8180	6285	6650	425	760	2471	3870	420
Temp. (°F)	−40	−60	−66	−25	−27	−32	−30	−19

20. (41) **Mountain Bikers** If the origin represents a parking lot, and one mountain biker bikes 2 units due north and 9 units due west, while a second mountain biker bikes 8 units due north and 5 units due east, how much farther from the parking lot is the second mountain biker than the first? Write the answer as a decimal to the nearest tenth of a mile, given that each unit represents 3 miles.

21. (27) **Estimate** Estimate the values of the x- and y-intercepts.



22. *(46)* **Multiple Choice** What is the value of the cotangent of $\angle A$ if the tangent of $\angle A$ is 0.75?

A 0.6 **B** 0.75 **C** 1.25 **D** $1.\overline{3}$

23. *(11)* **Multi-Step** Al's grandfather deposited the amounts shown in Bank A as gifts on Al's birthday during the years 2002 through 2005. His aunt deposited the amounts shown in Bank B. The value of the accounts on 7/2/06 is represented by these polynomials, where $x = 1 + r$, and r is the interest rate for both accounts:

	Bank A	Bank B
7/2/02	\$500	\$1000
7/2/03	\$1000	\$200
7/2/04	\$600	\$200
7/2/05	\$500	\$200

Bank A $500x^4 + 1000x^3 + 600x^2 + 500x$

Bank B $1000x^4 + 200x^3 + 200x^2 + 200x$

a. Write the polynomial that represents the total value of both accounts on 7/2/06.

b. What is the total value of both accounts on 7/2/06 if the interest rate is 5%?

24. *(36)* **Verify** Show that the equation of the line parallel to $y = mx + b$ through (x_1, y_1) is $y = mx + (y_1 - mx_1)$.

25. *(13)* **Write** Explain how to graph $y = \frac{3}{4}x + 9$ using a table. Be sure to show each step.

26. *(42)* **Probability** Two cards are drawn at random from a deck of 52 cards, without replacement. What is the probability that both cards drawn are aces?

27. *(23)* **Error Analysis** The factored form of $x^2 + 5x + 6$ is given as $(x - 1)(x + 6)$. Find and explain the error and give the correct factorization.

***28.** *(49)* **Geometry** The sides of a cube have length s. If each dimension of the cube is increased by 0.5, write a binomial expression for the volume of the cube. Expand the binomial.

29. *(43)* **Employment** A student earns \$8 per hour working for the recreation department and \$12 per hour at a candy store. She is not allowed to work more than 16 hours in a week, and her goal is to earn at least \$150 in a week. Let x represent a number of hours working for the recreation department and y represent a number of hours working at the candy store.

a. Write and graph a system of linear inequalities that represents the situation.

b. Describe the portion of the solution set that contains meaningful solutions for this situation.

c. Determine the vertices of the polygon you described in part **b.** Describe what each vertex represents.

***30.** *(49)* **Generalize** Write the expansion of $(a + b)^5$. Make a generalization about terms that have the same coefficient in a binomial expansion of the form $(a + b)^n$.

LAB
9

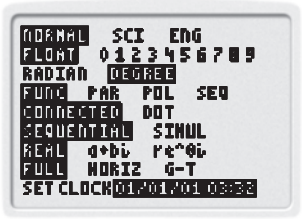
Using the Trigonometry Keys and Adjusting to a Trigonometric Window

Graphing Calculator Lab (*Use with Lessons 52, 67, 82, 86, 90, and 103*)

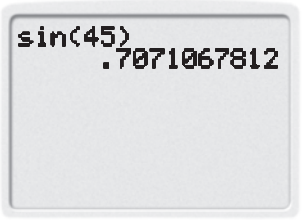
A graphing calculator can be used to calculate and graph trigonometric functions in both degrees and radians.

Using the Trigonometric Keys

1. Press MODE, and press the down arrow key twice ▼ ▼, and then press the right arrow key ▶ to highlight [**DEGREE**] and press ENTER. The calculator will now be in degree mode. Any trigonometric functions calculated will interpret input in terms of degrees.

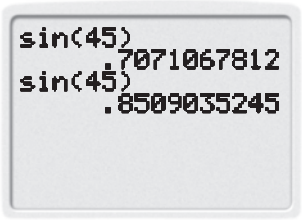


2. Press SIN 4 5) ENTER, to calculate sin(45°). Because the calculator is in degree mode, it will automatically assume that the input 45 is the same as 45°.



3. Press MODE ▼ ▼ ENTER, to change the mode back to [**RADIAN**] mode.

4. While the calculator is in radian mode, it will assume the input of a trigonometric function is in radians. Press SIN 4 5) again. This time the calculator will assume the input 45 is the same as 45 radians. Press ENTER. The value of sin(45) will display as two different values, because the first is calculated for 45° and the second is calculated for 45 radians.



5. The inverse trigonometric functions are also affected by the mode. The output of an inverse trigonometric function is always the measure of an angle. Therefore, when the calculator is in radian mode, the output (measure of the angle) will be in radians. Similarly, the output will be in degrees when the calculator is in degree mode.

Online Connection
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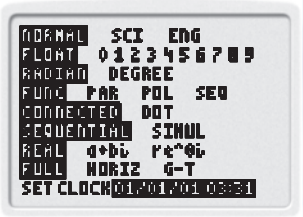
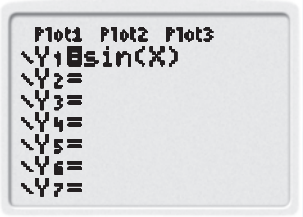
Adjusting to a Trigonometric Window

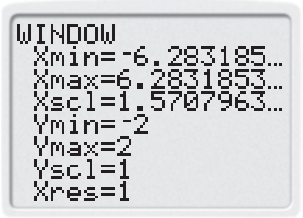
1. Make sure the calculator is in radian and function mode, by pressing MODE and verifying that [**RADIAN**] and [**FUNC**] are highlighted.



2. Press Y= to open the [**Y=**] editor. Make sure the equations are all empty, and move the cursor to $Y_1=$. Press SIN X,T,θ,n), to input the general sine function $y = \sin(x)$.



3. Press WINDOW to open the viewing window variables. To graph the sine function in radian mode, the window variables need to be changed accordingly.



   a. While the cursor is on Xmin, press (−) 2 2nd ^ to change the minimum x-value to -2π.

   b. Press ▼ to move the cursor to Xmax. Press 2 2nd ^ to change the maximum x-value to 2π.

   c. Press ▼ to move the cursor to Xscl. Press 2nd ^ ÷ 2 to change the x-scale to $\frac{\pi}{2}$.

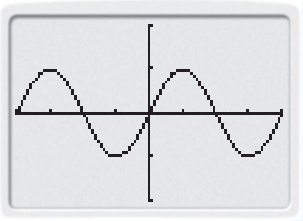
   d. Press ▼ to move the cursor to Ymin and press (−) 2.

   e. Press ▼ to move the cursor to Ymax and press 2.

   f. Press ▼ to move the cursor to Yscl and press 1.

   g. Press ▼ to move the cursor to Xres and press 1.

4. Press GRAPH. Two periods of the sine function are displayed in radian mode.



Graphing Calculator Tip

For help in adjusting the viewing window, see graphing calculator lab 4.

Lab Practice

a. Calculate $\cos(\pi)$ in radian mode.

b. Calculate $\cos(\pi)$ in degree mode.

c. Calculate $\cos(0)$ in degree and radian mode.

d. Graph $y = \cos(x)$, with the same viewing window as above.

LESSON
52

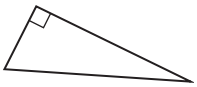
Using Two Special Right Triangles

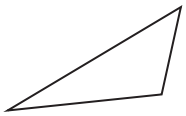
Warm Up

1. (46) **Vocabulary** In a right triangle, the ratio of the length of the opposite leg to the length of the adjacent leg is called the _____ of the angle.

2. (SB) Given that the measure of one acute angle of a right triangle is 24°, what is the measure of the other acute angle?

3. (SB) Which of these triangles is a right triangle?

A 

B 

C

4. (41) Find the unknown length for a right triangle with one leg that is 12 cm and a hypotenuse that is 13 cm.

New Concepts

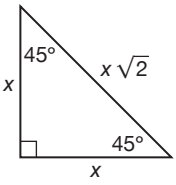
Some right triangles are used so frequently that it is helpful to remember some of their particular properties. These triangles, called **special right triangles,** are the 30°-60°-90° triangle and the 45°-45°-90° triangle. The name of each triangle gives the measures of its angles.

Math Reasoning

Verify Use the Pythagorean Theorem to show that the length of the hypotenuse of a 45°-45°-90° triangle is the length of one of the legs times $\sqrt{2}$.

Properties of a 45°-45°-90° Triangle: Side Lengths

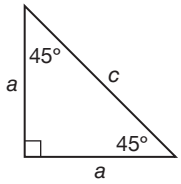
In a 45°-45°-90° right triangle, both legs are congruent and the length of the hypotenuse is the length of a leg multiplied by $\sqrt{2}$.



A triangle with sides 1-1-$\sqrt{2}$ is a 45°-45°-90° reference triangle. The properties of the 45°-45°-90° triangle can be used to find missing measures of a triangle if the length of one leg is known.

Example 1 Finding the Side Lengths in a 45°-45°-90° Triangle

Find the length of a when c is $5\sqrt{2}$ centimeters, using the properties of a 45°-45°-90° triangle.



SOLUTION

Since the length of the hypotenuse of a 45°-45°-90° triangle is $\sqrt{2}$ times the length of a its leg, $a = \frac{5\sqrt{2}}{\sqrt{2}}$ or 5 centimeters.

Check The hypotenuse of the given triangle is $5\sqrt{2}$ cm. Use the reference triangle and a graphing calculator to verify the answer.

$$(\sin 45°)(5\sqrt{2}) = 3$$

Online Connection
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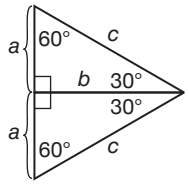
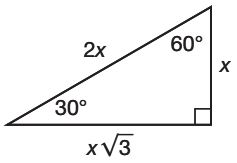
Math Reasoning

Verify Use the diagram and the Pythagorean Theorem to show that for a 30°-60°-90° triangle, the length of b is equal to a times $\sqrt{3}$.



Properties of 30°-60°-90° Triangles

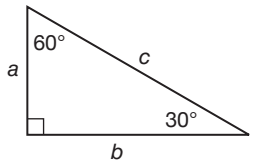
In a 30°-60°-90° triangle, the length of the hypotenuse is twice the length of the shorter leg, and the length of the longer leg is the length of the shorter leg times $\sqrt{3}$.



A triangle with sides 1-2-$\sqrt{3}$ is a 30°-60°-90° reference triangle. The properties of the 30°-60°-90° triangle can be used to find missing measures of a triangle if the length of one leg is known.

Example 2 Finding Side Lengths in a 30°-60°-90° Triangle

Find b when c is $\sqrt{6}$ centimeters, using the properties of a 30°-60°-90° triangle.



Math Reasoning

Verify Use the reference triangle and a graphing calculator to verify the answer.

SOLUTION

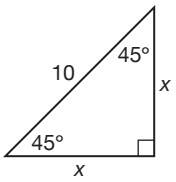
Since the length of the hypotenuse of a 30°-60°-90° triangle is 2 times the length of the shorter leg, $a = \frac{\sqrt{6}}{2}$. Since the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg, $b = \frac{\sqrt{6}\sqrt{3}}{2} = \frac{3\sqrt{2}}{2}$ or approximately 2.12 centimeters.

Example 3 Application: Distance

A 10-foot ladder leaning against a building makes a 45° angle with the ground. How far up the building does the ladder reach?

SOLUTION

Draw a sketch.



Since the length of the hypotenuse of a 45°-45°-90° is $\sqrt{2}$ times the length of a leg, then $10 = x\sqrt{2}$. Solve for x.

$$10 = x\sqrt{2}$$

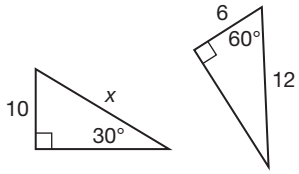
$$\frac{10}{\sqrt{2}} = x$$

$$5\sqrt{2} = x$$

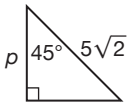
The ladder extends $5\sqrt{2}$, or about 7 feet up the side of the building.

Lesson Practice

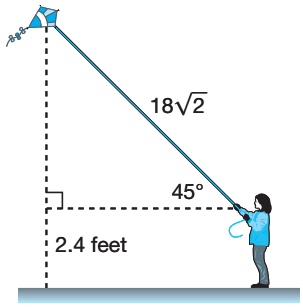
a. (Ex 1) Find the length of x.



b. (Ex 2) Find the length of p.



c. (Ex 3) The diagram shows a boy flying a kite. The kite forms a 45° angle as shown. Find the height of the kite from the ground. (Hint: The triangle is not touching the ground. Remember to add the distance between the triangle base and the ground to the height of the kite.)



Practice Distributed and Integrated

1. (36) Find the equation of the line parallel to the line $3y - x = 3$, and passing through the point $(2, -1)$.

Use elimination to solve.

2. (24)
$$8y - 3x = 22$$
$$2y + 4x = 34$$

3. (24)
$$3x + y = 11$$
$$3x - 2y = 2$$

4. (24)
$$3x + 4y = 20$$
$$-4x + 3y = 15$$

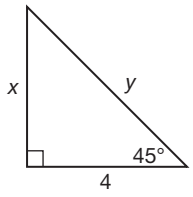
Add.

5. (37) $2 + \frac{a}{2x^2}$

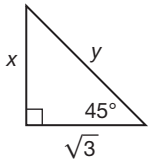
6. (37) $\frac{4}{cx} + c - \frac{3}{4c^2x}$

Use the properties of special right triangles to find x and y.

***7.** (52)



***8.** (52)



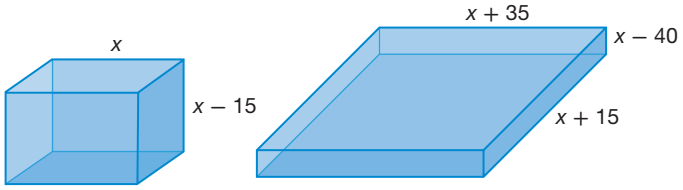
***9.** (51) Divide $(4x^4 + 21x^3 + 10x^2 + 25x) \div (x^2 + 5x)$ using polynomial long division.

10. (40) **Aviation** The radical expression $3.56\sqrt{A}$ can be used to approximate the distance in kilometers to the horizon from an altitude of A meters. What is the approximate distance to the horizon observed by a pilot flying a Boeing 747 at an altitude of 13,225 meters?

11. (31) What are all the values of x for which the multiplication expression $\frac{x+5}{x+2} \cdot \frac{x}{x^3+2x^2+x}$ is undefined?

***12.** (52) **Write** Explain why all 45°-45°-90° triangles are similar.

13. (38) **Physics** Through the process of annealing, metals are made more flexible, allowing a piece of metal to be flattened without breaking. Suppose a rectangular piece of metal with square ends goes through the annealing process and is flattened into a thin sheet. What is the ratio of the new surface area to the original?



14. (22) **Multiple Choice** A _____ function is a function with a connected graph.

A discrete **B** continuous **C** discontinuous **D** relation

***15.** (51) **Multi-Step**

a. Divide the polynomial $x^5 + 4x^4 - 13x^3 - 52x^2 + 36x + 144$ by $x^2 - x - 6$ using long division.

b. Divide the quotient of part **a** by $(x + 5)$ to find its quotient and remainder using synthetic division.

16. (46) **Monuments** A painter at the top of one of the two towers at the Tower Bridge in London throws a length of rope down from his position 63 meters above the ground to another painter who secures the rope to the ground so that it makes a 68° angle with the ground. To the nearest meter, what is the length of the rope from the ground to the top of the tower?

17. (29) **Error Analysis** Find and explain the error in solving the system. Correct the error and solve:

$$\begin{aligned} -9x + 4y + 5z &= -16 \\ x - 4y - 6z &= 25 \\ -x + y - z &= -3 \end{aligned}$$

Step 1:

$$\begin{array}{r} -9x + 4y + 5z = -16 \\ +\ x - 4y - 6z = 25 \\ \hline -8x \qquad -z = 9 \end{array}$$

Step 2:

$$\begin{array}{r} x - 4y - 6z = 25 \\ +\ -x + y - z = -3 \\ \hline -3y - 7z = 22 \end{array}$$

***18.** (52) **Estimate** The hypotenuse of a 30°-60°-90° triangle is 20 inches. Estimate the length of the side opposite the 60° angle. Give your answer to the nearest whole inch.

19. **Multiple Choice** What is the value of h in the vertex form of $f(x) = x^2 - 3$?
(27)

A -3 **B** 0 **C** 1 **D** 3

20. **Model** Model FOIL for the multiplication of any two binomials in the form $(a + b)(a - b)$.
(19)

21. **Predict** Describe how the graph $y = 0.25(x + 6)^2$ differs from the graph of the parent function.
(30)

22. **Geometry** Find the area of the shape formed by the linear inequalities $x \geq 1$, $y \geq 1$, $y \leq 5$, $y \leq -2x + 9$.
(43)

23. **Data Analysis** One day Irma hikes up a mountain. The table shows times of day and her distances from the base of the mountain. She camps overnight, and then hikes down the same path the next day.
(15)

Time of Day	8 am	10 am	12 pm	2 pm	4 pm
Distances 1st day (mi)	0	2	4	6	8
Distances 2nd day (mi)	8	5.75	3.5	1.25	0

a. Write equations that relate her distances from the bottom to the times of day. Represent the distance by d and the number of hours after 8 a.m. by t.

b. Graph those equations.

c. Is there a point on the path at which Irma arrives at the same time of day on both days? Explain.

24. **Error Analysis** A student expanded the binomial $(m - 2)^3$ as shown. Explain the error.
(49)

$$(m-2)^3 = {}_3C_0m^32^0 + {}_3C_1m^22^1 + {}_3C_2m^12^2 + {}_3C_3m^02^3$$
$$= m^3 + 3m^2(2) + 3m(2)^2 + (2)^3$$
$$= m^3 + 6m^2 + 12m + 8$$

25. **Architecture** The volume of a building is $5x^2 + 50x - 280$ yd^3. If the height of the building is 5 yd, find the length and width of the building.
(23)

***26.** **Multi-Step** Consider the function $y = 3x + 2$.
(50)

a. Find an equation for its inverse.

b. Graph the function and its inverse on the same grid.

c. How are the slopes of the function and its inverse related?

27. **Playing Cards** A deck of cards has 4 suits of 13 cards each for a total of 52 cards.
(42)

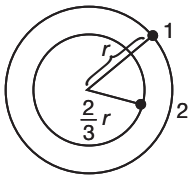
a. How many ways can one 5-card hand be dealt?

b. How many ways can one 5-card hand be dealt that includes 2 hearts and 3 cards from the other suits?

28. *(25)* **Model** The table shows the measurements in inches of various shipping boxes. Draw a box-and-whisker plot to represent the lengths of all edges of the boxes. Consider that a box has 12 edges.

Box	Length	Width	Height
1	11.875	3.375	13.625
2	11	8.5	5.5
3	7.5	5.125	14.375

29. *(37)* **Physics** A satellite has an orbital acceleration of $\frac{v^2}{r}$, where v is the speed of the satellite and r is the distance from the satellite to the Earth's center. Find the difference in acceleration between satellite 1 and satellite 2. Assume that each is traveling at the same speed, v.



***30.** *(45)* **Graphing Calculator** The table shows the average number of tropical storms per year compared to the average number of hurricanes per year. Use a graphing calculator to find the value of r and the equation for the line of best fit. Use it to predict the average number of hurricanes per year when there was an average of 9.5 tropical storms.

Tropical Storms	8.7	10.6	10.7	11.1	11.4	12.6	14.5
Hurricanes	5.3	6.1	6	6.2	6.3	6.8	7.8

LESSON 53

Performing Compositions of Functions

Warm Up

1. (4) **Vocabulary** A rule for using one number (an input) to calculate another number (an output) is called a _____.

2. (4) Write as a single simplified polynomial: $x(3x - 4) - 2(3x - 4)$.

3. (4) Evaluate the function $f(x) = x^2 - 3x$ at $x = -1$.

4. (4) Let $f(x) = x^2 - 5$. What are the domain and range of this function?

5. (23) Simplify $\frac{x^2 - x - 6}{x^2 - 4}$.

New Concepts

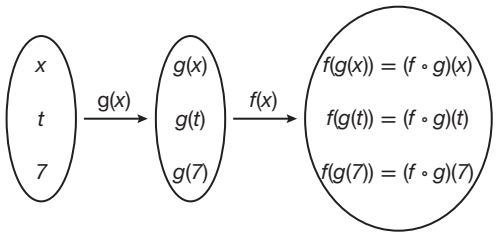
Sometimes two processes need to be combined into one using functions.

Reading Math

The composition $f(g(x))$ is read "f of g of x."

Composition of Functions

If $f(x)$ and $g(x)$ are functions, the **composite function** $f(g(x))$ uses output values from $g(x)$ as input values for $f(x)$. That is, the range values of $g(x)$ become the domain values for $f(x)$. The composition is written $f(g(x))$ or $(f \circ g)(x)$.



The **composition of functions** is not commutative. That is, for the functions $f(x)$ and $g(x)$, $f(g(x)) \neq g(f(x))$.

Example 1 Performing a Composition and Finding Its Domain and Range

Let $f(x) = 3x + 5$ and $g(x) = x^2$.

a. Find the composite function $f(g(x))$.

SOLUTION

$f(g(x)) = f(x^2)$ Substitute the output x^2 for $g(x)$.

$= 3x^2 + 5$ Use the output from $g(x)$ as input for $f(x)$.

b. Find the domain and range of the composite function from part **a**.

SOLUTION

The domain of $g(x)$ contains all real numbers. The range of $g(x)$ contains only the nonnegative real numbers. The range of $g(x)$ is the domain of $f(x)$. The range of the composite function $f(g(x))$ contains the real numbers greater than or equal to 5.

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Reading Math

The composite function $f \circ g$ is read "f of g."

Example 2 Evaluating a Composition

Let $f(x) = x^2 - 4x$ and $g(x) = 2x$. Evaluate the composition $(f \circ g)(3)$ using two different methods.

SOLUTION

Method 1: Use 3 as the input for the function $g(x) = 2x$.

$g(3) = 2(3) = 6$.

Now take 6 as the input for function $f(x) = x^2 - 4x$.

$$\begin{aligned} f(6) &= 6^2 - 4(6) \\ &= 36 - 24 \\ &= 12 \end{aligned}$$

So $(f \circ g)(3) = 12$.

Method 2: Replace the function $g(x)$ as the input for the function $f(x)$, then substitute 3 for the value of x.

$$\begin{aligned} f(g(x)) &= f(2x) && \text{Substitute } g(x) \text{ with } 2x. \\ &= (2x)^2 - 4(2x) && \text{Use the output from } g \text{ as the input for } f. \\ &= 4x^2 - 8x && \text{Simplify.} \end{aligned}$$

Now evaluate the function for $x = 3$: $(f \circ g)(3) = 4(3)^2 - 8(3) = 12$

Example 3 Determining Whether Function Composition Is Commutative

Let $f(x) = 2x - 3$ and $g(x) = 7x$.

a. Find the composition functions $(f \circ g)(x)$ and $(g \circ f)(x)$.

SOLUTION

$$\begin{aligned} (f \circ g)(x) &= f(7x) && \text{Substitute } g(x) \text{ with } 7x. \\ &= 2(7x) - 3 && \text{Use } 7x \text{ as the input for } f. \\ &= 14x - 3 && \text{Simplify.} \end{aligned}$$

On the other hand,

$$\begin{aligned} (g \circ f)(x) &= g(2x - 3) && \text{Substitute } f(x) \text{ with } 2x - 3. \\ &= 7(2x - 3) && \text{Use } 2x - 3 \text{ as the input for } g. \\ &= 14x - 21 && \text{Simplify.} \end{aligned}$$

b. Evaluate the composition functions $(f \circ g)(x)$ and $(g \circ f)(x)$ for $x = 3$.

SOLUTION $(f \circ g)(3) = 14(3) - 3 = 39$

$(g \circ f)(3) = 14(3) - 21 = 21$

Notice that the simplified expressions for the composition functions $(f \circ g)(x)$ and $(g \circ f)(x)$ are not equal. The values of the expressions at $x = 3$ are not equal either. Therefore the order of the composition of functions matters and is not commutative.

Math Reasoning

Verify If $f(x) = 7$ and $g(x) = 3x + 39$, then show that $f(g(x)) = 7$.

Example 4 Determining the Composition of Three Functions

Let $f(x) = -2x$, $g(x) = 4x - 5$, and $h(x) = x^2$. Find the composite function $f(g(h(x)))$.

SOLUTION

$$\begin{aligned} f(g(h(x))) &= f(g(x^2)) && \text{Substitute } h(x) \text{ with } x^2. \\ &= f(4(x^2) - 5) && \text{Use } x^2 \text{ as the input for } g(x). \\ &= f(4x^2 - 5) && \text{Simplify.} \\ &= -2(4x^2 - 5) && \text{Use } 4x^2 - 5 \text{ as the input for } f(x). \\ &= -8x^2 + 10 && \text{Simplify.} \end{aligned}$$

Example 5 Application: Sales and Discounts

A store is holding a sale. All items are being sold with a discount of 30%. Employees receive an additional 10% discount. Find a function that describes how much employees pay for an item with price p.

SOLUTION

Understand If an item's original price is \$10, the sale price is 30% off, or \$3 less, which is \$7. Employees receive a discount of 10%, which is \$0.70. So for an item with an original price of \$10, employees pay \$6.30. To find this result, we applied a 30% discount then the 10% employee discount.

Plan Notice that we use the output from the 30% sale function as the input to the 10% off employee discount function. The sale function can be written as $s(p) = p - 0.3p$, where p is the original price of an item without discounts. The employee discount function can be written as $d(p) = p - 0.1p$. Because two separate discounts are applied to the price p of the item, we need to find a composite function $d(s(p))$.

Solve

$$\begin{aligned} d(s(p)) &= d(p - 0.3p) && \text{Substitute } s(p) \text{ with } p - 0.3p. \\ &= (p - 0.3p) - 0.1(p - 0.3p) && \text{Substitute } p - 0.3p \text{ for the item price } p \text{ in the function } d(s) = p - 0.1p. \\ &= p - 0.3p - 0.1p + 0.03p && \text{Use the Distributive Property to simplify.} \\ &= 0.63p && \text{Simplify.} \end{aligned}$$

Employees pay 0.63 times the original price of an item.

Check Another way to solve the problem is to simplify first.

$s(p) = p - 0.3p = 0.7p$

$d(p) = p - 0.1p = 0.9p$

Then $d(s(p)) = d(0.7p) = 0.9(0.7p) = 0.63p$. This is the same function as above, so the solution checks.

Lesson Practice

a. (Ex 1) Let $f(x) = 2x - 9$ and $g(x) = (x + 1)^2$. Find the composite function $f(g(x))$.

b. (Ex 1) Let $f(x) = x^2 - 2$ and $g(x) = x - 4$. What are the domain and range of $(f \circ g)(x)$?

c. (Ex 2) Let $f(x) = 3x - 7$ and $g(x) = 4x + 5$. Evaluate the composition $(f \circ g)(7)$ using two different methods.

d. (Ex 3) Let $f(x) = x + 6$ and $g(x) = x^2$. Are $(f \circ g)(x)$ and $(g \circ f)(x)$ equal?

e. (Ex 3) Evaluate the composite functions $f(g(x))$ and $g(f(x))$ when $x = -1$ for the functions $f(x) = 2x - 1$ and $g(x) = -x$.

f. (Ex 4) Let $f(x) = -3x + 1$, $g(x) = 9x$, and $h(x) = -x^2$. Find the composite function $f(g(h(x)))$.

g. (Ex 5) A department store is advertising a sale in which all clothes are 15% off. On Saturday morning, there is an additional 20% discount. Find a function describing how much an item with price p will cost on Saturday morning.

Practice Distributed and Integrated

Solve.

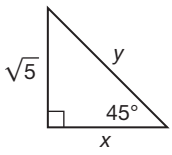
1. (7) $2\frac{1}{4}x - 3\frac{1}{2} = -\frac{1}{16}$

2. (7) $0.002x - 0.02 = 6.6$

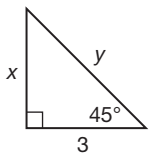
3. (7) $\frac{3 - 2x}{4} + \frac{x}{3} = 5$

Use the properties of special right triangles to find x and y.

***4.** (52)



***5.** (39)



Use $3x + 10 > 4y$.

6. (39) Find a point that is a solution.

7. (39) Find a point that is not a solution.

8. (49) How many terms does the binomial expansion of $(3r + s)^{17}$ contain?

***9.** (53) Let $x(t) = 3$ and $y(t) = -7t + 6$. Find $x(y(t))$ and $y(x(t))$.

***10.** (53) Let $p(x) = 3$ and $q(x) = -x^2 + 3x$. Find $p(q(x))$ and $q(p(x))$.

11. (33) **Analyze** Suppose there are three tiles in a bag, labeled A, B, and C. A tile is chosen at random, then a second tile is chosen at random, and then the third tile is chosen. Each tile that is chosen is not replaced in the bag. List all the outcomes in the sample space of this experiment. How many outcomes are in the sample space?

*12. (53) **Multi-Step** A bookstore gives you a 10% student discount, but you still must pay the 6% sales tax. Use composite functions to:

a. Determine how much you save if the discount is calculated first.

b. Determine how much you save if the tax is calculated first.

c. Determine which method will save you the most.

13. (44) **Multi-Step** A rectangle has an area of 60 square centimeters and a length of $\frac{3\sqrt{3}}{2}$ centimeters.

a. Find the width of the rectangle.

b. Find the perimeter of the rectangle.

14. (39) Find a point that is a solution of $-7x + 18 < 3y$ and a point that is not a solution.

15. (51) **Write** Determine the polynomial divisor, dividend, and quotient represented by the synthetic division shown.

$$\begin{array}{r|rrrrr} -1 & 10 & 5 & 4 & 0 & -9 \\ & 0 & -10 & 5 & -9 & 9 \\ \hline & 10 & -5 & 9 & -9 & 0 \end{array}$$

16. (40) **Error Analysis** Find and correct the error(s) the student made.

$$\frac{3}{\sqrt[3]{3}} \cdot \frac{\sqrt[3]{3}}{\sqrt[3]{3}} = \frac{3\sqrt[3]{3}}{\sqrt[3]{9}} = \frac{3\sqrt[3]{3}}{3} = \sqrt[3]{3}$$

*17. (47) **Graphing Calculator** At the start of the year, there are 23 rabbits living in a field. The growth of the rabbit population is modeled by the equation $P(x) = 1.2^x + 22$, where $P(x)$ is the number of rabbits in the field x weeks into the year. Use a graphing calculator to predict how many weeks it will take for there to be more than 100 rabbits.

*18. (53) **Multiple Choice** Let $h(x) = x^{-3} + x^3$ and $k(x) = x^{-1}$. What is the value of $h(k(2))$?

A 8 **B** $\frac{1}{8}$ **C** $8\frac{1}{8}$ **D** $\frac{8}{65}$

19. (46) **Geometry** The altitude of an isosceles triangle bisects its base. Find the length of the base of an isosceles triangle whose legs measure 20 inches and whose base angles measure 70°. Round to the nearest hundredth.

20. (35) **Gravity** The height of a free-falling object (neglecting air resistance) is given by the function $h(t) = -4.9t^2 + v_0t + h_0$, where h_0 is the initial height in meters, v_0 is the initial velocity in meters per second, and $h(t)$ is the height in meters at time t seconds. How long does it take for an object to hit the ground after it is thrown straight up at a speed of 4.9 meters per second from a height of 29.4 meters? (Hint: Factor −4.9 from the trinomial.)

*21. (53) **Analyze** Suppose $p(u) = 2u + 5$ with domain $\{1, 3, 5, 7\}$ and $r(u) = u - 3$ with domain $\{6, 8\}$. Then the domain of $p(u)$ limits the domain and range of the composite function. Can you change the domains of functions p and r so that the domain of $r(u)$ limits the domain and range of the composite function?

22. *(41)* **Cartography** A cartographer constructs a map of Montana over a grid system such that Great Falls is at (125, 150) and Conrad is at (86, 220). Find the distance between the cities to the nearest kilometer if each unit represents one kilometer.

23. *(49)* **Probability** Suppose the probability of you scoring a goal in a soccer game is $\frac{1}{3}$ for every shot taken. If you take 8 shots in a game, what is the probability that you will score 2 goals?

24. *(37)* **Analyze** What must be true of a and b if the following equation is true? Explain.

$$\frac{1}{x^2 + ax + bx + ab} = \frac{1}{x^2 - 1}$$

25. *(52)* **Irrigation** An irrigation pipeline starts from the top of a hill and has an angle of elevation of 45°. If the length of the pipe line is 720 meters, what is the vertical rise of the pipe? Round your answer to the nearest meter.

26. *(48)* **Generalize** Consider the following sequence of complex fractions.

$$1, \frac{1}{1+1}, \frac{1}{1+\frac{1}{1+1}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}}, \ldots$$

Simplify the first 6 terms. Describe any patterns you see. Can you predict the next term without writing and simplifying the complex fraction?

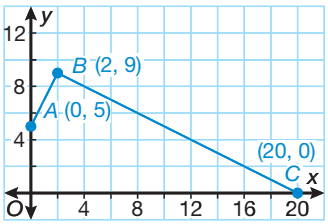
27. *(42)* **Multiple Choice** Evaluate $\frac{8!}{5!}$.

A 1.6 **B** 336 **C** 1680 **D** 6720

28. *(36)* **Speed** The speed of an object at time t is based on its initial speed v_o and its acceleration a: $v = v_o + at$.

a. Find the equations of the lines from A to B and B to C and describe the motion of the object.

b. What is the relationship between the equations of motion?



29. *(50)* **Population** The population p of a city for the years from 1998 is modeled by $p = 326{,}000 + 8526t$, where $t = 0$ represents the year 1998. Solve the equation for t and use it to predict the year in which the population of the city exceeds 450,000.

30. *(43)* **Ticket Prices** Two of the ticket prices for a football game were \$92 for a field seat and \$54 for an upper view seat. You decide to buy up to 15 tickets and to spend up to \$1000. Let x represent a number of field seats and y represent a number of upper view seats.

a. Write a system of linear inequalities that represents the situation.

b. Describe the points in the graph of the system that represent all the possible combinations of field seats and upper view seats you can buy.

LESSON
54

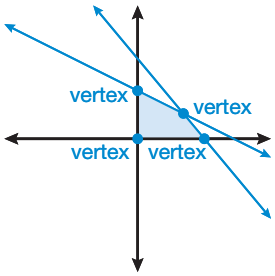
Using Linear Programming

Warm Up

1. (10) An expression that contains the symbols $<$ or $>$ is called a(n) _____.
2. (39) Is (5, 10) a solution of $y \geq 2x - 3$.
3. (4) Determine $f(-5)$ if $f(x) = 2x^2 - 3x$.

New Concepts

Linear programming is a mathematical technique that allows you to make optimal decisions from a complex set of choices. In linear programming, you are looking for the maximum or minimum value of an **objective function,** subject to certain **constraints.** The constraints are given as linear equations or inequalities.



In the graph, the shaded area is known as the **feasible region** where all possible solutions are found. When evaluating a solution, a key area to focus on are the corners of the feasible region, also known as vertices.

Math Language

Linear programming is useful in many fields. In business, linear programming may be used to minimize cost or optimize income.

Example 1 Graphing a Linear Programming Problem

A marketing manager is coordinating an ad campaign. TV ads are $2000 and reach 10,000 people, and radio ads are $500 and reach 5,000 people. The target size of the audience for this marketing campaign is at least 100,000 people. He wants to buy no more than 10 TV ads and no more than 15 radio ads. How can the marketing manager minimize his costs within these conditions?

SOLUTION

x = number of TV ads, y = number of radio ads,

Objective function: Cost $= 2000x + 500y$

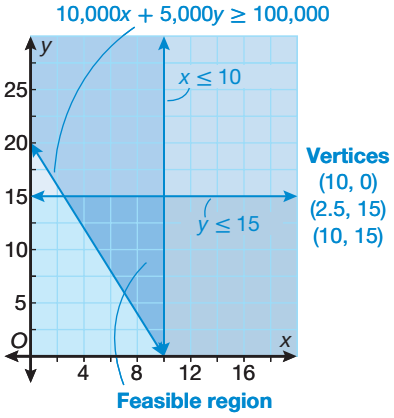
Constraints: $0 \leq x \leq 10$, $0 \leq y \leq 15$, $10{,}000x + 5{,}000y \geq 100{,}000$

Hint

The minimum (or maximum) value of the objective function occurs at a vertex of the graph of the system.

Find the points within the feasible region that result in the minimum value for the objective function. Evaluate the objective function for the coordinates of the vertices:

x	y	Cost $= 2000x + 500y$
10	0	20,000
2.5	15	12,500
10	15	27,500



The solution that leads to the least amount of marketing dollars spent is $x = 2.5$ and $y = 15$. Since it makes no sense to buy half a TV ad, the real solution is 3 TV ads and 15 radio ads, for a total cost of $13,500.

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Example 2 Finding the Minimum and Maximum Values for the Objective Function

A farmer needs to decide how much corn and soybeans to grow for the next harvest. The cost of corn seed is $80 per bag, while soybeans are $50 per bag. A bag of corn requires 2 acres of farmland, while soybeans require 1 acre. The farm's acreage is 5000 acres. It takes 3 hours to plant a bag of corn and 2 hours to plant soybeans. The farmer expects to need at least 2500 hours of planting. How can the farmer manage costs for the next harvest?

SOLUTION

Let x = the number of bags of corn

Let y = the number of bags of soybeans

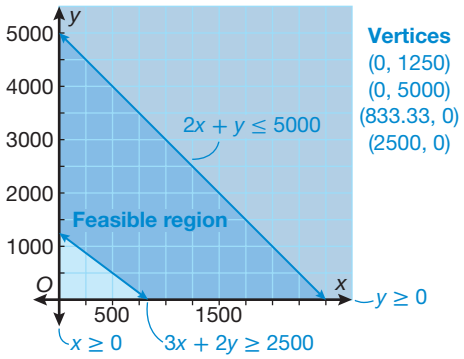
Objective function: Cost $= 80x + 50y$

The goal is to find the minimum and maximum cost subject to these constraints:

$x \geq 0, y \geq 0$

$2x + y \leq 5000$

$3x + 2y \geq 2500$



Evaluate the objective function for the coordinates of the vertices:

x	y	Cost $= 80x + 50y$
0	1250	62,500
0	5000	250,000
$833\frac{1}{3}$	0	66,666.67
2500	0	200,000

The farmer has several options.

- To keep costs at an absolute minimum, purchase 1250 bags of soybeans.
- To plant all 5000 acres at the lowest cost, purchase 2500 bags of corn.

Example 3 An Unbounded Feasible Region

A wildlife manager needs to ensure that Lake Katherine has a sufficient number of trout and bass for the fishing season. It costs 25 cents per trout and 75 cents per bass to stock the lake. About $\frac{1}{8}$ of the trout and $\frac{1}{4}$ of the bass stocked in the lake are caught during the fishing season. It is estimated that at least 10,000 fish will be caught this season. About 1% of the trout and 5% of the bass become the prey of birds and other animals. The attrition estimate for this season is at least 1,000 fish. How can the wildlife manager keep costs to a minimum?

Math Reasoning

Justify If you name the variables differently, that is, *y* would represent the trout and *x* the bass, would the solution be different? Explain.

SOLUTION

Let x = the number of trout

Let y = the number of bass

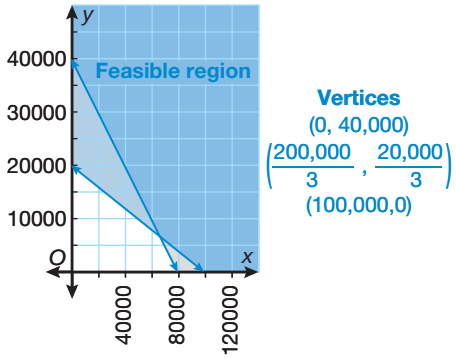
Objective function: Cost $= 0.25x + 0.75y$

The goal is to find the minimum and maximum cost subject to these constraints:

$$x \geq 0, y \geq 0$$

$$\frac{1}{8}x + \frac{1}{4}y \geq 10{,}000$$

$$0.01x + 0.05y \geq 1000$$



Math Reasoning

Analyze What kind of constraints would make the feasible region bounded? Provide some examples.

In this example, the feasible region is not enclosed. This means that there is no maximum value for the objective function. Evaluate the objective function for the coordinates of the vertices:

x	y	Cost $= 0.25x + 0.75y$
0	40,000	30,000
200,000/3	20,000/3	≈21,667
100,000	0	25,000

The wildlife manager has to consider the following:

- To keeps costs low, purchase about 66,667 trout and 6,667 bass.
- Since the feasible region is unbounded, there is no maximum cost. This happens because in the fishing season it is possible that a combination of heavy fishing and a larger predator population could severely cut back on the number of fish.
- It may be necessary to stock the lake with more fish during the fishing season. Thus, the minimum obtained from the objective function may not be the true minimum for stocking the lake.

Example 4 Application: Business

A company makes two models of MP3 players. The M20 takes 3 hours to manufacture and the M25 takes 1 hour. The company has multiple shifts and has a 20-hour day for manufacturing the players. The M20 generates a profit of $12, and the M25 a profit of $7. The M20 uses 15 special chips, while the M25 uses 10. For the next manufacturing cycle, consisting of 18 days, there are 3,000 chips available. How many of each type of MP3 player should the company produce to maximize its profits? What combination of MP3 players would result in the least amount of profit?

SOLUTION

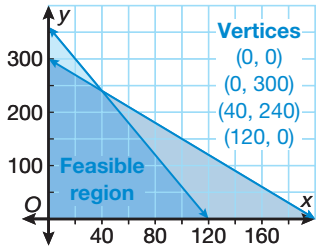
Let x = the number of M20 models

Let y = the number of M25 models

Objective function: Profit $= 12x + 7y$

The goal is to find the minimum and maximum profit with these constraints:

$$x \geq 0, y \geq 0$$
$$3x + y \leq 360$$
$$15x + 10y \leq 3000$$



Evaluate the objective function for the coordinates of the vertices:

x	y	Profit $= 12x + 7y$
0	0	0
0	300	2100
40	240	2160
120	0	1440

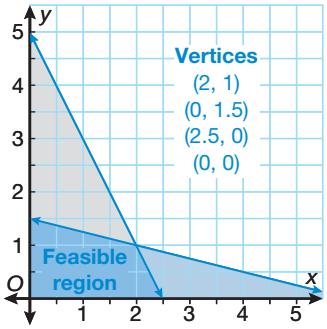
Aside from the trivial solution of (0, 0), the solution that results in the least profit is to solely produce 120 M20s. The solution that offers the greatest profit is to produce 40 M20s and 240 M25s.

Math Reasoning

Justify How many hours will be spent manufacturing the players in 18 days? Explain.

Lesson Practice

a. *(Ex 1)* Given the objective function $C = 25x + 30y$ and the graph and vertices shown, find the minimum and maximum values.



b. (Ex 2) A caterer needs to prepare two different meals for a conference dinner. One is a chicken dish and the other is a seafood dish. The chicken dish costs \$2.50 to prepare and the seafood dish costs \$3.75. There will at most be 1000 people attending. Usually, twice as many people order chicken than order fish. How can the caterer keep costs to a minimum while still making sure that most people get their meal choice?

c. (Ex 3) Parade organizers are expecting at least 100,000 people to attend the annual parade. For every group of 100, the organizers would like to have a security officer available along the parade route. The cost for each officer is \$100. How can they keep costs to a minimum?

d. (Ex 4) Using the information from Example 4, suppose that for the next manufacturing cycle, consisting of 36 days, there are 9,000 chips available. How many of each type of MP3 player should the company produce to maximize its profits?

Practice Distributed and Integrated

Given the objective function *P* and the constraints, find the minimum and maximum values for the objective function.

***1.** (54) $P = 2x + 3y$ given $x \geq 0$, $y \geq 0$, $0.4x + 0.9y \leq 25$

***2.** (54) $P = 5x + 9y$ given $x \geq 0$, $y \geq 0$, $25x + 35y \leq 25$

Simplify.

3. (48) $\dfrac{a^2x - \frac{a}{x}}{ax - \frac{4}{x}}$

4. (48) $\dfrac{\frac{m}{m+x}}{\frac{x}{m+x}}$

5. (48) $\dfrac{\frac{s}{a+b} + \frac{x}{b}}{\frac{a}{s+x}}$

6. (22) **Multiple Choice** A point of discontinuity is a point at which the graph

A increases. **B** is disconnected. **C** ends. **D** decreases.

Identify which equations are direct variations and which are inverse variations.

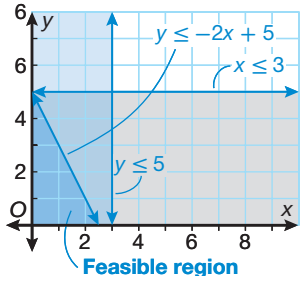
7. (12) $y = \frac{1}{2}x$

8 (12) $xy = 15$

9 (12) $y = -\frac{1}{2}x + 5$

10. (18) **Saving Money** Jason has 300 nickels in his money jar. How many dollars does Jason have in nickels?

***11.** (54) **Coordinate Geometry** Find the coordinates of the vertices for the feasible region.



12. (54) Give an example of a problem situation where the feasible region might include negative values.

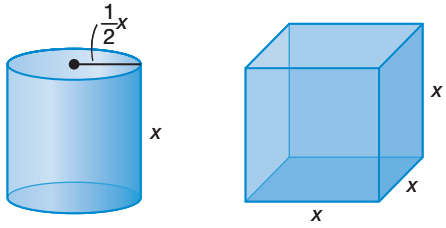
13. (21) **Error Analysis** Two students were asked which of these lines were intersecting. Their answers are below. Which student is correct? What is the relationship between the lines that the incorrect student answered?

Equation 1: $3y + 4x = 5$; Equation 2: $8x = 10 - 6y$; Equation 3: $6y - 10 = 8x$

Dietrich: Equations 1 and 2 intersect.

Shelby: Equations 1 and 3 intersect, so equations 2 and 3 intersect as well.

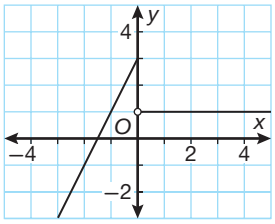
14. (11) **Aquarium Design** Two aquariums each require the same amount of space on a shelf. The shapes of the aquariums are given in the picture. Write a polynomial that represents how much less water the cylindrical aquarium holds than the aquarium in the shape of a cube. (*Hint:* The formula for the volume of a cylinder is $V = \pi r^2 h$.)



***15.** (53) **Write** Describe how to find a composite function $f(g(x))$ if the functions $f(x)$ and $g(x)$ are known.

16. (52) **Analyze** A chord in a circle makes an angle of 45° with the radius. If the chord is 4 cm long, what is the radius of the circle?

17. (22) **Verify** Explain how to show that the graph is an example of a function. Is the function continuous, discontinuous, and/or discrete?



18. (25) **Rainfall** Annual rainfall amounts in inches for Austin, Texas, for 2005 are listed in the table below. If the mean amount of rainfall in Austin during that year was 1.861 inches, find x.

2005	J	F	M	A	M	J	J	A	S	O	N	D
Amount (inches)	2.25	2.21	x	0.72	3.13	0.89	2.75	2.44	1.44	1.78	0.33	0.09

19. (51) **Error Analysis** Jenny and Haley each tried to divide the polynomial $2x^4 - 6x^3 + x^2 - 3x - 3$ by $x - 3$. Who is correct? Explain your reasoning.

Jenny

3	2	−6	1	−3	−3
	0	6	0	3	0
	2	0	1	0	−3

Haley

−3	2	−6	1	−3	−3
	0	−6	36	−111	342
	2	−12	37	−114	339

20. (30) **Justify** Max said that it is possible to write the standard form of a quadratic equation given its vertex and y-intercept. Show that Max is correct by writing the equation of a quadratic function in standard form whose vertex is (1, 9) and y-intercept is 13.

21. *(46)* **Generalize** Tell how to find the cotangent of an angle when given the tangent of the angle. Give an example.

22. *(29)* **Multiple Choice** Circle the letter that represents the number and type of solutions of the system

$$-2x - 10y - 6z = -2$$
$$x + 5y + 3z = -16$$
$$5x + 25y + 15z = 5$$

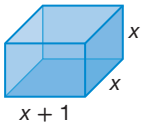
A No solution, Inconsistent **B** One solution, Inconsistent

C One solution, Consistent **D** Infinitely many solutions, Consistent

***23.** *(14)* **Graphing Calculator** Use a graphing calculator to solve the following.

$$\det \begin{bmatrix} 9 & 3 & 0 \\ 6 & 12 & 1 \\ -3 & -9 & 15 \end{bmatrix}$$

24. *(31)* **Geometry** Write a rational expression in simplest form for the ratio of the rectangular prism's surface area to its volume.



***25.** *(53)* Let $a(t) = t$ and $b(t) = 3.1t^2 - 2.0t + 4.3$. What is $a(b(t))$?

***26.** *(53)* **Cost of Lobster** At a restaurant on the Maine coast, the supply of fresh new shell lobsters during the season (July through October) is given by the function $L(t) = -0.006t^2 + 0.733t$, where t is the number of days after July 1. If the restaurant has x lobsters for a day, it charges $C(x) = 37 - x$ dollars for each one. What function gives the restaurant's price per lobster in terms of the number of days after July 1?

27. *(44)* **Pendulums** The time t in seconds a pendulum takes to complete one full swing is given by the equation $t = 2\pi\sqrt{\frac{l}{9.8}}$, where l is the length of the pendulum in meters. Suppose a pendulum has a length of 5 meters. How much time will the pendulum take to complete one full swing?

28. *(45)* **Multi-Step**

a. Make a scatter plot of the data set.

b. Sketch a line of best fit.

x	20	30	42	58	64	75	83	90
y	1	4	2.5	3	6	4.5	7	6.5

29. *(23)* Factor $5x^2 + 4x^3 - x^4$ completely. Always factor the GCF as the first step.

30. *(Inv 5)* Give an example of how to use a number cube to make a binomial experiment.

LESSON
55

Finding Probability

Warm Up

1. (33) **Vocabulary** The total number of possible events is known as the _____.

2. (33) If there are three models of bicycles that come in four different colors, how many different bicycles are there to choose from?

3. (SB) **Multiple Choice** What is $\frac{3}{8}$ written as a percent?

A 3.8% **B** 3.75%

C 37.5% **D** 375%

4. (42) Evaluate ${}_3P_2$.

5. (42) Evaluate ${}_3C_2$.

New Concepts

An event is an outcome, or set of outcomes, of a sample space. The **probability** of an event is the **likelihood** of that event occurring. It is given as a number on a continuous scale from 0 to 1, where 0 means that the event is impossible and 1 means that the event will definitely occur. Probabilities can be written as fractions, decimals, or percents, so for an event that is just as likely to occur as not to occur, the probability, P, is $\frac{1}{2} = 0.5 = 50\%$.

A **probability experiment** is an occurrence whose outcome is uncertain. An event in a probability experiment can be called a **random event** because it occurs at **random,** or by chance. For example, the event of getting heads in the experiment of flipping a coin is a random event because it cannot be predicted.

Math Reasoning

Analyze What is the probability of an event that is made up of all the outcomes in the sample space? Why?

Theoretical Probability

For equally likely outcomes, the **theoretical probability** P of an event occurring is the ratio of the number of favorable outcomes to the total number of outcomes in the sample space: $P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$, where a **favorable outcome** is an outcome in the event.

In theoretical probability there must be **equally likely outcomes.** That is, each outcome has the same probability of occurring. In the experiment of rolling a number cube, the likelihood of the six outcomes is the same. $P(\text{even number}) = \frac{3}{6} = \frac{1}{2}$ because there are three favorable outcomes (2, 4, 6) out of six total outcomes (1, 2, 3, 4, 5, and 6). This probability is a *theoretical probability* because it says what should happen. In theory, half of all the rolls should yield an even number, but actual experiments may yield different results.

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Geometric probability is a theoretical probability based on geometric measurements such as length, area, or volume.

Example 1 Calculating Probability

a. A calculator is programmed to generate random integers from 1 to 100 inclusive when the enter button is pressed. What is P(multiple of 20) when the enter button is pressed one time?

SOLUTION Each outcome (numbers 1 to 100) is equally likely. Use the formula $P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$. There are five favorable outcomes (20, 40, 60, 80, and 100) and 100 total outcomes.

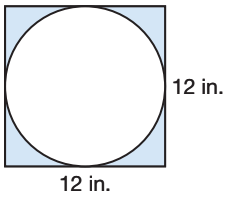
$$P(\text{multiple of } 20) = \frac{5}{100} = \frac{1}{20} = 0.05$$

Hint

For mutually exclusive events A and B, $P(A \text{ or } B) = P(A) + P(B)$.

Notice that the event of getting a multiple of 20 is made up of the mutually exclusive events of getting 20, 40, 60, 80, and 100. The probability can also be found by adding $P(20)$, $P(40)$, $P(60)$, $P(80)$, and $P(100)$: $0.01 + 0.01 + 0.01 + 0.01 + 0.01 = 0.05$.

b. Estimate the probability that a point chosen randomly inside the figure is in the shaded region.



SOLUTION

$$P(\text{shaded region}) = \frac{\text{area of shaded region}}{\text{area of square}}$$

$$= \frac{\text{area of square} - \text{area of circle}}{\text{area of square}}$$

$$= \frac{s^2 - \pi r^2}{s^2} = \frac{12^2 - \pi 6^2}{12^2}$$

$$= \frac{144 - 36\pi}{144} \approx 0.2146$$

The probability that the point is in the shaded region is about 21.46%.

Example 2 Calculating Probability from Permutations and Combinations

a. All of the parking spaces in a lot are labeled with two different digits so that each parking space is labeled differently. What is the probability that a parking spot chosen at random is labeled with two even digits?

SOLUTION The total number of outcomes is a permutation because space 14 is different from space 41. The number of outcomes is ${}_{10}P_2 = 90$.

The outcomes that make up the event are numbers that begin with 0, 2, 4, 6, and 8. Each of these can be followed by an even number other than the number that makes up the first digit, so each of 0, 2, 4, 6, and 8 can be followed by four other digits. The number of favorable outcomes is therefore $5 \cdot 4 = 20$.

$$P(\text{2 even digits}) = \frac{20}{90} = \frac{2}{9} \approx 0.22 \text{ or about } 22\%$$

b. A set of 22 books (volumes 1 through 22) are in a crate. If a student randomly chooses 5 books from the crate, what is the probability that the student picks either the first five volumes or the last five volumes?

SOLUTION The total number of outcomes is a combination because choosing volumes 1, 2, 3, 4, and 5 is the same as choosing volumes 5, 4, 3, 2, and 1. The number of outcomes is ${}_{22}C_5 = 26{,}334$.

Out of the possible 26,334 possible groupings of 5 books, there are two favorable groupings, the one with the first five volumes and the one with the last five volumes.

$$P(\text{first five or last five}) = \frac{2}{26{,}334} = \frac{1}{13{,}167} \approx 0.00008$$

When two events are independent, the occurrence of one event does not affect the probability of the other. When events are dependent, the occurrence of one event does affect the probability of the other.

Math Language

Conditional probability
$P(B|A)$ is the probability of event B given that event A has occurred.

Probability of Independent and Dependent Events
For independent events A and B, $P(A \text{ and } B) = P(A) \cdot P(B)$. For dependent events A and B, $P(A \text{ and } B) = P(A) \cdot P(B\|A)$, where $P(B\|A)$ is the probability of B given that A has already occurred.

Example 3 Calculating the Probability of Independent and Dependent Events

a. A calculator is programmed to generate random integers from 1 to 100 inclusively when the enter button is pressed. If the enter button is pressed twice, what is the probability of getting a multiple of 20 and then a number greater than 90?

SOLUTION Because the occurrence of a multiple of 20 on one press does not affect the probability of getting a number greater than 90 on the next, the events are independent. Multiply the individual probabilities.

$$P(\text{multiple of 20, } > 90) = \frac{5}{100} \cdot \frac{10}{100} = \frac{1}{20} \cdot \frac{1}{10} = \frac{1}{200}$$

b. A bag contains 2 red and 6 blue marbles. What is the probability of picking a red marble, keeping it out of the bag, and then picking a blue marble?

SOLUTION Keeping the first marble out of the bag affects the probability of picking the next marble because the sample space has one less outcome. Whereas $P(\text{blue})$ would be $\frac{6}{8} = \frac{3}{4}$ if the events were independent, it is now $\frac{6}{7}$ because the events are dependent.

$$P(\text{red then blue}) = \frac{2}{8} \cdot \frac{6}{7} = \frac{12}{56} = \frac{3}{14}$$

Like probability, the odds in favor of an event indicate the likelihood of the event occurring.

Odds of an Event
$\text{odds in favor of event} = \dfrac{\text{number of favorable outcomes}}{\text{number of unfavorable outcomes}}$

Odds are sometimes written as a ratio with the word *to* or with a colon.

number of favorable outcomes : number of unfavorable outcomes

Example 4 Calculating Odds

a. A bag contains 2 red and 6 blue marbles. What are the odds in favor of picking a red marble?

SOLUTION $\text{odds in favor of red} = \dfrac{\text{number of red marbles}}{\text{number of non-red marbles}} = \dfrac{2}{6} = \dfrac{1}{3}$

b. The odds in favor of an event are 5 : 2. What is the probability of the event?

SOLUTION The total number of outcomes is the sum of the favorable and unfavorable outcomes: $5 + 2 = 7$.

$$P(\text{event}) = \frac{\text{favorable outcomes}}{\text{all outcomes}} = \frac{5}{7}$$

A probability model shows the probability of each outcome in a sample space. The probabilities can be theoretical or experimental (based on actual data). **Experimental probability** is the ratio of the number of times an event occurs to the number of trials, or times, that an activity is performed. The sum of the probabilities in a probability model should sum to 1 (or near 1 if rounding).

Example 5 Application: Meteorology

The table shows the number of tropical storms in the Atlantic, Caribbean, and Gulf of Mexico from 1851 through 2006. Construct an experimental probability model.

Season	Jan–Apr	May–July	Aug–Oct	Nov–Dec
Number of Storms	5	198	1081	70

SOLUTION Divide the number of storms in each season by the total number of storms. The total is $5 + 198 + 1081 + 70 = 1354$. Round as needed.

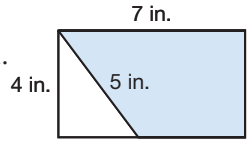
Season	Jan–Apr	May–July	Aug–Oct	Nov–Dec
***P*(tropical storm)**	0.00369	0.14623	0.79838	0.05170

Check The sum of the probabilities, $0.00369 + 0.14623 + 0.79838 + 0.05170$, is 1.

Lesson Practice

a. (Ex 1) A calculator is programmed to generate random integers from 1 to 10 inclusively when the enter button is pressed. What is P(perfect square) when the enter button is pressed one time?

b. (Ex 1) Estimate the probability that a point chosen randomly inside the rectangle is in the shaded region.



c. (Ex 2) Five of seven athletes of varying heights will be chosen for a photo. If their positions are randomly chosen, what is the probability that they are lined up from shortest to tallest from left to right?

d. (Ex 2) A teacher has 6 papers to grade. If she randomly chooses 4 papers to grade after school, what is the probability that she chooses the papers of the four students who usually score the highest?

e. (Ex 3) A calculator is programmed to generate random numbers from 1 to 100 inclusively when the enter button is pressed. If the enter button is pressed twice, what is the probability of getting a number less than 10 and then a number greater than 50?

f. (Ex 3) A bag contains 15 red and 10 blue marbles. What is the probability of picking a blue marble, keeping it out of the bag, and then picking another blue marble?

g. (Ex 4) What are the odds of rolling a six on a number cube?

h. (Ex 4) The probability of an event is 10%. What are the odds in favor of the event?

i. (Ex 5) The table shows the number of wins, losses, and overtime losses of the Buffalo Sabres in the 2006–2007 regular season. Construct an experimental probability model.

Buffalo Sabres

Wins	Losses	Overtime Losses
53	22	7

Practice Distributed and Integrated

Simplify.

1. (44) $\dfrac{1}{-2\sqrt{24}}$

2. (44) $\dfrac{\sqrt{5}+3}{\sqrt{2}+\sqrt{45}}$

3. (44) $\dfrac{-1}{\sqrt{12}+3}$

4. (40) $5\sqrt{5}\left(2\sqrt{5}-3\sqrt{10}\right)$

5. (40) $3\sqrt{200}-5\sqrt{18}+7\sqrt{50}$

6. (40) $2\sqrt{3}\cdot 2\sqrt{2}\left(6\sqrt{6}-3\sqrt{2}\right)$

7. (43) **Write** Explain why all solutions of the system $\begin{matrix} x \geq 1 \\ y > x \end{matrix}$ lie in quadrant I.

***8.** (54) Given the objective function P and the constraints, find the minimum and maximum values for the objective function. $P = 7x + 8y$ given $x \geq 0$, $y \geq 0$, $2x + 3y \leq 100$, $4x + 5y \geq 75$

9. (37) **Error Analysis** Two students are subtracting the rational expressions $\frac{2x}{2x^2+7x-15}$ and $\frac{1}{x+5}$ but get different results. Which student made the mistake? Explain.

Student A	Student B
$\frac{2x}{2x^2+7x-15}-\frac{1}{x+5}$ $=\frac{2x}{(2x-3)(x+5)}-\frac{(2x-3)}{(x+5)(2x-3)}$ $=\frac{3}{2x^2+7x-15}$	$\frac{2x}{2x^2+7x-15}-\frac{1}{x+5}$ $=\frac{2x}{(2x-3)(x+5)}-\frac{(2x-3)}{(x+5)(2x-3)}$ $=\frac{4x-3}{2x^2+7x-15}$

10. (49) **Multi-Step** Suppose one term in the expansion of $(a+b)^n$ is $1716a^7b^6$.

a. To what power was $(a+b)$ raised?

b. Which term is $1716a^7b^6$?

11. (41) **Geometry** Write the length of the diagonal of a 3 ft by 3 ft square as a simplified radical expression.

***12.** (55) **Pet Shelter** In a pet shelter, there are two pairs of related kittens among 14 other kittens. What is the probability that someone randomly selecting two kittens will choose either pair of the related kittens?

13. (45) **Multiple Choice** Which value of r describes the set of points which are *farthest* from forming a line?

A $r=-0.99$ **B** $r=-0.4$ **C** $r=0.22$ **D** $r=0.81$

14. (50) **Doppler Effect** When something making a sound moves in relation to a stationary listener, the frequency heard by the listener is not the same as the actual frequency of the sound at its source. In normal atmospheric conditions, if f_l is the frequency (in hertz, or cycles per second) heard by the listener, f_s is the frequency at the source, and v is the velocity of the sound source (in miles per hour), then $f_l=\frac{740f_s}{740-v}$.

Suppose you are standing still and a police car is moving away from you with its siren blaring at 1800 hertz. You estimate its speed at about 6000 feet per minute. What frequency do you hear?

***15.** (54) **Analyze** Mobile-1 has two cell phone plans, the Star Plan and the Silver Plan. The cost for each plan is shown in the table. A small company is comparing the two plans. The company needs at least 10 cell phones, one for each sales rep, and each rep will use at least 800 minutes per month. What options does this company have?

Monthly Phone Rates

	Star Plan	Silver Plan
Start-up fee	\$75	\$125
Free minutes	2200	1500
Per minute rate above free minutes	7 cents	5 cents
Monthly service charge	\$25 × the number of cell phones	\$30 × the number of cell phones

*16. (47) **Graphing Calculator** Plot the graphs of $y = 3^x$ and $y = 3^x + 5$ on the same axes on a graphing calculator. Compare the domain and the range.

17. (42) **Multi-Step** How many 6-letter words (including nonsense words) can be formed from the letters **A B C D E F**?

a. If the order of the six letters doesn't matter?

b. If the letter **B** must be in the first place?

c. Is the answer in part **a** different from the answer for part **b?** Explain.

18. (33) **Analyze** Suppose there are three tiles in a bag, labeled A, B, and C. A tile is chosen at random, then a second tile is chosen at random, and then a third tile is chosen at random. Each tile that is chosen is replaced in the bag. List all the outcomes in the sample space of this experiment. How many outcomes are in the sample space?

19. (46) **Coordinate Geometry** The vertices of a triangle are $A(-3, 6)$, $B(2, -4)$, and $C(-3, -4)$. Find the value of the cotangent of $\angle B$.

*20. (55) **Error Analysis** The odds in favor of an event are 3 to 5. Which student made an error in finding the probability of the event? What is the error?

Student A	Student B
$P(\text{event}) = \frac{3}{3+5}$ $= \frac{3}{8} = 37.5\%$	$P(\text{event}) = \frac{3}{5}$ $= 60\%$

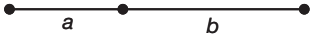
*21. (53) **Compound Interest** You deposit p dollars into a savings account. If the account earns interest at a rate of 1% per quarter, then after one quarter your account is worth $f(p) = (1.01)p$ dollars.

a. What does the function $f(f(p))$ represent?

b. What's the value of $f(f(p))$?

c. How much is your account worth after a year?

22. (23) **Architecture** Many famous works of architecture have sides whose proportions follow the Golden Ratio. To derive the Golden Ratio, look at the diagram to the right.



a. The line segment is divided into two segments, one of length a and one of longer length b. Write the ratio of the shorter segment to the longer segment.

b. Now write the ratio of the longer segment to the sum of the lengths of both segments.

c. The Golden Ratio is for values of a and b such that both ratios are equal to each other. Write this as an equation.

d. Multiply both sides by the expression $b(a + b)$.

e. If $a = 1$ and $b = x$, find the roots of this quadratic equation using the quadratic formula.

f. Which root can you ignore? Why?

g. To the nearest thousandth, what is the value of the Golden Ratio?

*23. (55) **Multiple Choice** The probability of an event is $\frac{x}{y}$. Which shows the odds in favor of the event?

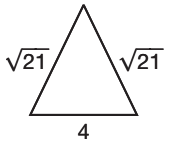
A x to y **B** y to x **C** x to $(x - y)$ **D** x to $(y - x)$

24. (50) **Temperature** The formula to convert degrees Fahrenheit to degrees Celsius is $C = \frac{5}{9}(F - 32)$.

Find the formula to convert degrees Celsius to degrees Fahrenheit by finding the inverse for this function.

25. (52) **Explain** Explain why all 30°-60°-90° triangles are similar.

26. (40) Find the area of this isosceles triangle. Dimensions are in inches.



Divide using synthetic division.

27. (51) $x^3 - 2$ by $x - 5$

28. (51) $2x^3 - 1$ by $x - 2$

29. (26) Find the equation of the line that passes through $(6, 0)$ and $(-3, -3)$.

30. (51) **Predict** The result of dividing one polynomial by another is $x^2 - 6x + 8 - \frac{2}{x-4}$. What two polynomials might have been divided?

LESSON
56

Finding Angles of Rotation

Warm Up

1. (41) **Vocabulary** To find the length of the hypotenuse of a right triangle, given the lengths of the legs, use the _____.

2. (46) Sin A is $\frac{5}{13}$ and cos A is $\frac{12}{13}$. Find csc A and sec A.

3. (3) Find the product of matrix A and matrix B.

$$A = \begin{bmatrix} 2 & 7 \\ 3 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

New Concepts

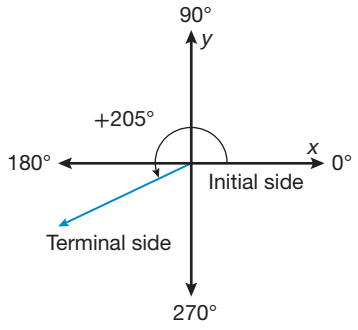
Rotations can be important in many applications such as flying aircraft, designing views for computer games, and describing patterns in art.

Angles of Rotation

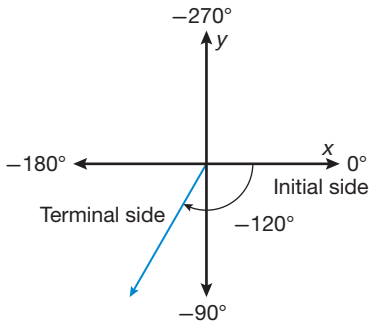
An angle is said to be in **standard position** when its vertex is at the origin and one ray is on the positive x-axis. The **terminal side** of the angle is the ray of the angle that is rotated relative to the x-axis. **The initial side** of the angle is located directly on the x-axis.

An angle formed by rotating the terminal side and keeping the initial side in place is called the **angle of rotation.** The terminal side can be rotated more than 360°.

Positive Rotation



Negative Rotation



Caution

To measure positive rotation, rotate *counterclockwise* from the initial side. To measure negative rotation, rotate *clockwise* from the initial side.

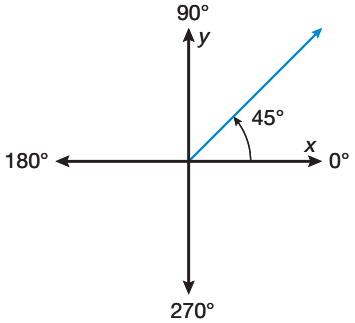
Example 1 Drawing Angles of Rotation in Standard Position

Draw each of the indicated angles in standard position.

a. 45°

SOLUTION

Start at the initial side and rotate the terminal side counterclockwise to 45°.



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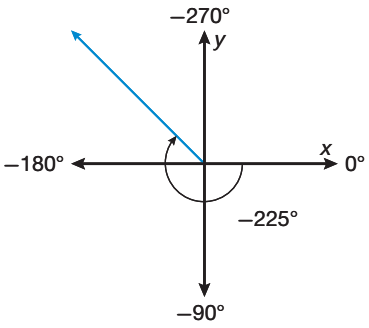
b. $-225°$

SOLUTION

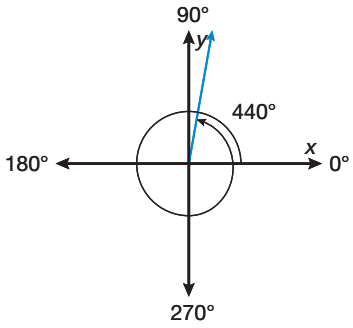
Start at the initial side and rotate the terminal side clockwise to $-225°$.



c. $440°$

SOLUTION

$440°$ can be rewritten as $360° + 80°$. The terminal angle will rotate $80°$ counterclockwise from the initial side.



Hint

Rotating an angle 360° means the terminal side and the initial side coincide. That is, they are the same. To measure an angle larger than 360°, express it as a sum of $360° + x$. If $x > 360°$, continue expressing it as $360° + \ldots + 360° + x$ until $x < 360°$.

Coterminal angles are angles that share the terminal side. For example, the angles $90°$ and $450°$ are coterminal. Positive and negative angles may also be coterminal. The angle $45°$ is coterminal to the angle $-315°$. The number of coterminal angles is infinite, as any angle with measurement $x°$ can be represented as $x° + 360n$, where n is the number of times the angle rotates back to the initial side.

Example 2 Finding the Measures of Coterminal Angles

Find the measure of one positive and one negative coterminal angle to the angles given.

a. $68°$

SOLUTION

To find a positive coterminal angle, add $360°$. To find a negative coterminal angle, subtract $360°$.

$$68° + 360° = 428° \qquad 68° - 360° = -292°$$

Two possible coterminal angles to $68°$ are $428°$ and $-292°$.

b. $515°$

SOLUTION

To find a positive coterminal angle, subtract $360°$ from $515°$.

$515° - 360° = 155°$

To find a negative coterminal angle, subtract $360°$ from $155°$, which was already determined to be a coterminal angle to $515°$.

$155° - 360° = -205°$

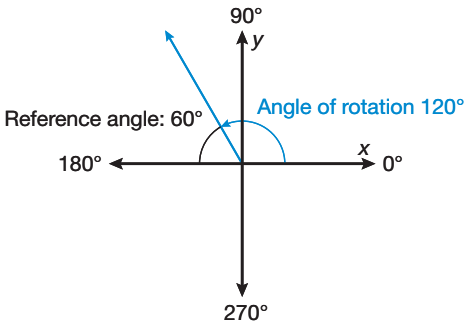
Two possible coterminal angles to $515°$ are $155°$ and $-205°$.

Hint

Another coterminal angle can be found by adding $360°$ to $515°$.

$515° + 360° = 875°$

An angle with measurement θ in standard position forms an angle with the x axis. The acute angle formed by the terminal side of the angle of rotation and the x axis is called a **reference angle.** Reference angles always measure less than 90°.



Hint

To find reference angles for θ, $\theta < 360°$ with terminal side in:

Quadrant II - $180° - \theta$

Quadrant III - $\theta - 180°$

Quadrant IV - $360° - \theta$

Example 3 Finding Reference Angles

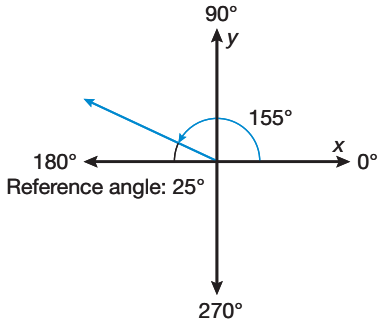
Find the measure of the reference angle for each given angle.

a. $\theta = 155°$

SOLUTION

$180° - 155° = 25°$

The reference angle for 155° is 25°.

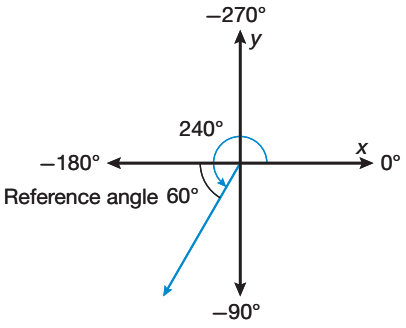


b. $\theta = 240°$

SOLUTION

$240° - 180° = 60°$

The reference angle for 240° is 60°.



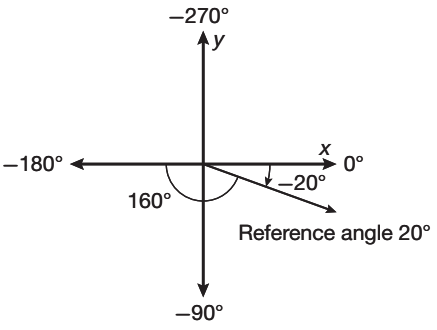
c. $\theta = -20°$

SOLUTION

Find the positive coterminal angle between 0° and 360°.

$-20° + 360° = 340°$

$360° - 340° = 20°$



Angles in standard position and reference angles can be used to find the value of trigonometric functions for an angle with measurement θ. The distance r from a point Q with coordinates (x, y) located on the terminal side of the angle to the origin is given by $\sqrt{x^2 + y^2}$.

Math Reasoning

Analyze The formula for distance is $d = \sqrt{(x - x_1)^2 + (y - y_1)^2}$.

How does this formula relate to the formula $r = \sqrt{x^2 + y^2}$ given for the distance between Q and the origin?

Trigonometric Functions

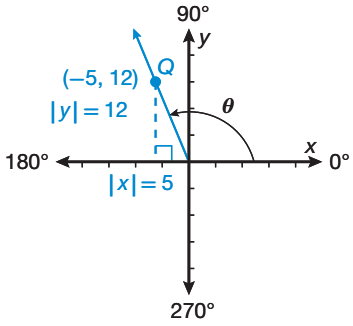
For a point $Q(x, y)$ on the terminal side of θ in standard position and $r = \sqrt{x^2 + y^2}$,

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\ x \neq 0$$

Example 4 Finding Values of Trigonometric Functions

Point Q is located on the terminal side of angle θ and has coordinates $(-5, 12)$. Find the exact value of the 6 trigonometric functions for θ.

SOLUTION



Step 1: Plot point Q and draw the angle in standard form. Then sketch a right triangle.

Step 2: Find the value of r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Step 3: Use the values of x, y, and r to find the values of the sine, cosine, and tangent of angle θ.

$$\sin\theta = \frac{12}{13} \qquad \cos\theta = -\frac{5}{13} \qquad \tan\theta = -\frac{12}{5}$$

Step 4: Use reciprocals to find $\csc\theta$, $\sec\theta$, and $\tan\theta$.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{13}{12} \qquad \sec\theta = \frac{1}{\cos\theta} = -\frac{13}{5} \qquad \cot\theta = \frac{1}{\tan\theta} = -\frac{5}{12}$$

Example 5 Application: Air Traffic Control

Math Language

A vector can also be considered to be the terminal side of an angle of rotation. The coordinates (r, θ) are the polar coordinates of a point, and give the magnitude r of the vector forming an angle with measurement θ.

Air traffic control tells the pilot of a passenger plane to circle an airport. When the plane begins to circle, its polar coordinates (with respect to the airport) are $(9, 35°)$, meaning 9 miles away and at 35°. Ten minutes later, the plane's polar coordinates are $(7, 305°)$. What is the measure of the plane's angle of rotation about the airport?

SOLUTION

The change in magnitude doesn't affect the angle of rotation. The information is not given of how many times the plane circled. The measure of the angle of rotation is some multiple of 360 degrees plus $305 - 35$, or 270, degrees. That is, the rotation is through $270 + 360n$ degrees for some integer n.

Lesson Practice

Draw each of the indicated angles in standard position.

(Ex 1)

a. 60° **b.** −120° **c.** 385°

Find the measure of one positive and one negative coterminal angle to the angles given.

(Ex 2)

d. 94° **e.** 730°

Find the measure of the reference angle for each given angle.

(Ex 3)

f. 115° **g.** 200° **h.** −145°

i. (Ex 4) Point P is located on the terminal side of angle θ and has coordinates $(-15, -8)$. Find the exact values of the 6 trigonometric functions for θ.

j. (Ex 5) A plane is located at (10, 50°). The first coordinate gives the miles from the airport and the second coordinate gives the angle. After 20 minutes, the plane's polar coordinates are (8, 355°). What is the measure of the plane's angle of rotation about the airport?

Practice Distributed and Integrated

Determine the kind of variation, if any, for each equation.

1. (12) $y = ax^2 + bx + c$, for $a, b, c \neq 0$

2. (12) $y = ax^2 + bx + c$, for any values of a, b, and c

Point P is located on the terminal side of angle θ. Find the exact values of the 6 trigonometric functions for θ with the given coordinates for P.

***3.** (56) $P(7, -24)$

***4.** (56) $P(-\sqrt{5}, 2)$

5. (11) Identify the leading coefficient and the constant term in the polynomial $-x^3 + 5x - 2$.

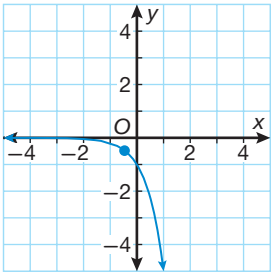
6. (49) Find the sixth term of the expansion of $(a + b)^9$.

7. (20) Given $f(x) = 2x - 7$ and $g(x) = -5x + 8$. Graph the sum $(f + g)(x)$.

8. (54) The odds in favor of an event are 4:12. What is the probability of the event happening?

9. (23) **Justify** A square room has an area of 196 square feet. What are the measurements of the room? Use a polynomial to explain.

10. (47) **Error Analysis** Consider the graph shown. Donnie says the equation of this graph is $y = -5^x$. Chris disagrees; he says it is $y = (-5)^x$. Who is correct? Explain the error.



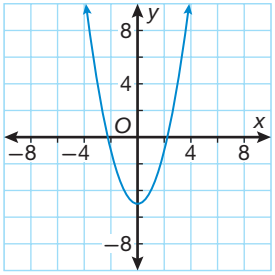
11. (18) **Exercise** Kellie's last three walks were 2.25 mi, 3.29 mi, and 2.318 mi. Use significant digit to find the sum of Kellie's walking distances.

***12.** (55) **Multi-Step** A brown bag contains 12 peaches and 8 nectarines.

a. Which has the greater probability: randomly selecting 2 peaches or 2 nectarines?

b. How much greater is the probability?

13. (27) **Multiple Choice** What is the range of the function?



A all real numbers between −4 and 4

B all real numbers less than −5

C all real numbers greater than −5

D all real numbers

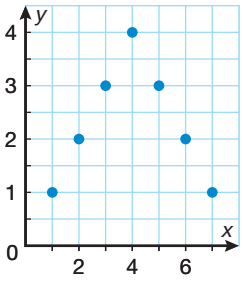
14. (49) Find the seventh term of the expansion of $(g + 2)^{12}$.

15. (46) **Sea Level** From an observation point 435 feet above sea level, a tourist spots a buoy at an angle of depression of 57° and another in the same line of sight at an angle of depression of 13°. To the nearest foot, how far apart are the buoys?

16. (15) **Investments** A couple has saved $20,000. An advisor suggests that they invest this amount in treasury bonds and in certificates of deposit (CDs). The couple can get a one-year CD that pays 5.42% (annual) interest, and a 2-year treasury bond that will yield 4.18% annually. If the couple wants to earn $1000 per year from the investment, how much should they invest in treasury bonds?

17. (39) **Food Preparation** Antonio is making a dip for a school function. Each tablespoon of olive oil has 15 grams of fat and each tablespoon of mayonnaise has 5 grams of fat. He wants the snack to have less than 75 grams of fat. Graph the inequality that represents the possible amounts of each ingredient in the dip. Choose a point in the shaded area. Tell what it means.

18. (45) **Error Analysis** A student said that the correlation coefficient for the data graphed is 1 because it follows a perfect pattern. Why is the student incorrect?



Draw each of the indicated angles in standard position.

***19.** (56) 70°

***20.** (56) −200°

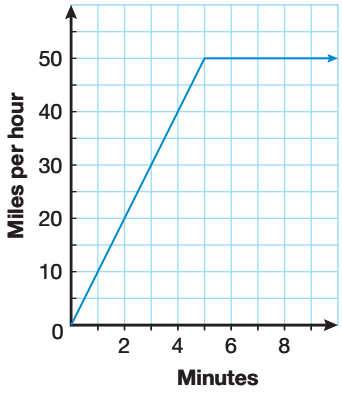
***21.** (56) −90°

22. (48) **Verify** Verify that the value of the complex fraction $3 + \frac{1}{7 + \frac{1}{15 + 1}}$ is a good approximation of π.

23. (14) **Analyze** Create two 2 × 2 matrices that are inverses of each other. Calculate and compare their determinants. Explain your findings.

*24. (54) **Graphing Calculator** Describe the feasible region for the set of inequalities $x \leq 4$, $y \geq 2$, $y \leq -0.25x + 5$

25. (22) **Acceleration** A car increases its speed from 0 to 50 miles per hour. It then drives 50 miles per hour for the rest of the trip. Examine the graph shown. Does this graph represent a continuous, discontinuous, and/or discrete function? Justify your answer.



*26. (55) **Probability** In 2006, the U.S. Census Bureau estimated there were 18,089,888 people living in Florida and 299,398,484 living in the United States.

a. Estimate the probability that a person randomly selected in the United States lives in Florida.

b. Estimate the odds in favor that a person randomly selected in the United States lives in Florida.

27. (42) **Multi-Step** A bag of marbles contains 6 red marbles, 5 blue marbles, and 1 green marble. Two marbles are selected at random. Which is more likely: selecting 2 red marbles or 1 red and 1 blue marble?

28. (30) **Write** Describe how the graph of $y = -2.5(x - 5)^2 + 2$ differs from the graph of the parent function.

*29. (54) A community center wants to build a rectangular playground and fence around it. The cost of fencing is $25 per foot. The cost for painting the fence is $14 per foot. The center cannot spend more than $2500 on fencing. The playground cannot be any longer than 40 ft and needs to be at least 20 ft wide. What is the maximum area, in square feet, the center can get for the size of the playground?

30. (53) **Geometry** Suppose that function $f(x)$ rotates every point on the plane clockwise 180 degrees about the origin, and $g(x)$ translates every point to the right 2 units.

a. What is the effect of $(g \circ f)(x)$ on the point with coordinates (0, 1)?

b. What is the effect of $(f \circ g)(x)$ on that same point?

LESSON 57

Finding Exponential Growth and Decay

Warm Up

1. (47) **Vocabulary** In an _____ function the independent variable is found in the exponent.

2. (50) Suppose that $f(x)$ has an inverse $g(x)$. What is $f(g(x))$?

3. (3) Evaluate 2^{-1}.

4. (47) True or False. A function of the form $y = ab^x$ is an exponential function if x is a real number, $a = 0$, $b > 0$, and $b \leq 1$.

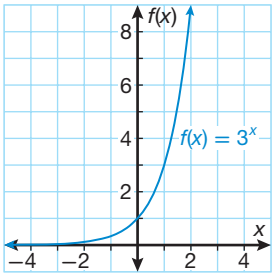
New Concepts

Math Language

In the expression ab^x, b is a **growth factor** if $b > 1$. If $0 < b > 1$, then it is a decay factor.

A function is **increasing** if the output values increase as the input values increase. Its graph rises from left to right. **Exponential growth** functions are increasing functions used to model situations where a quantity always increases by the same percent for a given time period. It is modeled by the function $f(x) = ab^x$, where $a > 0$ and $b > 1$.

A function is **decreasing** if the output values decrease as the input values increase. Its graph falls from left to right. Exponential functions that are decreasing model **exponential decay.** Exponential decay is used to model situations where a quantity always decreases by the same percent in a given time period. It is modeled by the function $f(x) = ab^x$, where $a > 0$ and $0 < b < 1$.



One common application of exponential growth is calculating the amount in an account that earns interest. Recall from Lesson 47 that the formula to find the amount A in an account with an initial deposit of P, interest rate r, years of interest t, and times compounded per year n is $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

Reading Math

A decreasing function can also be written in the form $f(x) = ab^{-x}$, where $a > 0$, $b > 1$.

Example 1 Calculating Simple Interest

A savings account earns interest at an annual rate of 4%, compounded quarterly. If the account begins with a principal amount of $1,000, what will its value be after 3 years?

SOLUTION

Use the formula for interest. The rate, r, is 0.04. The initial deposit is $1000. It is compounded 4 times per year, and the time, t, is 3 year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 1000\left(1 + \frac{0.04}{4}\right)^{4(3)}$$

$$A = 1126.83$$

So the account will be worth approximately $1126.83.

Online Connection
www.SaxonMathResources.com

Math Language

The constant e is equal to $\left(1+\frac{1}{n}\right)^n$ as the value of n goes to infinity.

Banks frequently compound interest continuously, meaning that n is infinite. To do this, they use the mathematical constant e, which is approximately equal to 2.718.

The formula to find the value of an account when interest is compounded continuously is $A = Pe^{rt}$.

Example 2 Modeling Exponential Decay Calculating Continuous Interest

a. A savings account earns interest at an annual rate of 4%, compounded continuously. If the account begins with a principal amount of $1000, what will its value be after 3 years?

SOLUTION

Use the formula for continuously compounded interest, $A = Pe^{rt}$.

$A = Pe^{rt}$

$A = (1000)e^{(0.04)(3)}$

$A = 1127.50$

The value of the account will be approximately $1,127.50.

b. After 5 years of earning continuously compounded interest at the rate of 3.5%, a savings account is worth $3200. What was its value initially?

SOLUTION

This time, solve for P.

$A = Pe^{rt}$

$3200 = (P)e^{(0.035)(5)}$

$A = 2686.26$

The initial value of the account was $2686.26.

To solve problems using exponential growth and decay, first decide whether a situation exhibits the properties of growth or decay. Then, substitute values of x and y into the equation to solve for a and b.

Remember that x will generally represent the number of time intervals that pass in an exponential equation, and y will usually represent the total quantity of something after x periods of time.

Math Language

The **half-life** of a substance is the time it takes for one-half of the substance to decay.

Example 3 Finding Exponential Functions

The element Nobelium has a half-life of 58 minutes. Write an exponential equation to find the portion of a mass of Nobelium that is left after x minutes and find what percentage of an amount of Nobelium is left after 15 minutes.

Hint

The starting amount is arbitrary. Choose an amount that allows easy computation and remember to substitute half of that amount for y.

SOLUTION

The equation will be in the form $y = ab^x$. Since the question is asking for a percentage, a, which is the starting amount of material, is simply 1. We know that half of the mass decays in 58 minutes. Use this data point to find the equation.

$$y = b^x$$
$$0.5 = b^{58}$$
$$b = \sqrt[58]{0.5}$$

So the equation is $y = \left(\sqrt[58]{0.5}\right)^x$. Now substitute 15 for x.

$$y = \left(\sqrt[58]{0.5}\right)^{15}$$
$$y \approx 0.84$$

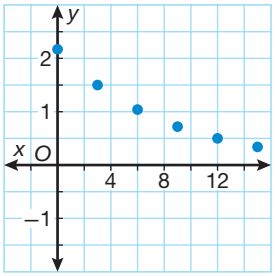
So approximately 84% of the Nobelium is left after 15 minutes.

Example 4 Modeling Exponential Decay

An experimenter measures the maximum distance a large pendulum travels from its resting point at each swing. The measurements are recorded in this table:

Swing	0	3	6	9	12	15
Distance (meters)	2.17	1.5	1.04	0.72	0.50	0.34

Using the table of values and the graph, write an exponential equation that relates the number of swings, x, and the distance of each swing, y.



SOLUTION

The graph shows that this problem is a case of exponential decay, so the equation will be in the form $y = ab^x$, with $a > 0$ and $0 < b < 1$.

Substitute the first point from the table of values, (0, 2.17) to find the value of a.

$$y = ab^x$$
$$2.17 = ab^0$$
$$2.17 = a$$

So the equation is in the form $y = 2.17b^x$. Use the second point, (3, 1.5), to find b.

$$y = 2.17b^x$$
$$1.5 = (2.17)b^3$$
$$b^3 = \frac{1.5}{2.17}$$
$$b = \sqrt[3]{\frac{1.5}{2.17}}$$
$$b \approx 0.88$$

So the data is modeled by the equation $y = (2.17)0.88^x$.

Example 5 Application: Population Growth

The 1990 US census found that the population of the USA was approximately 248.7 million people. In 2000, the new census put the total population at 281.4 million people. Assuming the population of the USA shows exponential growth, predict what the population will be in the 2010 census.

SOLUTION

Use the two data points given in the problem to write an exponential equation. Let $x =$ the number of years since 1990. So, $x = 0$ in 1990, and $x = 10$ in 2000. The two points are (0, 248.7) and (0, 281.4).

$$y = ab^x$$
$$248.7 = ab^0$$
$$a = 248.7$$

Now use the second point to find b.

$$y = 248.7b^x$$
$$281.4 = 248.7b^{10}$$
$$b^{10} = \frac{281.4}{248.7}$$
$$b = \sqrt[10]{\frac{281.4}{248.7}}$$
$$b \approx 1.0124$$

Now use the equation $y = (248.7)1.0124^x$ to find y when $x = 20$, which corresponds to 2010.

$$y = (248.7)1.0124^x$$
$$y = (248.7)1.0124^{20}$$
$$y \approx 318.21$$

So the population in 2010 should be approximately 318.21 million people.

Lesson Practice

a. *(Ex 1)* A savings account earns interest at an annual rate of 3.5%, compounded monthly. If the account begins with a principal amount of \$5000, what will its value be after 4 years?

b. *(Ex 2)* A savings account earns interest at an annual rate of 3.5%, compounded continuously. If the account begins with a principal amount of \$5000, what will its value be after 4 years?

c. *(Ex 2)* After 4 years, a savings account with continuously compounded interest at a rate of 4% is worth \$15,320. What was the initial amount deposited into the savings account?

d. *(Ex 3)* A mass of carbon-14 atoms decays in such a way that half disappear every 5730 years. What portion of a 10 kg mass of atoms will remain after 9000 years?

e. (Ex 4) Use the table of values to write an exponential equation and predict the value of y when $x = 15$ and $x = 45$. Round the values in your equation to four decimal places.

x	0	10	15	45
y	803	412		

f. (Ex 5) The population of Phoenix, Arizona was 983,403 in 1990. In 2000, it was 1,321,045. Predict the population of Phoenix in 2010.

Practice Distributed and Integrated

1. (29) Is the ordered triple $(4, 0, -1)$ a solution of the system $\begin{aligned} 2x - y + z &= 7 \\ -3x + 5y - 3z &= -9 \\ 4x - 2y + 2z &= -14 \end{aligned}$?

Simplify.

2. (40) $4\sqrt{3} \cdot 3\sqrt{12} \cdot 2\sqrt{3}$

3. (40) $3\sqrt{75} - 4\sqrt{48}$

4. (40) $2\sqrt{5}\left(5\sqrt{5} - 3\sqrt{15}\right)$

5. (31) What are all the values of x for which the division expression $\frac{x}{x-2} \div \frac{6x-2}{3x+6}$ is undefined?

Given the objective function *P* and the constraints, find the minimum and maximum values.

***6.** (54) $P = 15x + 7y$ given $x \geq 0$, $y \geq 0$, $1.5x + y \leq 35$, $0.25x + 5y \leq 15$

***7.** (54) $P = 50x + 75y$ given $x \geq 0$, $y \geq 0$, $2x + y \geq 30$, $0.75x + y \leq 20$, $0.5x + y \leq 18$

8. (11) If a trinomial in standard form is added to a binomial in standard form and the result is written in standard form, how many terms can be in the result?

9. (53) **Error Analysis** A student says: If $f(x) = \frac{5}{x}$ and $g(x) = \frac{-3}{x}$, then $f(g(x)) = \frac{5}{\frac{-3}{x}} = -\frac{5}{3}x$. Explain the reasoning.

10. (11) **Multi-Step** **a.** Complete the table of values for the polynomial function $f(x) = x^3 - 5x + 8$.

x	-3	-2	-1	0	1	2
$f(x)$						

b. Using only integer values of x, what is the smallest x-interval whose associated portion of the graph contains a relative maximum? What is the smallest x-interval whose associated portion of the graph contains a relative minimum? Justify your answers.

*11. (56) **Skateboarding** In the 1999 X Games, skateboarder Tony Hawk became the first to land a 900—a rotation of 900° in the air from a half-pipe. Find an angle with measure between 0° and 360° that is coterminal with 900°. Describe the skateboarding move.

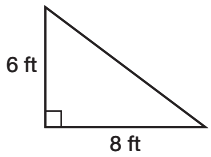
12. (41) **Cartography** A student constructs a map of Montana over a grid system such that Lewiston is at (1, 4) and Havre is at (0.5, 10). Find the distance between the cities to the nearest kilometer if each unit represents 25 kilometers.

13. (22) $f = \{(3, 4), (6, 8), (2, 1), (3, 6)\}$ represents a _____.

A continuous function **B** discrete function

C continuous relation **D** discrete relation

14. (13) **Geometry** The height of a right triangle is 6 ft and the base is 8 ft. Find the slope of the hypotenuse of the right triangle.



15. (39) **Write** Describe two ways to graph $2y - x < 6$.

16. (23) **Error Analysis** The factored form of $9x^2 - 6x - 15$ is given as $3x(3x - 7)$. Find and explain the error and give the correct factorization.

Determine whether the function shows exponential growth or decay.

*17. (57) 1.4^x

*18. (57) $10(0.4)^x$

19. (56) **Analyze** Find all the angle measures between 0° and 360° for which $\tan\theta$ is undefined.

20. (55) **Coordinate Geometry** A point is randomly chosen in one of the four quadrants of the coordinate plane. Construct a theoretical probability model to show the probabilities of the point being in each quadrant.

21. (32) **Error Analysis** Given matrices $A = \begin{bmatrix} 2 & -3 \\ 5 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$, a student solved $AX = B$ as follows:

$X = BA^{-1} = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 8 & -3 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -4 \\ 28 & -11 \end{bmatrix}$. What is the error? Find the correct solution.

22. (24) **Multiple Choice** To solve $\begin{matrix} 6x - 7y = -1 \\ 3x + 2y = 5 \end{matrix}$ by elimination, what should the second equation be multiplied by in order to eliminate the x-terms?

A -6 **B** -3 **C** -2 **D** 2

23. (42) **Write** Identify some mathematical patterns found in Pascal's Triangle. For example, the third diagonal has the **Triangular Numbers.**

24. (34) **Model** Graph the equation $2x + 3y = 3$.

25. (40) **Electricity** Ohm's Law states that the current I, in amps, equals $\frac{\sqrt{P}}{\sqrt{R}}$ where P is power, measured in watts, and R is resistance measured in ohms. Simplify the expression.

26. (14) **Analyze** If you apply expansion by minors to the 2×2 matrix $\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$, you get $3|1| - 4|-2|$. How must you define the determinant of a single number so that the result agrees with the formula for the determinant of a matrix this size?

***27.** (57) **Formulate** Write the equation of an exponential function whose graph passes through the points (0, 2) and (2, 18). Determine whether the function is increasing or decreasing.

28. (43) **Catering** A caterer is ordering salmon and shrimp for an event. The salmon costs \$7.50 per pound and the shrimp costs \$15 per pound. She needs at least as much shrimp as salmon, and the most she can spend is \$300. Let x represent the number of pounds of salmon and y represent the number of pounds of shrimp.

a. Write and graph a system of linear inequalities that represents the situation.

b. Determine the point of intersection of the boundary lines in the graph. Round to the nearest hundredth.

c. Assume that the salmon and shrimp can be ordered in quarter-pound increments. Determine the point closest to the point of intersection of the boundary lines that represents a possible purchase. Describe what it means.

***29.** (57) **Life Insurance** From 1950 through 1980, the total amount, l (in billions), of life insurance in the United States is shown in the table below.

Year	1950	1955	1960	1965	1970	1975	1980
Amount	234	373	586	901	1402	2314	3541

Write an exponential model that approximates this data. How much money does this model predict will be invested in life insurance by the year 2010?

***30.** (27) **Graphing Calculator** Use a graphing calculator to determine the y-intercept of the quadratic function $y - 1 = -3x^2 - 6x$.

LESSON
58

Completing the Square

Warm Up

1. (11) **Vocabulary** In the trinomial $ax^2 + bx + c$, c is the _____ term.
2. (23) Factor the expression $x^2 + 2x + 1$.
3. (3) True or False. If a number x is a perfect square, then $\sqrt{x}$ is a whole number.

New Concepts

Recall that a perfect square trinomial can be factored into a squared binomial.

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

The Square Root Property can be used in solving quadratic equations in the form of a binomial square.

Square Root Property
If $x^2 = a$, where $a > 0$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$. In general, if $x^2 = a$, then $x = \pm\sqrt{a}$ for any $a > 0$.

Example 1 Solving Quadratic Equations That Are Perfect Squares

a. Solve $x^2 - 10x + 25 = 9$.

SOLUTION

Step 1: Factor the perfect square trinomial.

$$x^2 - 10x + 25 = 9$$
$$x^2 - 2(5)x + 5^2 = 9$$
$$(x - 5)^2 = 9$$

Step 2: Apply the Square Root Property.

$$(x - 5)^2 = 9$$
$$x - 5 = \pm\sqrt{9}$$
$$x - 5 = \pm 3$$

Step 3: Solve for x.

$$x - 5 = 3 \qquad\qquad x - 5 = -3$$
$$\underline{+5 \quad +5} \qquad\qquad \underline{+5 \quad +5}$$
$$x = 8 \qquad\qquad x = 2$$
$$x = 8 \quad \text{or} \quad x = 2$$

Caution

Do not forget that the Square Root Property yields exactly **two** solutions.

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b. Solve $x^2 + 24x + 144 = 5$.

SOLUTION

Step 1: Factor the perfect square trinomial.

$$x^2 + 24x + 144 = 5$$
$$x^2 + 2(12)x + 12^2 = 5$$
$$(x + 12)^2 = 5$$

Step 2: Apply the Square Root Property.

$$(x + 12)^2 = 5$$
$$x + 12 = \pm\sqrt{5}$$

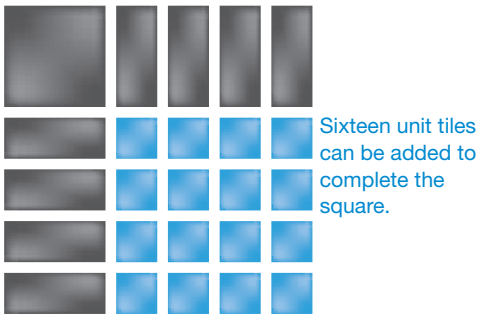
Since 5 is not a perfect square, it can be left as $\sqrt{5}$.

Step 3: Solve for x.

$$x + 12 = \pm\sqrt{5}$$
$$-12 \quad -12$$
$$x = -12 \pm \sqrt{5}$$
$$x = -12 + \sqrt{5} \quad \text{or} \quad x = -12 - \sqrt{5}$$

Some quadratic equations cannot be solved by factoring. In these cases the method of completing the square can be used. To **complete the square,** a term can be added to a quadratic expression of the form $x^2 + bx$ to form a perfect square trinomial.

Exploration Modeling Completing the Square

The model represents the quadratic binomial $x^2 + 8x$. $x^2 + 8x$	How many more unit tiles should be added to the model to create a square? $x^2 + 8x + 16$  Sixteen unit tiles can be added to complete the square.
The result is a perfect square trinomial.	$x^2 + 8x + 16 = (x + 4)^2$

Draw a model of $x^2 + 10x$. Use your model to find a perfect square trinomial and its factors.

Completing the Square

Given a quadratic of the form $x^2 + bx$, add to it the square of half the coefficient of x, $\left(\frac{b}{2}\right)^2$, to create a perfect square trinomial.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 2 Completing the Square

Complete the square and factor the resulting perfect square trinomial

$x^2 + 18x$.

SOLUTION Add the square of half the coefficient of x to the binomial. The coefficient of x is 18. Add $\left(\frac{18}{2}\right)^2 = 9^2$ to the binomial to create a perfect square trinomial.

$x^2 + 18x + 9^2 = x^2 + 18x + 81 = (x + 9)^2$

When using the method of completing the square to solve a quadratic equation, keep in mind that when you add a constant term to one side of an equation you must also add that constant to the other side of the equation.

Example 3 Solving Quadratic Equations by Completing the Square

Solve $x^2 - 14x - 8 = 0$ by completing the square.

SOLUTION

Step 1: $x^2 - 14x - 8 + 8 = 0 + 8$

$x^2 - 14x = 8$ Isolate the binomial $x^2 - 14x$.

Step 2: Add the square of half the coefficient of x to both sides of the equation.

The coefficient of x is -14; $\left(\frac{-14}{2}\right)^2 = (-7)^2 = 49$.

$x^2 - 14x + 49 = 8 + 49$

$x^2 - 14x + 49 = 57$

Step 3: $x^2 - 14x + 49 = 57$

$x^2 - 2(7)x + 7^2 = 57$ Factor the perfect square trinomial.

$(x - 7)^2 = 57$

Step 4: $x - 7 = \pm\sqrt{57}$ Apply the Square Root Property.

Step 5: $x - 7 + 7 = \pm\sqrt{57} + 7$

$x = 7 \pm \sqrt{57}$ Solve for x.

Math Reasoning

Estimate Without using a calculator, estimate the decimal approximation for the solutions to Example 3.

Example 4 Application: Projectile Motion

During a baseball game a player hits a fly ball. The height $f(t)$, in feet, of the ball at time t, in seconds, can be described by the function $f(t) = -16t^2 + 64t + 4$. Write the equation in vertex form by completing the square. Then determine the maximum height of the ball and the time it takes to reach the maximum height.

SOLUTION

Complete the square for the trinomial expression that describes the height of the ball.

$$f(t) = -16t^2 + 64t + 4$$

$= (-16t^2 + 64t) + 4$	Group the binomial $-16t^2 + 64t$
$= -16(t^2 - 4t) + 4$	Factor out -16 from $(-16t^2 + 6t)$.
$= -16(t^2 - 4t + 4) + 4 + \boxed{64}$	Complete the square.
$= -16(t - 2)(t - 2) + 68$	Factor. Then simplify.
$f(t) = -16(t - 2)^2 + 68$	Write the function in vertex form.

Caution

When completing the square, be sure to keep the equation equivalent to the original equation. Notice $(-16)(4)$ was added to complete the square, so $+64$ was added to keep the equation equivalent to the original equation.

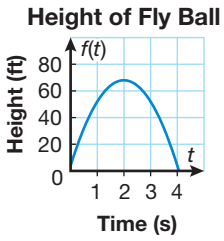
To determine the maximum height of the ball use the vertex form of a function. Recall the vertex form $f(x) = a(x - h)^2 + k$ of a quadratic function. The point (h, k) is the vertex.

For the function $f(t) = -16(t - (+2)^2 + 68$, the vertex is $(2, 68)$. Therefore, the ball reaches a maximum height of 68 feet 2 seconds after the ball was hit.

Check Use a table to graph the function. Then estimate the vertex of the parabola on the graph.

x	0	1	2	3	4
$f(x)$	4	52	68	52	4

The graph verifies that the vertex occurs at $(2, 68)$



Lesson Practice

a. *(Ex 1)* Solve $x^2 - 2x + 1 = 36$.

b. *(Ex 1)* Solve $x^2 + 16x + 64 = 6$.

c. *(Ex 2)* Complete the square and factor the resulting perfect square trinomial

$$x^2 - 20x.$$

d. *(Ex 3)* Solve $x^2 + 26x - 2 = 0$ by completing the square.

e. *(Ex 4)* A pebble is thrown into the air from ground level with an initial velocity of 20 feet per second. The height of the pebble h at any given time t can be described by the equation

$$h = -16t^2 + 20t.$$

Find the time at which the height of the pebble is 4 feet. Round your answer to the hundredths place.

Practice Distributed and Integrated

Solve by using either substitution or elimination.

1. (24) $2y - 2x = 8$, $y + x = -2$

2. (24) $y - 2x = 1$, $y = -2$

3. (24) $3x + 2y = 12$, $5x - 4y = 8$

4. (10) Solve the inequality $-3 < d + 2$ and $d + 2 < 12$.

5. (10) Solve and graph the inequality $-4 \leq 4x - 8 < 12$.

Determine the kind of variation, if any, for each equation.

6. (12) $y = \frac{1}{x}$, for $x \neq 0$

7. (12) $z = \frac{1}{xy + a}$, for $x, y \neq 0$ and any value of a

Solve.

***8.** (58) $x^2 - 8x + 16 = 3$

***9.** (58) $x^2 + 26x + 169 = 81$

10. (30) **Physics** An object dropped from a height 4 meters above the ground after x seconds is modeled by the function $y = -4.9x^2 + 4$. If measured in feet, the height is modeled by $y = -16x^2 + 1.2$. Without graphing, tell how the graphs of each function compare to each other.

11. (22) **Error Analysis** Sam listed the following points $\{(2, 3), (1, 6), (5, -3), (1, 2)\}$ when giving an example of a discrete function. Correct his example.

12. (51) **Multiple Choice** What is the result of dividing $x^3 - 7x + 3$ by $x - 2$?

A $x^2 + 2x + 3 - \frac{3}{x - 2}$

B $x^2 + 2x - 3$

C $x^2 + 2x - 3 - \frac{3}{x - 2}$

D $x^2 + 2x$

***13.** (58) **Multi-Step** Use $x^2 + 104x - 97 = 4$.

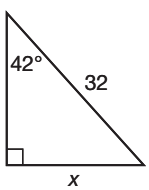
a. What is the first step before you can complete the square?

b. What constant can be added to the binomial to create a perfect square trinomial?

c. What is the factored form of the perfect square trinomial?

d. Solve the equation for x.

14. (46) **Justify** Show that it is possible to solve for x by using either the sine or cosine function.



*15. (57) **Formulate** Formulate an expression for the amount Stu will have in his bank account at the end of a year if he invests a principal of $1,850 at an annual interest rate of 2.4%, compounded monthly.

*16. (58) **Multiple Choice** Chose the letter that best represents the value of x in the equation $x^2 - 42x + 441 = 2$.

A $x = -21 \pm \sqrt{2}$ **B** $x = 20, 22$

C $x = 21 \pm \sqrt{2}$ **D** $x = -20, -22$

17. (26) **Geometry** Miriam is going to create a rectangular vegetable garden along the side of her house and is going to enclose it within a fence on three sides. If the total amount of fencing she purchases is 80 feet, write an equation in standard form for finding all possible lengths and widths for the garden.

18. (19) **Write** Explain how to multiply $(2x + 8)(3x - 6)$ using the FOIL method.

19. (45) **Formulate** Describe the correlation coefficient r using a compound inequality and absolute value symbols.

*20. (55) **Pass Codes** A computer generates temporary pass codes to users. Each pass code is exactly 3 letters long and it can have repeating letters. What is the probability of getting a pass code with 3 vowels (A, E, I, O, or U)?

21. (23) **Deck Building** The area of a triangular section of a deck is $72x^2 - 98$ ft^2. If the base of the section of deck is $12x + 14$ ft, find the height of the section.

22. (52) **Agriculture** Lee and his brother own a piece of land that has the shape of an equilateral triangle with side length 200 m. They divide the land between them into two equal parts along the altitude of the triangle by putting a fence. What is the length of the fence? What area of land does each of them own? Use a graphing calculator and round your answers to the nearest hundredth.

*23. (54) **Graphing Calculator** Describe the feasible region for the set of inequalities

$$x \geq 0,\ y \geq 0,\ y \leq -0.75x + 8.$$

24. (24) **Error Analysis** Find and explain the error in solving the system $\begin{aligned} x - 10y &= 10 \\ 2x - 20y &= 23 \end{aligned}$ in the step below.

$$\begin{array}{ll} -2(x - 10y = 10) & -2x + 20y = -20 \\ \underline{+2x - 20y = 23} & \underline{+2x - 20y = 23} \\ & 40y = 3 \end{array}$$

25. (50) **Pendulums** The period T (in seconds) of a simple pendulum as a function of its length l (in meters)is given by the equation $T = 2\pi\sqrt{\frac{l}{9.8}}$. Express the length l as a function of the time t.

*26. (57) **Multi-Step** The half-life of Uranium-234 is 245,000 years.

a. Formulate an exponential function representing the amount of a grams Uranium-234 remaining after t years.

b. Use the function to find how much of a 150-gram sample of Uranium-234 will be left after 182,000 years.

27. (53) Let $f(x) = 3x$ and $g(x) = 3x + 2$. Find the composite function $g(f(x))$.

28. (40) **Justify** A student saw his friend write that $-\sqrt{81} = -9$, and he told his friend that it is wrong because a square root cannot yield a negative number. Why is the student wrong and his friend correct?

29. (44) **Generalize** How might you rationalize the denominator for the expression $\frac{1}{\sqrt[3]{5}}$, when there is a *cube* root in the denominator?

*30. (58) **Projectile Motion** The world record for the World Championship Punkin Chunkin, held each year in the state of Delaware, is held by an air cannon named "2nd Amendment." The height of a pumpkin t seconds after being shot from this machine can be described by the equation $h = -16t^2 + 880t + 70.7$. After how many seconds will a pumpkin shot from this machine land on the ground? Round your answer to the tenths place.

LESSON
59

Using Fractional Exponents

Warm Up

1. (3) **Vocabulary** In the expression $x^{(a+b)}$ the term $(a + b)$ is the _____.

2. (3) Simplify. $\dfrac{x^2y^4}{x^{-1}y^3}$

3. (40) Solve $\sqrt{x} = 4$.

New Concepts

Exponents of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$, are called **rational exponents.** Rational exponents can be written as radical expressions.

Rational Exponent Property
For m and n integers and $n \neq 0$: $a^{\frac{1}{n}} = \sqrt[n]{a}$ $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

Example 1 Writing Rational Exponents as Radical Expressions

Write the rational exponents as radical expressions.

a. $3^{\frac{1}{5}}$

SOLUTION Apply the Rational Exponent Property:

$a^{\frac{1}{n}} = \sqrt[n]{a}$.

In this case, $a = 3$ and $n = 5$.

$3^{\frac{1}{5}} = \sqrt[5]{3}$

b. $9^{\frac{3}{4}}$

SOLUTION Apply the Rational Exponent Property:

$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$.

In this case, $a = 9$, $m = 3$, and $n = 4$.

$9^{\frac{3}{4}} = \sqrt[4]{9^3}$ or $(\sqrt[4]{9})^3$

Radical expressions of the form $\sqrt[n]{a}$ are called ***n*th roots.** In this form n is the index and a is the radicand. Similar to square roots, $\sqrt[n]{a} = b$ if $b^n = a$.

When nth roots have an odd index, there is only one real root that maintains the sign of the radicand. For example, $\sqrt[3]{8} = 2$ and $\sqrt[3]{-8} = -2$. When nth roots have an even index and a positive radicand, there are two real roots, one positive and one negative. For example, $\sqrt[4]{16} = \pm 2$. An nth root with an even index and a negative radicand has no real roots.

Reading Math

Square roots have an index of 2 although the 2 is not written with the radical in this case.

Properties of *n*th Roots
For $a > 0$ and $b > 0$, **Product Property:** $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ **Quotient Property:** $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

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Example 2 Simplifying Radical Expressions

Simplify each expression. Let $x > 0$ for each problem.

a. $\sqrt[5]{243x^{10}}$

SOLUTION Factor the radicand into perfect fifths.

$$\sqrt[5]{243x^{10}} = \sqrt[5]{3^5 \cdot x^5 \cdot x^5}$$

Apply the Product Property.

$$\sqrt[5]{3^5 \cdot x^5 \cdot x^5} = \sqrt[5]{3^5} \cdot \sqrt[5]{x^5} \cdot \sqrt[5]{x^5}$$

Evaluate the radical.

$$\sqrt[5]{3^5} \cdot \sqrt[5]{x^5} \cdot \sqrt[5]{x^5} = 3 \cdot x \cdot x$$

Simplify.

$$3 \cdot x \cdot x = 3x^2$$

b. $\sqrt[5]{\frac{x^5}{2}}$

SOLUTION Apply the Quotient Property.

$$\sqrt[5]{\frac{x^5}{2}} = \frac{\sqrt[5]{x^5}}{\sqrt[5]{2}}$$

Evaluate the numerator.

$$\frac{\sqrt[5]{x^5}}{\sqrt[5]{2}} = \frac{x}{\sqrt[5]{2}}$$

Rationalize the denominator.

$$\frac{x}{\sqrt[5]{2}} \cdot \frac{\sqrt[5]{2}}{\sqrt[5]{2}} \cdot \frac{\sqrt[5]{2}}{\sqrt[5]{2}} \cdot \frac{\sqrt[5]{2}}{\sqrt[5]{2}} \cdot \frac{\sqrt[5]{2}}{\sqrt[5]{2}}$$

Apply the Product Property.

$$\frac{x}{\sqrt[5]{2}} \cdot \frac{\sqrt[5]{2}}{\sqrt[5]{2}} \cdot \frac{\sqrt[5]{2}}{\sqrt[5]{2}} \cdot \frac{\sqrt[5]{2}}{\sqrt[5]{2}} \cdot \frac{\sqrt[5]{2}}{\sqrt[5]{2}} = \frac{x\sqrt[5]{2^4}}{\sqrt[5]{2^5}}$$

Simplify.

$$\frac{x\sqrt[5]{2^4}}{\sqrt[5]{2^5}} = \frac{x\sqrt[5]{16}}{2}$$

Hint

When a radical is in the denominator you must always rationalize the denominator.

The properties of integer exponents covered in Lesson 3 hold true for rational exponents.

Properties of Rational Exponents

For all nonzero real numbers a and b and rational numbers m and n,

Negative Exponent Property: $a^{-n} = \frac{1}{a^n}$

Zero Exponent Property: $a^0 = 1$

Product of Powers Property: $a^m \cdot a^n = a^{m+n}$

Quotient of Powers Property: $\frac{a^m}{a^n} = a^{m-n}$

Power of a Power Property: $(a^m)^n = a^{m \cdot n}$

Power of a Product Property: $(ab)^m = a^m \cdot b^m$

Power of a Quotient Property: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Caution

Notice the difference between the Product of Powers Property and the Power of a Power Property.

Example 3 Simplifying Rational Exponents

Simplify each expression.

a. $64^{\frac{7}{16}} \cdot 64^{\frac{1}{16}}$

SOLUTION Apply the Product of Powers Property.

$$64^{\frac{7}{16}} \cdot 64^{\frac{1}{16}} = 64^{\frac{7}{16}+\frac{1}{16}} = 64^{\frac{8}{16}} = 64^{\frac{1}{2}}$$

Apply the Rational Exponent Property $a^{\frac{1}{n}} = \sqrt[n]{a}$ and simplify.

$$64^{\frac{1}{2}} = \sqrt{64} = 8$$

b. $\dfrac{16^{\frac{6}{4}}}{16^{\frac{7}{4}}}$

SOLUTION Apply the Quotient of Powers Property.

$$\frac{16^{\frac{6}{4}}}{16^{\frac{7}{4}}} = 16^{\frac{6}{4}-\frac{7}{4}} = 16^{-\frac{1}{4}}$$

Apply the Negative Exponent Property.

$$16^{-\frac{1}{4}} = \frac{1}{16^{\frac{1}{4}}}$$

Apply the Rational Exponent Property $a^{\frac{1}{n}} = \sqrt[n]{a}$ and simplify.

$$\frac{1}{16^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{16}} = \frac{1}{2}$$

c. $(12^{10})^{\frac{1}{5}}$

SOLUTION Apply the Power of a Power Property.

$$(12^{10})^{\frac{1}{5}} = 12^{10 \cdot \frac{1}{5}} = 12^{\frac{10}{5}} = 12^2$$

Evaluate the exponent.

$$12^2 = 144$$

d. $(27 \cdot 125)^{\frac{1}{3}}$

SOLUTION Apply the Power of a Product Property.

$$(27 \cdot 125)^{\frac{1}{3}} = 27^{\frac{1}{3}} \cdot 125^{\frac{1}{3}}$$

Evaluate the exponents and simplify.

$$= 27^{\frac{1}{3}} \cdot 125^{\frac{1}{3}} = 3 \cdot 5 = 15$$

Hint

Recall that the product of a fraction and its reciprocal is 1.

The Power of a Power Property can be used to help solve equations with rational exponents. To cancel a rational exponent the exponent must be multiplied by its reciprocal.

Example 4 Solving Equations with Rational Exponents

Solve each equation for the value of x.

a. $x^{\frac{1}{3}} = 6$

SOLUTION To cancel out the rational exponent, raise the exponential expression to the reciprocal power. The reciprocal of $\frac{1}{3}$ is 3.

$$\left(x^{\frac{1}{3}}\right)^3 = (6)^3$$

$$x^{\frac{1}{3} \cdot 3} = (6)^3$$

$$x^1 = 6^3$$

$$x = 216$$

b. $x^{\frac{4}{5}} = 16$

SOLUTION To cancel out the rational exponent, raise the exponential expression to the reciprocal power. The reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$.

$$\left(x^{\frac{4}{5}}\right)^{\frac{5}{4}} = (16)^{\frac{5}{4}}$$
$$x^{\frac{4}{5}\cdot\frac{5}{4}} = (16)^{\frac{5}{4}}$$
$$x^1 = \left(\sqrt[4]{16}\right)^5$$
$$x = (2)^5 = 32$$

c. $x^{\frac{4}{3}} = 7$

SOLUTION To cancel out the rational exponent, raise the exponential expression to the reciprocal power. The reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.

$$\left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} = (7)^{\frac{3}{4}} \rightarrow x^{\frac{4}{3}\cdot\frac{3}{4}} = (7)^{\frac{3}{4}} \rightarrow x^1 = \sqrt[4]{7^3} \rightarrow x = \sqrt[4]{343}$$

Example 5 Application: Cubic House

In the Netherlands, the city of Rotterdam has a complex of cubic homes designed by the Dutch architect Piet Blom. Each home has a side length of 7.5 meters. Use the equation $y = \sqrt[3]{V}$, where y is the side length and V is the volume, to find the volume of a Cube-House.

SOLUTION

Understand The side length of the cube is 7.5 meters. The equation $y = \sqrt[3]{V}$ models the side length of a cube, y, in terms of the volume of the cube, V.

Plan Substitute the side length of the Cube-House into the equation.

$$7.5 = \sqrt[3]{V}$$

Solve Solve the equation for the value of x.

$$7.5 = \sqrt[3]{V}$$
$$7.5 = V^{\frac{1}{3}}$$
$$(7.5)^3 = \left(V^{\frac{1}{3}}\right)^3$$
$$421.875 = V$$

Check Substitute the value found for V back into the original equation and use a calculator to check.

$$y = \sqrt[3]{421.875}$$
$$y = 7.5$$

The volume of a Cube-House is 421.875 square meters.

Graphing Calculator Tip

If your calculator does not have a function for *n*th roots, write the radical expression with rational exponents.

Lesson Practice

Write the rational exponent as a radical expression.

(Ex 1)

a. $7^{\frac{1}{7}}$ **b.** $4^{\frac{5}{3}}$

Simplify. Assume that all variables are positive.

(Ex 2)

c. $\sqrt[6]{64x^{18}}$ **d.** $\sqrt[3]{\frac{x^{24}}{6}}$

Simplify.

(Ex 3)

e. $49^{-\frac{5}{4}} \cdot 49^{\frac{3}{4}}$ **f.** $\frac{15^{\frac{17}{6}}}{15^{\frac{5}{6}}}$

g. $\left(625^{\frac{1}{16}}\right)^4$ **h.** $(1{,}024 \cdot 32)^{\frac{1}{5}}$

Solve.

(Ex 4)

i. $x^{\frac{1}{9}} = 3$ **j.** $x^{\frac{7}{6}} = 128$

k. $x^{\frac{5}{2}} = 4$

l. (Ex 5) The Apple Store on Fifth Ave. in New York City has a 32-foot-tall cube entrance way. Use the equation $y = \sqrt[3]{x}$, where y is the side length and x is the volume, to find the volume of the Apple Store entrance.

Practice Distributed and Integrated

1. (49) Find the probability of getting exactly 0 heads if a coin is tossed 3 times.

Simplify.

2. (40) $3\sqrt{\frac{7}{5}} + 2\sqrt{\frac{5}{7}}$ **3.** (40) $\frac{2}{5\sqrt{18}}$ **4.** (40) $\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{2}}$

Complete the square and factor the resulting perfect square trinomial.

***5.** (58) $x^2 + 3x$ ***6.** (58) $x^2 - 14x$

7. (36) Find the equation of the line that is perpendicular to $2x + y = 4$ and passes through the point $(-2, -1)$.

Write the rational exponent as a radical expression.

***8.** (59) $27^{\frac{1}{3}}$ ***9.** (59) $16^{\frac{3}{4}}$ ***10.** (59) $64^{\frac{1}{6}}$

11. (23) **Verify** Use FOIL to verify the perfect square trinomial formulas.

a. $a^2 + 2ab + b^2 = (a + b)^2$ **b.** $a^2 - 2ab + b^2 = (a - b)^2$

*12. (59) **Music** Frets are small metal bars placed across the neck of a guitar so that the guitar can produce notes of a specific scale. To find the distance a fret should be placed from the bridge, multiply the length of the string by $2^{-\frac{n}{12}}$, where n is the number of notes higher than the string's root note. Where should a fret be placed to produce a G note on the A string (2 notes higher), if the length of the string is 66 cm, rounded to the nearest tenth place?

13. (39) **Multiple Choice** Which ordered pair is a solution of the system below?

$$y \leq x$$
$$x > -4$$
$$2x - 5y > 5$$

A $(-3, -3)$ **B** $(-3, 3)$ **C** $(-5, -5)$ **D** $(-5, 5)$

*14. (27) **Coordinate Geometry** Graph the function $f(x) = \frac{1}{2}x^2 - 1$ by making a table.

*15. (39) **Graphing Calculator** Graph $y \leq 2x + 3$ on your graphing calculator. Find a point that is a solution and is not on the boundary line and then find a point that is a solution and is on the boundary line.

*16. (59) **Verify** Explain why radical expressions with even indices and negative radicands have no real roots. Use $\sqrt[4]{-256}$ to demonstrate your explanation.

17. (38) **Analyze** Suppose that the expression $\frac{(x^2 - a^2)}{(x - d)}$, for integers a and d, has no remainder. What must be true of a and d?

18. (42) **Decorative Tiles** A 4 in. by 8 in. picture is formed using square tiles that are 2 in. on each side. If there are 8 tiles of different colors, how many different patterns can be created?

*19. (58) **Multi-Step** Use $x^2 - 90x - 15 = -2037$.

a. What is the first step before you can complete the square?

b. What constant can be added to the binomial to create a perfect square trinomial?

c. What are the factors of the perfect square binomial?

d. Solve the equation for x.

20. (51) **Carpentry** A carpenter plans to use wooden plywood to make a wardrobe. He needs a total area of $A = 20x^2 - 23x - 21$ and a minimum width of $5x + 3$. Find the length of the rectangular plywood the carpenter would use to make the wardrobe.

21. (50) **Geometry** The formula for the circumference of a circle is $C = 2\pi r$. Find the inverse of the function. Use the inverse to find the radius of the circle whose circumference is 44 square inches.

22. (45) **Geometry** Find the area and perimeter of each rectangle. Then sketch a scatter plot comparing the area and perimeter. Describe the correlation.

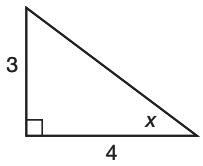
Dimensions	1 by 6	4 by 4	2 by 9	3 by 10	1 by 1	2 by 3	5 by 5
Area (unit^2)							
Perimeter (unit)							

*23. (57) **Marine Biology** The estimated worldwide population $P(t)$ of humpback whales, in thousands, is shown in the table below for various years.

Year	1900	1910	1920	1930	1940
Number (in thousands)	150	100	66	44	29

Model the data with an exponential function, and use the model to estimate the humpback population in 1950.

24. (46) **Error Analysis** Explain and correct the error made: $\tan x = \frac{4}{3} = 1.\overline{3}$.



25. (55) **Multiple Choice** What is the probability that 3 out of 5 books randomly chosen from a box where all the titles are different are chosen in alphabetical order?

A about 0.02% **B** about 1% **C** about 1.7% **D** about 10%

26. (52) **Justify** Explain the following statement: The triangles formed by the origin, the point (x, x), and the point $(x, 0)$, for any real x, are similar.

27. (55) **Write** What is a conditional probability? Give an example.

28. (44) Order the numbers $\frac{3}{\sqrt{5}}$, $\frac{3\sqrt{5}}{2\sqrt{2}}$, and $\frac{6}{\sqrt{10}}$ from least to greatest, without using a calculator.

29. (53) **Error Analysis** A student is working with function $m(t) = t^{-3}$ and $n(t) = t^{-2}$. The student claims that the domain of $m(n(t))$ consists of all real numbers. Explain the error.

30. (56) Name two different angles that have a reference angle of 15°.

LESSON
60

Distinguishing Between Mutually Exclusive and Independent Events

Warm Up

1. (33) **Vocabulary** The number of ways objects can be arranged when order matters is called a _____.

2. (33) Calculate ${}_8C_3$.

3. (47) If a number cube is rolled, what is the probability of rolling an odd number?

New Concepts

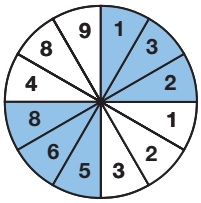
A **compound event** is an event that is made up of two or more simple events. Events that have at least one outcome in common are **inclusive events.** **Disjoint events** are events that have no outcomes in common.

Math Language

Inclusive events may also be described as overlapping. Disjoint events are also called mutually exclusive.

Example 1 Disjoint Events and Inclusive Events

When spinning the number wheel to the right the possible outcomes are {B1, B2, B3, B5, B6, B8, W1, W2, W3, W4, W8, W9}. Determine whether the compound events described are disjoint events or inclusive events.



a. The spinner landing on a blue sector and the spinner landing on a 4.

SOLUTION Blue: {B1, B2, B3, B5, B6, B8}, 4: {W4} The two sets have no outcomes in common; therefore, they are disjoint events.

b. The spinner landing on a blue sector and the spinner landing on a 3.

SOLUTION Blue: {B1, B2, B3, B5, B6, B8}, 3: {B3, W3} The two sets intersect at B3; therefore, the two events are inclusive.

To calculate the probability of one or the other of two inclusive events use the formula below.

Inclusive Events
For two inclusive events A and B: $$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Since events that are mutually exclusive, or disjoint, have no outcomes in common, $P(A \text{ and } B) = 0$.

Therefore,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = P(A) + P(B) - 0 = P(A) + P(B).$$

Mutually Exclusive Events
For two mutually exclusive events A and B: $$P(A \text{ or } B) = P(A) + P(B)$$

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Example 2 Probability of Mutually Exclusive and Inclusive Events

A jar contains 10 marbles: 5 red, 4 blue, and 1 green. The red marbles are numbered 1–5, the blue are numbered 6–9, and the green is numbered 10. A single marble is drawn from the jar.

a. Find the probability that the marble is blue or has an even number.

SOLUTION It is possible to select a blue marble with an even number. The events are inclusive and the formula $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ can be used. Let A be the event that the marble is blue and B the event that the marble is numbered evenly.

$$P(A) = \frac{4}{10} = \frac{2}{5} \qquad P(B) = \frac{5}{10} = \frac{1}{2} \qquad P(A \text{ and } B) = \frac{2}{10} = \frac{1}{5}$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = \frac{2}{5} + \frac{1}{2} - \frac{1}{5} = \frac{7}{10} = 0.7$$

b. Find the probability that the marble is red or has a two digit number.

SOLUTION It is not possible to choose a red marble with a two digit number. The events are mutually exclusive and the formula $P(A \text{ or } B) = P(A) + P(B)$ can be used. Let A be the event that the marble is red and B the event that the marble has a two digit number.

$$P(A) = \frac{5}{10} = \frac{1}{2} \qquad P(B) = \frac{1}{10}$$

$$P(A \text{ or } B) = P(A) + P(B) = \frac{1}{2} + \frac{1}{10} = \frac{3}{5} = 0.6$$

Math Reasoning

Connect Think of an example of two events that are mutually exclusive but *not* independent.

Example 3 Dependent and Independent Events

A jar contains 9 white marbles and 7 green marbles. Determine whether the listed events are dependent or independent.

a. A white marble is selected and replaced. Then a green marble is selected.

SOLUTION The white marble is replaced so the first outcome has no effect on the selection of the green marble. Thus the events are independent.

b. A white marble is selected. Then without replacement another white marble is selected.

SOLUTION The first white marble is not replaced so it alters the possibilities of the next outcome. Thus the events are dependent.

The formula below can be used to calculate the probability of independent events.

Probability of Independent Events
For the probability of two independent events A and B: $$P(A \text{ and } B) = P(A) \cdot P(B)$$ In general, for n independent events, $A_1, A_2, \ldots A_n$: $$P(A_1, A_2, \ldots, \text{and } A_n) = P(A_1) \cdot P(A_2) \cdot \ldots \cdot P(A_n)$$

Example 4 Probability of Independent Events

Two fair coins are flipped three times. Find the probability that both coins come up one heads and one tails all three times.

SOLUTION Since each flip does not affect the next, the events are independent. The formula $P(A_1, A_2, \ldots, \text{and } A_n) = P(A_1) \cdot P(A_2) \cdot \ldots \cdot P(A_n)$ can be used.

Let A_1 be the event of one heads and one tails on the first flip, A_2 the event of one heads and one tails on the second flip, and A_3 the event of one heads and one tails on the third flip.

$$P(A_1) = \frac{1}{2} \qquad P(A_2) = \frac{1}{2} \qquad P(A_3) = \frac{1}{2}$$

$$P(A_1, A_2, \text{and } A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} = 0.125$$

Math Reasoning

Verify Show the work to find the probability of one heads and one tails in one flip of two coins.

Example 5 Application: Giant Panda Population

The survival rate of giant panda cubs born in captivity is 59.4%. Find the probability that out of 4 giant panda cubs born consecutively into captivity the first two will survive and the last two will not. Round to the ten thousandths place.

SOLUTION

The survival of one panda cub does not affect the survival of the next cub, so the events are independent. The formula $P(A_1, A_2, \ldots, \text{and } A_n) = P(A_1) \cdot P(A_2) \cdot \ldots \cdot P(A_n)$ can be used.

Let A_1 be the event of one cub's survival, A_2 the event of the next cub's survival, A_3 the event of the third cub not surviving, and A_4 the event of the fourth cub not surviving.

$$P(A_1) = 0.594 \qquad P(A_2) = 0.594 \qquad P(A_3) = 0.406 \qquad P(A_4) = 0.406$$

$$P(A_1, A_2, A_3, \text{and } A_4) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4)$$

$$= 0.594 \cdot 0.594 \cdot 0.406 \cdot 0.406 \approx 0.05816 \approx 0.0582$$

Lesson Practice

(Ex 1) **a.** A number cube is rolled once. Are the events that the number rolled is odd and the number rolled is greater than 5 inclusive or mutually exclusive?

(Ex 1) **b.** A number cube is rolled once. Are the events that the number rolled is even and the number rolled is divisible by 3 inclusive or mutually exclusive?

(Ex 2) **c.** A jar contains 10 green marbles and 10 orange marbles. The green marbles are numbered 1–5 and 11–15 and the orange marbles are numbered 6–10 and 16–20. Find the probability that a marble drawn from the jar is odd or green.

d. (Ex 2) A jar contains 10 green marbles and 10 orange marbles. The green marbles are numbered 1–5 and 11–15 and the orange marbles are numbered 6–10 and 16–20. Find the probability that a marble drawn from the jar is orange or less than 6.

e. (Ex 3) A number cube is rolled and a fair coin flipped. Are the events that a number less than 3 is rolled and heads is flipped dependent or independent?

f. (Ex 3) A bag contains 26 tiles lettered A–Z. Two tiles are drawn without replacement. Are the events that a vowel and a consonant are selected dependent or independent?

g. (Ex 4) A jar contains 12 marbles and 7 blocks. An object is drawn from the jar then replaced. Find the probability that in 3 draws a marble, a block, then a marble is drawn. Round to the ten thousandths place.

h. (Ex 5) **Birth Trends** The National Vital Statistics Report on births in 2004 shows that approximately 48.8% of live births are females. Find the probability that women would give birth to three daughters followed by a son. Round to the ten thousandths place.

Practice Distributed and Integrated

1. (36) Find the equation of the line that passes through the point (2, 2) and is perpendicular to the line $y = 3x - 5$.

2. (54) Given the objective function $P = 111x + 95y$ and the constraints $x \leq 4$, $y \geq 0$, $x + 2y \leq 50$, $3x + y \geq 15$, find the minimum and maximum values for the objective function.

3. (41) Find the distance between $(4, -2)$ and $(-3, -5)$.

Divide using synthetic division.

4. (51) $x^3 - 7$ by $x - 5$

5. (51) $x^3 + 3x^2 + 7x + 5$ by $x + 1$

Divide using long division.

6. (38) $x^3 + 2$ by $x + 1$

7. (38) $x^4 - 2$ by $x + 1$

***8.** (60) **Multiple Choice** A jar contains 19 black blocks and 25 gray blocks. The black blocks are numbered 1–19 and the gray blocks are numbered 20–44. Choose the answer that best represents the probability of selecting a black block or a number greater than 21.

A 0.8864 **B** 0.9545 **C** 0.9773 **D** 0.9318

9. (43) **Error Analysis** A student stated that (1, 3) is a solution of the system below. What is the error?

$y \leq x + 2$

$y > -x + 4$

10. **Generalize** Suppose that $ax^2 + bx + c$ is divisible by $x + d$ with no remainder, for real numbers a, b, c, and d. What can you conclude about the graph of the parabola? *(38)*

11. **Coordinate Geometry** Matrix multiplication can represent transforming a geometric figure. For example, suppose the three vertices of a triangle are given by the three rows of the matrix A. Then multiplying on the right by the matrix T gives a result that represents a new triangle. *(14)*

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 3 & 3 \\ -3 & 3 \end{bmatrix}$$

a. Calculate the product of these matrices.

b. Graph this new triangle.

c. Calculate the areas of the original and the transformed triangles.

d. How do these areas relate to the determinant of matrix T?

12. **Estimate** The hypotenuse of a 30°-60°-90° triangle is 16 inches. Estimate the length of the side adjacent to the 30°. Give your answer to the nearest whole inch. *(52)*

***13.** **Multi-Step** A number cube is rolled once and a fair coin is flipped. *(60)*

a. Are the events, a number less than 5 and heads, dependent or independent? Explain why.

b. Find the probability of the number cube showing less than 5. Round to the hundredths place.

c. Find the probability of the coin showing heads. Round to the hundredths place.

d. Use the probabilities found in parts **b** and **c** to find the probability that a roll and flip will show a number less than 5 and heads. Round to the hundredths place.

14. **Chemistry** Seventy percent of a compound is sodium chloride. If 660 grams of other chemicals are in the compound, what is the total weight of the compound? *(53)*

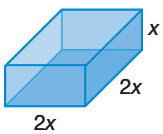
15. **Physics** The formula $t = \sqrt{\frac{2s}{g}}$ can be used to determine the time t, in seconds, it takes an object initially at rest to fall s meters. In this formula, g stands for the acceleration due to gravity, which on Earth is about 9.8 m/s^2. Use the formula to find out how long it takes a rock initially at rest to fall 110 meters. Write your answer in radical form. *(44)*

***16.** **Graphing Calculator** Use a graph to estimate the solution of $\begin{aligned} 3.1a + 2.8b &= -12.2 \\ -0.9a - 1.1b &= 3.85 \end{aligned}$. *(15)*

17. **Explain** Can synthetic division be used to simplify $\frac{x^3 - 5x + 3}{x^2 + 2}$? *(51)*

***18.** **Write** Two number cubes are tossed once. Explain why the events, one cube shows 2 and the sum of the two cubes is greater than 5, are dependent. *(60)*

19. **Geometry** Write a rational expression in simplest form for the ratio of the rectangular prism's surface area to its volume. *(24)*



*20. (58) **Projectile Motion** Gravity on the planet Jupiter is almost three times stronger than on Earth. If an object is thrown into the air, with an initial velocity of 90 feet per second, from the surface of Jupiter, the height, h, of the object in feet after t seconds can be approximated by the equation $h = -42t^2 + 90t$. Find the time at which the object will return to the surface. Round your answer to the tenths place.

21. (41) **Justify** Given the lengths of three sides of a right triangle, how do you know which side is the hypotenuse?

22. (29) **Multiple Choice** Circle the letter that represents the number and type of solutions of the system of three equations: $\begin{aligned} -4x + 12y - 16z &= -28 \\ 3x - 9y + 12z &= 21 \\ -x + 3y - 4z &= -7 \end{aligned}$.

A No solution, inconsistent **B** One solution, inconsistent

C One solution, consistent **D** Infinitely many solutions, consistent

23. (58) **Error Analysis** Find and explain the error in the work below. Then find the correct solution.

$$\begin{aligned} x^2 - 34x - 15 &= 15 \\ x^2 - 34x &= 30 \\ x^2 - 34x + 289 &= 30 \end{aligned} \qquad \begin{aligned} (x - 17)^2 &= 30 \\ x - 17 &= \pm\sqrt{30} \\ x &= 17 \pm \sqrt{30} \end{aligned}$$

*24. (59) **Statistics** Given that the variance of a data set is $425\frac{5}{3}$ find the standard deviation for the data set. (Hint: standard deviation $= \sqrt{\text{variance}}$)

25. (46) **History** The Red Pyramid is an Egyptian pyramid with a square base length of 220 meters. The slope is approximately 43°. Approximate the length of an edge of the pyramid to the nearest tenth.

26. (57) **Medicine** If a patient ingests medicine containing a grams of a drug, the amount left in the patient's bloodstream after t hours is given by the equation $y = ab^t$. If one-fourth of the medicine is left after 3 hours, what portion is left after 4 hours?

27. (45) **Verify** Use the following data to verify that exchanging the x and y values does not change the value of r for the data set.

x	1	2	3	4
y	67	56	84	70

28. (53) **Investment** You invest in a business. The first month your investment gains 5%. The second month your investment loses 5%. Write a composition function to show whether the investment equals the original investment after two months.

*29. (59) **Verify** Show the steps needed to write $\sqrt{3} \cdot \sqrt[5]{3}$ as $\sqrt[10]{2{,}187}$.

30. (55) **Geography** The table shows the number of states in the continental U.S. that fall into each time zone. The total exceeds 48 because some states are in two time zones. What is the probability that a randomly chosen state is not in the Pacific time zone?

Time Zone	Eastern	Central	Mountain	Pacific
Number of States	22	20	13	5

INVESTIGATION

6

Deriving the Quadratic Formula

Model The general form of a quadratic equation is written as

$$f(x) = ax^2 + bx + c$$

The roots of the equation are found by solving the equation

$$ax^2 + bx + c = 0$$

The most common way of solving a quadratic equation is to use the **quadratic formula**

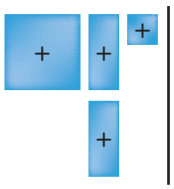
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

But where does this equation come from? Is there a way of deriving this equation?

In this activity, you will use algebra tiles to model completing the square. You will need a quantity of these three models:

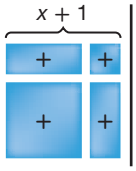
x^2 term	x term	Constant term (one tile represents 1, two tiles represent 2, etc.)

1. Use the tiles to model the following quadratic equation.



$x^2 + 2x + 1 = 0$

2. This quadratic equation is the perfect square $(x + 1)^2 = 0$. Rearrange the tiles, as shown below, to show the square.



$(x + 1)^2 = 0$

3. Use tiles to show that the following quadratic equations are perfect squares.

a. $x^2 + 4x + 4 = 0$ **b.** $x^2 + 6x + 9 = 0$ **c.** $x^2 + 8x + 16 = 0$

Math Reasoning

Analyze In the algebra tiles model, the vertical line represents the = sign. How is zero modeled?

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4. Complete the table below with the values of c and b for each of the perfect squares.

Perfect Square	b	c
$(x + 1)^2$		
$(x + 2)^2$		
$(x + 3)^2$		
$(x + 4)^2$		

Math Reasoning

Generalize Expand the perfect square $(x + n)^2 = 0$. Since this is also a quadratic equation, what are the values of a, b, and c?

5. Write an equation that shows the relationship between c and b for perfect squares.

Not all quadratic equations are perfect squares, but they can be turned into perfect squares.

6. **Multi-Step** Use algebra tiles.

a. Model the equation $x^2 + 6x + 1 = 0$.
b. What needs to be added to both sides of the equation to get a perfect square on the left side?
c. Write the new equation with the perfect square on the left side.
d. Solve the equation.

To solve the general quadratic equation, you can complete the square using the relationship between b and c you determined in problem 5.

$$ax^2 + bx + c = 0$$

7. To simplify solving the equation, divide both sides of the equation by a. Write the new form of the equation.

8. Subtract the constant term from each side of each equation. Write the new form of the equation.

9. Using the relationship between c and b you discovered in question 5, determine what needs to be added to the left side to complete the square.

10. Add the amount determined in problem 9 to both sides of the equation to complete the square.

11. Write the left side as the square of a binomial.

12. Simplify the right side of the equation.

13. Apply the Square Root Property.

14. Simplify the right side of the equation.

15. Solve for x:

16. Solve the quadratic equation $2x^2 + 6x - 3 = 0$ by substituting the values for a, b, and c.

Investigation Practice

Use algebra tiles to determine which of the following quadratic equations represent perfect squares.

a. $x^2 + 10x + 25 = 0$

b. $2x^2 + 6x + 3 = 0$

c. $4x^2 + 4x + 1 = 0$

Solve by completing the square and solve by using the quadratic formula.

d. $x^2 + 4x + 1 = 0$

e. $x^2 + 6x + 2 = 0$

LESSON 61

Understanding Advanced Factoring

Warm Up

1. (11) **Vocabulary** A polynomial of degree one with two terms is a _____.
2. (23) True or False: The GCF of $-2x^4y^5 - xy^2 + 2$ is $2xy^2$.
3. (51) Divide $4x^2 - 2x + 5$ by $x - 1$ using synthetic substitution.

New Concepts

Recall that synthetic division is used to divide a polynomial $P(x)$ by a linear binomial of the form $x - a$ and that the last number in the bottom row is the remainder and also the value of $P(a)$. If the remainder is 0, the linear binomial is a factor of the polynomial. This is known as the Factor Theorem.

Factor Theorem
For polynomial $P(x)$, $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.

Hint

$x + 2 = x - (-2)$, so the value of a is -2.

For example, $x + 2$ is a factor of $x^2 + 10x + 16$ and

$$\begin{aligned} P(-2) &= (-2)^2 + 10(-2) + 16 \\ &= 4 - 20 + 16 \\ &= 0. \end{aligned}$$

Example 1 Determining Whether a Linear Binomial is a Factor

Determine whether the linear binomial is a factor of the polynomial.

a. $P(x) = 2x^3 - x^2 - 43x + 60$, $x - 4$

SOLUTION Use synthetic division with $a = 4$.

$$\begin{array}{r|rrrr} 4 & 2 & -1 & -43 & 60 \\ & & 8 & 28 & -60 \\ \hline & 2 & 7 & -15 & \textcircled{0} \end{array}$$

Because $P(4) = 0$, $x - 4$ is a factor of $P(x) = 2x^3 - x^2 - 43x + 60$.

b. $P(x) = x^3 + x^2 - 24x + 36$, $x + 3$

SOLUTION Use synthetic division with $a = -3$.

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -24 & 36 \\ & & -3 & 6 & 54 \\ \hline & 1 & -2 & -18 & \textcircled{90} \end{array}$$

Because $P(-3) \neq 0$, $x + 3$ is not a factor of $P(x) = x^3 + x^2 - 24x + 36$.

When the remainder is zero, the quotient of $P(x)$ and $(x - a)$ is another factor of the polynomial. This factor may be able to be further factored by other factoring methods. For instance, in Example 1a, the quotient $2x^2 + 7x - 15$ factors into $(2x - 3)(x + 5)$. Therefore, $P(x)$ factors into $(x - 4)(2x - 3)(x + 5)$.

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If the remaining quotient is a cubic, it may be factorable by one of the following factoring methods.

Sum and Difference of Cubes
Sum of Two Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Two Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Math Reasoning

Justify Tell why $3x - 1$ is not a difference of cubes.

Example 2 Factoring the Sum and Difference of Two Cubes

Factor each expression.

a. $d^3 + 125$

SOLUTION The expression is a sum of two cubes, where $a = d$ and $b = 5$.

$d^3 + 125$

$(d + 5)(d^2 - 5d + 25)$

b. $27x^6 + y^9$

SOLUTION The expression is a sum of two cubes, where $a = 3x^2$ and $b = y^3$.

$27x^6 + y^9 = (3x^2)^3 + (y^3)^3$

$(3x^2 + y^3)(9x^4 - 3x^2y^3 + y^6)$

c. $3m^5 - 24m^2$

SOLUTION First factor out the GCF, $3m^2$. Then factor the difference of the two cubes, where $a = m$ and $b = 2$.

$3m^5 - 24m^2$

$3m^2(m^3 - 8)$

$3m^2(m - 2)(m^2 + 2m + 4)$

Another method of factoring is factoring by grouping. To factor by grouping, group the terms evenly so that each group has a common factor. Factor out the GCF from each group. Then factor out the common polynomial factor.

Example 3 Factoring by Grouping

Factor each expression.

a. $x^3 + 5x^2 + 3x + 15$

SOLUTION The first two terms have a common factor as do the last two terms.

$(x^3 + 5x^2) + (3x + 15)$

$x^2(x + 5) + 3(x + 5)$ Factor x^2 from the first group and 3 from the second.

$(x + 5)(x^2 + 3)$ Factor out the GCF of $(x + 5)$.

Neither factor can be factored further.

Math Reasoning

Analyze Can a polynomial with five terms be factored by using the grouping method? Explain.

b. $x^4 + 4x^3 - x - 4$

SOLUTION Factor x^3 from the first two terms and -1 from the last two terms.

$x^4 + 4x^3 - x - 4$

$x^3(x + 4) - 1(x + 4)$

$(x + 4)(x^3 - 1)$ Factor out the GCF of $(x + 4)$.

$(x + 4)(x - 1)(x^2 + x + 1)$ Factor the difference of cubes.

c. $x^6 + 3x^5 + 2x^4 + x^2 + 3x + 2$

SOLUTION Use two groups of three terms each.

$(x^6 + 3x^5 + 2x^4) + (x^2 + 3x + 2)$

$x^4(x^2 + 3x + 2) + 1(x^2 + 3x + 2)$

$(x^2 + 3x + 2)(x^4 + 1)$ Factor out the GCF of $(x^2 + 3x + 2)$.

$(x + 1)(x + 2)(x^4 + 1)$ Factor the trinomial.

Example 4 Application: Three-Dimensional Figures

The volume of a rectangular solid with a length of $x - 5$ feet has a volume of $x^3 - 23x^2 + 170x - 400$ cubic feet. Factor the expression for the volume completely.

SOLUTION Since volume is the product of the length, width, and height, the expression for the length, $x - 5$, is a factor of the polynomial. Use synthetic division to find the quotient of the polynomial and $x - 5$.

5	1	−23	170	−400
		5	−90	400
	1	−18	80	0

The expression for the volume factors into $(x - 5)(x^2 - 18x + 80)$. But the trinomial can still be factored: $(x - 10)(x - 8)$.

The expression for the volume factors into $(x - 5)(x - 10)(x - 8)$.

Lesson Practice

Determine whether the linear binomial is a factor of the polynomial.

a. (Ex 1) $P(x) = 3x^3 + 19x^2 + 32x + 16$, $x + 3$

b. (Ex 1) $P(x) = 4x^3 - 33x^2 + 56x - 12$, $x - 6$

Factor each expression.

c. (Ex 2) $64 - h^3$

d. (Ex 2) $s^6t^{12} - 125r^3$

e. (Ex 2) $\frac{8}{27}x^7 + x^4$

f. (Ex 3) $xy - 9y + 5x - 45$

g. (Ex 3) $4m^2n + 12m^2 - n - 3$

h. (Ex 3) $x^5 + 12x^4 + 36x^3 + 5x^2 + 60x + 180$

i. (Ex 4) The volume of a rectangular solid with a length of $x - 15$ feet has a volume of $x^3 + 5x^2 - 600x + 4500$ cubic feet. Factor the expression for the volume completely.

Practice Distributed and Integrated

Graph.

1. (13) $x = -5$

2. (13) $2x - y = 4$

3. (13) $7y + 7x = 21$

4. (Inv 3) Find the intercepts of the equation $5x - 4y + 7z = 21$.

Simplify.

5. (59) $-16^{-\frac{1}{2}}$

6. (59) $27^{-\frac{1}{3}}$

7. (59) $9^{\frac{3}{2}}$

Find an inverse for the equations.

8. (50) $y = \frac{2}{3}x - 3$

9. (50) $y = 5x + 15$

10. (53) Let $K(u) = 17$ and $L(u) = -13$. Find $K\big(L(u)\big)$ and $L\big(K(u)\big)$.

***11.** (54) Minimize $P = 4x + 3y$ for the constraints $x \le 4$, $y \le 6$, $x + y \ge 7$

***12.** (57) Mary deposited \$980 at 7.0% compounded continuously. How much money will she have after 9 years?

13. (55) **Meteorology** The number of tornadoes in the United States in 2006 is shown in the table.

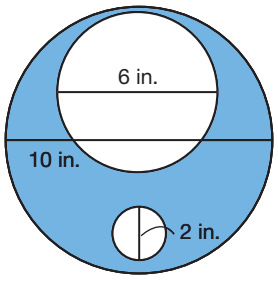
Season	Jan-Feb	Mar-Apr	May-Jun	Jul-Aug	Sep-Oct	Nov-Dec
Number of Tornadoes	59	395	258	151	160	82

What is the experimental probability that a tornado occurs in the first six months of the year?

***14.** (61) **Write** Tell how factoring a sum of cubes is similar to factoring a difference of cubes.

15. (45) **Probability** A student graphed a set of data points and found that the correlation coefficient was -0.73. What is the probability that the slope of the line of best fit is greater than $\frac{1}{2}$? Explain.

***16.** (60) **Geometry** The probability that a randomly selected point lies within a region can be found using the ratio of the area the point would lie in to the total area. Find the probability that a random point within the region lies within one of the smaller circles.



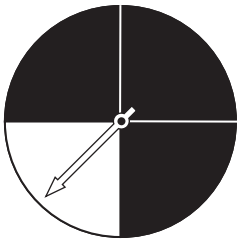
17. (46) **Buildings** One building is 310 feet tall and a nearby building is 374 feet tall. From the top of the taller building, you can spot the top of the shorter building at an angle of depression of 30°. How far apart are the two buildings? Round to the nearest foot.

18. (53) **Walking Shadows** While walking the dog one night, you notice how the length of your shadow changes as you approach each lamppost. The length of the shadow seems to be $\frac{1}{5}$ of your distance to the lamppost. Your dog is dragging you along at about 3 feet per second, aiming for a lamppost. Show how to write a composite function that gives the length of your shadow in terms of time as you approach the lamppost.

***19.** (59) **Error Analysis** Is the work below correct? If not, find and explain the error, and give the correct solution.

$$x^{\frac{3}{8}} = 3 \longrightarrow \left(x^{\frac{3}{8}}\right)^{\frac{3}{8}} = (3)^{\frac{3}{8}} \longrightarrow x^1 = 3^{\frac{3}{8}} \longrightarrow x = \sqrt[8]{27}$$

20. (49) **Analyze** Suppose you spin the spinner shown 8 times. What is the probability that you will spin black 5 times?



21. (58) **Verify** Use the form $x^2 + 2cx + c^2$ for a quadratic perfect square trinomial to verify the method of completing the square.

***22.** (54) **Graphing Calculator** Describe the feasible region for the inequalities:

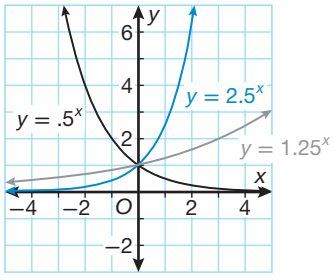
$$x \geq 0,\ y \geq 0,\ y \leq -0.75x + 8$$

23. (45) **Multiple Choice** Which value of r, the correlation coefficient, describes the set of points that are *closest* to forming a line?

A $r = -0.75$ **B** $r = 0$ **C** $r = 0.6$ **D** $r = 0.65$

***24.** (60) **Commuting to Work** According to data collected in the 2005 American Community Survey, approximately 10.67% of workers carpool to work, 4.66% use public transportation, and 2.47% walk. Find the probability that a randomly selected worker carpools or uses public transportation.

25. (47) **Write** A few different equations of the form $y = b^x$ are graphed to the right. Describe in your own words how the shape of the graph depends on the value of b.



*26. (59) **Multi-Step** Use $\frac{5^{\frac{3}{5}} \cdot 5^{\frac{3}{5}}}{5^{\frac{4}{5}}}$.

a. Which property of rational exponents should be applied first to simplify the expression? Explain.

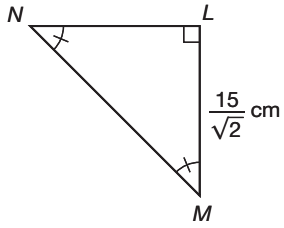
b. Apply the property from part a.

c. Which property of rational exponents should be applied next?

d. Apply the property from part c.

e. If possible, finish simplifying the expression from part d.

27. (52) **Multiple Choice** In the diagram to the right, find the length of the hypotenuse of triangle LMN.



A 7.5 **B** $\sqrt{2}$

C 15 **D** $15\sqrt{3}$

*28. (61) **Multi-Step** The area of a twin size mattress is $x^2 - 275 - 11x + 25x$ square inches.

a. Factor the polynomial by grouping.

b. Find the area given that $x = 50$.

29. (56) **Amusement Parks** The Giant Swing at Silver Dollar City in Branson, Missouri, rotates through an angle measuring 230°. Find the reference angle for this angle.

*30. (61) **Verify** Show that there are two different ways to group the terms in $3d + 6f + d^2 + 2df$ and that both lead to the same factorization.

LESSON
62

Using Complex Numbers

Warm Up

1. (1) **Vocabulary** Any point on the number line is a _____ number.
2. (2) Simplify $(5 + 3x) - (6 - 2x)$.
3. (58) Solve $5x^2 - 125 = 0$.
4. (40) Simplify $\sqrt{150}$.

New Concepts

The equation $x = \pm\sqrt{-16}$ has no real solutions because there are no real numbers that, when squared, equal -16. But the equation does have solutions when a new set of numbers, outside of the real numbers, is introduced. These numbers are the **imaginary numbers.**

Imaginary Numbers

An imaginary number is written in the form bi where b is a real number and i is the **imaginary unit.**

i is a solution of $x^2 = -1$.

Because $i^2 = -1$, $i = \sqrt{-1}$.

The square root of a negative number is an imaginary number.

$$\sqrt{-16} = \sqrt{16 \cdot -1} = \sqrt{16}\sqrt{-1} = 4i$$

Math Reasoning

Generalize For any positive real n, $\sqrt{-n} =$ _____.

Example 1 Simplifying Square Roots of Negative Numbers

a. Simplify $-2\sqrt{-49}$.

SOLUTION

$-2\sqrt{-49}$	
$-2\sqrt{49 \cdot -1}$	Write -49 as the product of 49 and -1.
$-2\sqrt{49}\sqrt{-1}$	The square root of a product is the product of the square roots.
$-2 \cdot 7 \cdot i$	$\sqrt{49} = 7$ and $\sqrt{-1} = i$.
$-14i$	Multiply the real numbers.

b. Simplify $\sqrt{-117}$.

SOLUTION

$\sqrt{-117}$	
$\sqrt{9 \cdot 13 \cdot -1}$	Write -117 as a product with a perfect square.
$\sqrt{9}\sqrt{13}\sqrt{-1}$	The square root of a product is the product of the square roots.
$3\sqrt{13}(i)$	$\sqrt{9} = 3$ and $\sqrt{-1} = i$.
$3i\sqrt{13}$	Commutative Property of Multiplication

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Example 2 Solving a Quadratic Equation with Imaginary Numbers

Solve $x^2 + 25 = 0$. Write the solutions in terms of i. Check the answers.

SOLUTION $x^2 + 25 = 0$

$x^2 = -25$	Subtract 25 from each side.
$x = \pm\sqrt{-25}$	Take the square root of each side.
$x = \pm 5i$	Simplify the square root.

The solutions are the conjugate pairs $5i$ and $-5i$.

Check

$x^2 + 25 = 0$

$(5i)^2 + 25 \stackrel{?}{=} 0$

$25i^2 + 25 \stackrel{?}{=} 0$

$25(-1) + 25 \stackrel{?}{=} 0$

$0 = 0$ ✓

$x^2 + 25 = 0$

$(-5i)^2 + 25 \stackrel{?}{=} 0$

$25i^2 + 25 \stackrel{?}{=} 0$

$25(-1) + 25 \stackrel{?}{=} 0$

$0 = 0$ ✓

Math Language

Complex solutions of quadratic equations always occur in **conjugate pairs** of the form $a + bi$ and $a - bi$.

An imaginary number bi is part of a **complex number.**

Complex Numbers

A complex number is a number that can be written in the form $a + bi$, where a and b are real numbers.

In a complex number, a is called the **real part** and bi is called the **imaginary part.** If $b = 0$, then the imaginary part is 0 and the number is a real number. If $a = 0$ and $b \neq 0$, then the real part is 0 and the number is an imaginary number.

Complex	Real	Imaginary
$2 + 3i$	$2 + 0i = 2$	$0 + 3i = 3i$

Math Reasoning

Analyze Are real numbers a subset of complex numbers or are complex numbers a subset of real numbers?

To add or subtract complex numbers, add or subtract the real parts and then add or subtract the imaginary parts.

Add or Subtract Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Example 3 Simplifying Expressions with Complex Numbers

Add or subtract. Write the answer in the form $a + bi$.

a. $(7 + 4i) + (-2 - 5i)$

SOLUTION $(7 + (-2)) + (4 + (-5))i = 5 + (-1)i = 5 - i$

b. $(5 - 3i) - (5 - 11i)$

SOLUTION $(5 - 5) + (-3 - (-11))i = 0 + 8i = 8i$

Example 4 Solving Quadratic Equations

Solve each equation. Write the solutions in the form $a + bi$.

a. $(x - 3)^2 = -4$

SOLUTION $(x - 3)^2 = -4$

$x - 3 = \pm\sqrt{-4}$	Take the square root of each side.
$x - 3 = \pm 2i$	Simplify the square root.
$x - 3 = 2i$ or $x - 3 = -2i$	Solve both equations.
$x = 3 + 2i$ or $x = 3 - 2i$	

The solutions are $3 + 2i$ and $3 - 2i$.

b. $x^2 + 8x + 18 = 0$

SOLUTION

$x^2 + 8x = -18$	Subtract 18 from each side.
$x^2 + 8x + 16 = -18 + 16$	Complete the square.
$(x + 4)^2 = -2$	Write the left side as a perfect square.
$x + 4 = \pm\sqrt{-2}$	Take the square root of each side.
$x + 4 = \pm i\sqrt{2}$	Simplify the square root.
$x = -4 \pm i\sqrt{2}$	Subtract 4 from each side.

The solutions are $-4 + i\sqrt{2}$ and $-4 - i\sqrt{2}$.

Lesson Practice

a. (Ex 1) Simplify $\frac{1}{2}\sqrt{-100}$.

b. (Ex 1) Simplify $3\sqrt{-450}$.

c. (Ex 2) Solve $0 = 196 + x^2$. Write the solutions in terms of i. Check the answer.

d. (Ex 2) Solve $-6x^2 = 216$. Write the solutions in terms of i.

Add or subtract. Write the answer in the form $a + bi$. (Ex 3)

e. $(-2 + i) + (14 + 4i)$

f. $(10 - 6i) - (-4 + 3i)$

Solve each equation. Write the solutions in the form $a + bi$. (Ex 4)

g. $(x + 9)^2 = -9$

h. $x^2 - 2x + 2 = 0$

Practice Distributed and Integrated

Factor.

***1.** (61) $125x^4 + 375x^3 - x - 3$

***2.** (61) $64x^{12} - 125y^9$

Add or subtract.

3. (37) $a + \frac{a}{b}$

4. (37) $\frac{ax^2}{m^2p} - c + \frac{2}{m}$

5. (37) $-\frac{gh^2}{x^3} + xh + \frac{g^2h^2}{gx^3}$

Factor the greatest common factor.

6. (23) $35x^7y^5m - 7x^5m^2y^2 + 14y^7x^4m^2$

7. (23) $6x^2ym^5 - 2x^2ym + 4xym$

Simplify.

8. (59) $\frac{1}{81^{\frac{-3}{4}}}$

9. (59) $(-27)^{\frac{-2}{3}}$

10. (59) $\frac{1}{-3^{-2}}$

11. (49) **School** It is estimated that 15% of students walk to school. What is the probability that exactly 2 out of 5 randomly selected students will walk to school?

12. (50) Find an equation for the inverse of $y = \frac{1}{5}x^3 + 6$.

13. (55) What are the odds in favor of rolling a 3 on a number cube?

***14.** (62) **Multiple Choice** Which is a real number?

A $15i$ **B** $0 + 15i$ **C** $5 + 0i$ **D** $15 + 15i$

***15.** (37) **Verify** Show that the expression $\frac{1}{1 + \frac{1}{x+1}}$ is equivalent to $\frac{x+1}{x+2}$.

***16.** (62) **Electrical Engineering** A circuit has a current of $(4 - 2i)$ amps, and a second circuit has a current of $(6 - 6i)$ amps. Find the sum of the currents.

17. (55) **Analyze** Explain why the odds in favor of an event with a probability of 50% are 1 to 1.

18. (58) **Projectile Motion** The Leaning Tower of Pisa measures 180.4 feet. If an object is thrown into the air, with an initial velocity of 80 feet per second, from the top of the lower side, the height, h, of the object in feet after t seconds can be described by the equation $h = -16t^2 + 80t + 180.4$. Find the time at which the object will reach a height of 200 feet. Round your answer to the tenths place.

***19.** (60) A number cube is rolled once. Are the events a number greater than 3 or a multiple of 2 mutually exclusive or inclusive? Explain why.

***20.** (61) **Transportation** The approximate volume of a moving truck, in cubic feet, can be given by $x^3 - 6x^2 + 8x + 5x^2 - 30x + 40$. Factor this expression by grouping. Then find the volume when $x = 10$.

21. (56) **Multiple Choice** Which of the following angles is *not* coterminal with 225°?

A 585° **B** −135° **C** 935° **D** −495°

22. (57) **Multi-Step** The population of an insect colony is growing at a rate of 1.5% per week.

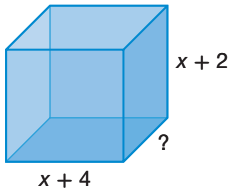
a. Write an expression for the function $N(P)$ that gives the population one week after the population is P.

b. What composite function gives the population three weeks after the population is P?

*23. (61) **Graphing Calculator** Factor $2x^5 - x^4 - 2x + 1$. Then enter the original expression for Y1 and the factored expression for Y2 on a graphing calculator. Access the Table function and study the table of values for the expressions. What is true? What does it mean?

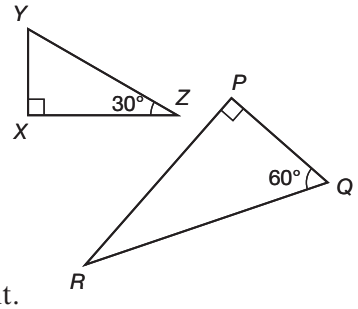
24. (59) **Write** Explain why radical expressions with even indices and positive radicands have two real roots. Use $\sqrt[6]{64}$ to demonstrate your explanation.

*25. (51) **Geometry** An expression for the volume of the rectangular prism is $V = 2x^3 + 7x^2 - 14x - 40$. Find an expression for the missing dimension.



26. (57) **Analyze** Suppose the graph of $y = b^x + k$ passes through the points $(3, -3)$ and $(5, 0)$. Determine whether the equation models exponential growth or decay.

27. (52) **Error Analysis** A student made an error while studying the relation between the two triangles shown. Explain the error.



Find the missing angles.

$m\angle Y = 90° - 30° = 60°$ $m\angle R = 90° - 60° = 30°$

$m\angle X = 90°$ $m\angle Y = 60°$ $m\angle Z = 30°$

$m\angle P = 90°$ $m\angle Q = 60°$ $m\angle R = 30°$

The corresponding angles are congruent. So, the triangles are congruent.

28. (58) **Probability** A sales clerk has a weekly quota of 15 sales. The probability, p, that the sales clerk will meet this quota after working d days, in a 5-day work week, can be approximated by $p = -0.04d^2 + 0.4d$. How many days a week should the sales clerk work for the probability of making 15 sales to be 0.8?

*29. (62) **Verify** Show that $11i$ and $-11i$ are solutions of $x^2 = -121$.

30. (54) **Cell Phones** Mobile-U has two cell phone plans, the Metropolitan Plan and the Continental Plan. The cost for each plan is shown in the table. A company is comparing the two plans. The company needs at least 12 cell phones, one for each sales representative. Each representative will use at least 500 minutes per month. What options does this company have?

	Continental Plan	Metropolitan Plan
Start-up Fee	\$50	\$100
Free minutes	2000	1200
Per minute rate above free minutes	10 cents	7 cents
Monthly charge	\$20 × # of cell phones	\$25 × # of cell phones

LESSON
63

Understanding the Unit Circle and Radian Measures

Warm Up

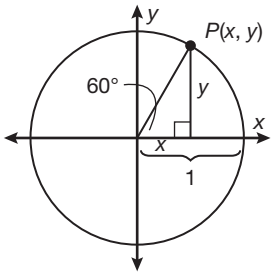
1. (SB) **Vocabulary** Half the diameter of a circle is the _____ of the circle.
2. (41) If both legs of a right triangle have length 1, then the length of the hypotenuse is _____.
3. (52) If the shortest leg of a 30°–60°–90° triangle has length 1, then the lengths of the other two sides of the triangle are _____ and _____.

New Concepts

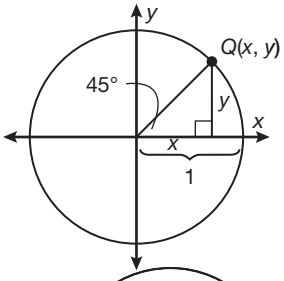
A **unit circle** is a circle with a radius of 1 unit.

Exploration Exploring the Unit Circle

Use trigonometric ratios to determine the coordinates of points on a unit circle that is centered at the origin.



1. The figure shows a unit circle and a 60° angle in standard position. What is the length of the hypotenuse of the 30°-60°-90° triangle? How do you know?
2. Use what you know about special right triangles to find the lengths of the legs of the 30°-60°-90° triangle.
3. What are the coordinates of point P?
4. What are the exact values of cos 60° and sin 60°?
5. How are the values of cos 60° and sin 60° related to the coordinates of point P?



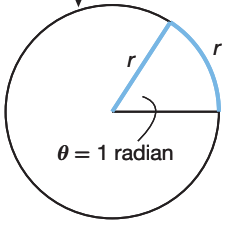
6. Describe how you can use a similar method to find the coordinates of point Q.
7. Explain how the values of cos 45° and sin 45° are related to the coordinates of point Q.

Reading Math

Remember the symbol θ is read "theta."

Angles can be measured in degrees or in radians.

The measure of an angle in **radians** is based on arc length, the distance between two points on a circle. In a circle with radius r, the measure of a central angle θ is one radian when it intercepts an arc that has a length equal to the radius.



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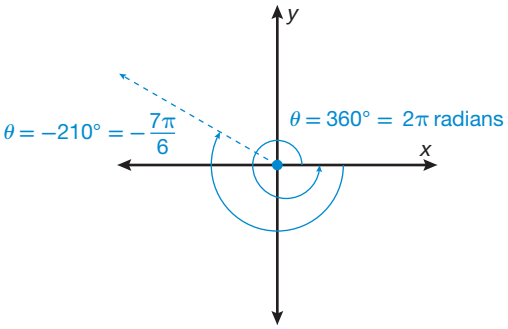
There are 360°or 2π radians in a circle. Thus the conversion factor $\frac{2\pi}{360°}$, which is equivalent to 1, can be used to convert measures between degrees and radians.



All angles are measured clockwise or counterclockwise from the positive x-axis. Measured clockwise, they are negative angles. In the figure, the $-210°$ angle is equivalent to an angle measure of 150°.

Converting Angle Measures	
Degrees to Radians	**Radians to Degrees**
Multiply the number of degrees by $\left(\frac{\pi \text{ radians}}{180°}\right)$.	Multiply the number of radians by $\left(\frac{180°}{\pi \text{ radians}}\right)$.

Math Language

Angle measures in radians often appear without the label radians. For example, $-\frac{\pi}{6}$ radian is often written simply as $-\frac{\pi}{6}$.

Example 1 Converting Between Degrees and Radians

Convert each measure from degrees to radians or from radians to degrees.

a. $-30°$

SOLUTION

$$-\cancel{30°}\left(\frac{\pi \text{ radians}}{\cancel{180°}}\right) = -\frac{\pi}{6} \text{ radians} \qquad \text{Multiply by } \left(\frac{\pi \text{ radians}}{180°}\right).$$

b. $\frac{3\pi}{4}$ radians

SOLUTION

$$\left(\frac{3\cancel{\pi}}{4}\cancel{\text{radians}}\right)\left(\frac{180°}{\cancel{\pi}\cancel{\text{radians}}}\right) = 135° \qquad \text{Multiply by } \left(\frac{180°}{\pi \text{ radians}}\right).$$

For every point $P(x, y)$ on the unit circle, the value of r is 1. Therefore, if θ is an angle in standard position whose terminal side passes through point (x, y) on the unit circle, then:

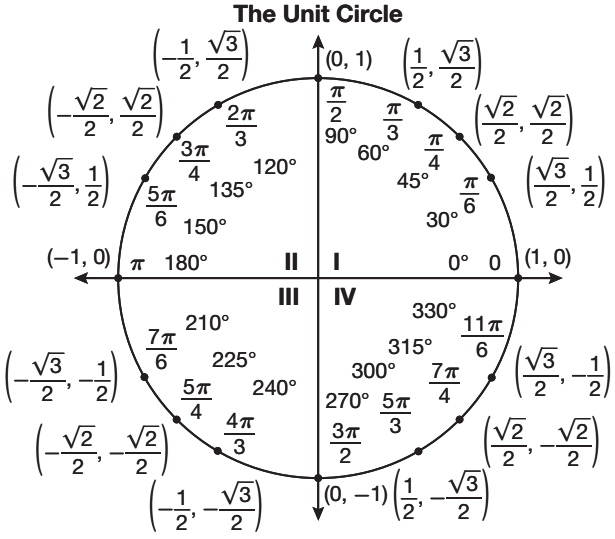
$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$$

$$\tan\theta = \frac{y}{x}$$

So, the coordinates of P can be written as $(\cos\theta, \sin\theta)$.

The Unit Circle



The diagram shows the equivalent degree and radian measures of special angles, and the corresponding x- and y-coordinates of points on the unit circle.

Example 2 Using the Unit Circle to Evaluate Trigonometric Functions

Use the unit circle to find each trigonometric function value. Find exact values.

a. $\sin 120°$

SOLUTION

The terminal side of 120° passes through the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle.

$\sin \theta = y$

$\sin 120° = \frac{\sqrt{3}}{2}$

b. $\tan \frac{11\pi}{6}$

SOLUTION

The terminal side of $\frac{11\pi}{6}$ passes through the point $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

$\tan \theta = \frac{y}{x}$

$$\tan \frac{11\pi}{6} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= -\frac{1}{2} \cdot \frac{2}{\sqrt{3}}$$ To divide by a fraction, multiply by its reciprocal.

$$= -\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$ Rationalize the denominator.

$$= -\frac{\sqrt{3}}{3}$$

Hint

The exact value of $\sin 120°$ is irrational. An approximate value is 0.866.

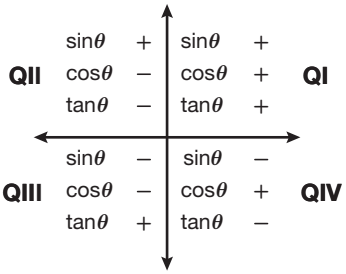
$\sin 120° = \frac{\sqrt{3}}{2} \approx 0.866$

exact approximate

You can use reference angles and the portion of the unit circle in Quadrant I to determine trigonometric function values.

Evaluate Trigonometric Functions using Reference Angles
To find the sine, cosine, or tangent of θ:
Step 1: Determine the measure of the reference angle of θ.
Step 2: Use the portion of the unit circle in Quadrant I to find the sine, cosine, or tangent of the reference angle.
Step 3: Use the quadrant of the terminal side of θ in standard position to determine the sign of the sine, cosine, or tangent.

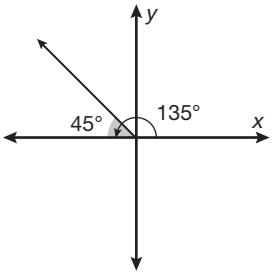
The diagram shows how the signs of the trigonometric functions are determined by the quadrant that contains the terminal side of θ in standard position.



Example 3 Using Reference Angles to Evaluate Trigonometric Functions

Find the sine, cosine, and tangent of 135°. Find exact values.

SOLUTION



Step 1: Find the measure of the reference angle.

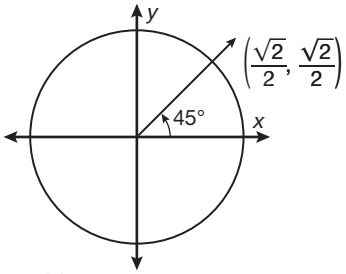
The measure of the reference angle is 45°.

Step 2: Find the sine, cosine, and tangent of the reference angle.

$\sin 45° = \frac{\sqrt{2}}{2}$ Use $\sin \theta = y$.

$\cos 45° = \frac{\sqrt{2}}{2}$ Use $\cos \theta = x$.

$\tan 45° = 1$ Use $\tan \theta = \frac{y}{x}$.



Step 3: Determine the sign.

$\sin 135° = \frac{\sqrt{2}}{2}$ In Quadrant II, $\sin \theta$ is positive.

$\cos 135° = -\frac{\sqrt{2}}{2}$ In Quadrant II, $\cos \theta$ is negative.

$\tan 135° = -1$ In Quadrant II, $\tan \theta$ is negative.

Hint

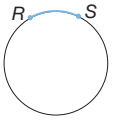
Recall that a reference angle is an acute angle. So for $\theta < 360°$, the measure of θ is:

QII $-$ $180° - \theta$
QIII $-$ $\theta - 180°$
QIV $-$ $360° - \theta$

Math Language

An **arc** is an unbroken part of a circle consisting of two points on the circle, called endpoints, and all the points on the circle between them.

This arc is named $\overset{\frown}{RS}$.

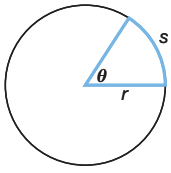


The arc length of a circle intercepted by the central angle is related to the central angle.

Arc Length Formula

For a circle of radius r, the **arc length** s intercepted by a central angle θ (measured in radians) is given by the following formula.

$$s = r\theta$$

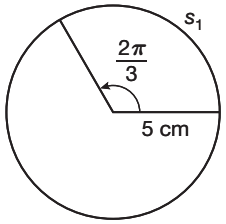


The arc length s of a circle is a portion of the circumference of a circle. This portion is the ratio between the measure of the central angle and the measure of the entire circle multiplied by the circumference of the circle. Thus, $s = \frac{\theta}{2\pi} \cdot 2\pi r$ or $s = \theta r$.

Example 4 Finding Arc Lengths

a. Find the length of arc s_1. Approximate to the nearest tenth.

SOLUTION



$s = r\theta$ Write the formula.

$s_1 = 5 \cdot \frac{2\pi}{3}$ Substitute s_1 for s, 5 for r, and $\frac{2\pi}{3}$ for θ.

$= \frac{10\pi}{3}$ Simplify.

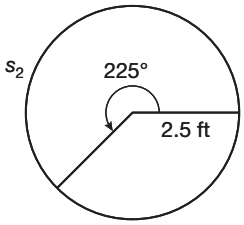
≈ 10.5 cm Use a calculator to approximate.

b. Find the length of arc s_2. Approximate to the nearest tenth.

SOLUTION

Step 1: Convert 225° to radians.

$$\cancel{225^\circ}\left(\frac{\pi \text{ radians}}{\cancel{180^\circ}}\right) = \frac{5\pi}{4} \text{ radians}$$



Step 2: Use the formula for arc length.

$s = r\theta$ Write the formula.

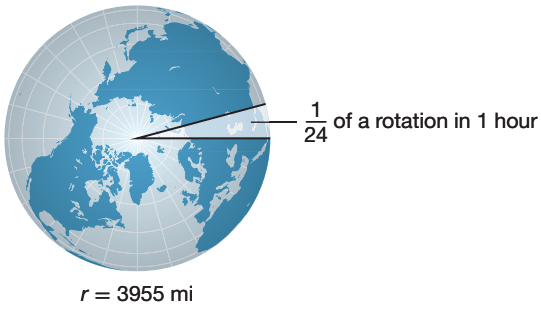
$s_2 = 2.5\left(\frac{5\pi}{4}\right)$ Substitue s_2 for s, 2.5 for r, and $\frac{5\pi}{4}$ for θ.

$= 3.125\pi$ Simplify.

≈ 9.8 ft Use a calculator to approximate.

Example 5 Application: Planet Rotation

Earth makes one complete rotation in 24 hours, and its radius is approximately 3955 miles. If an object is fixed on Earth's equator, how far does it travel in 1 hour due to Earth's rotation?



SOLUTION

The central angle that corresponds to a complete rotation is 2π radians. In 1 hour, Earth makes $\frac{1}{24}$ of a complete rotation. So, $\theta = \frac{1}{24} \cdot 2\pi = \frac{\pi}{12}$.

$s = r\theta$

$= 3955 \cdot \frac{\pi}{12}$

≈ 1035

The object travels about 1035 miles in 1 hour.

Check Find the approximate circumference of Earth: $2\pi r \approx 2\pi \cdot 3955 \approx 24{,}850$ miles.

$\frac{1}{24}$ of 24,850 is slightly greater than 1000, so 1035 miles is reasonable.

Lesson Practice

a. (Ex 1) Convert 150° to radians.

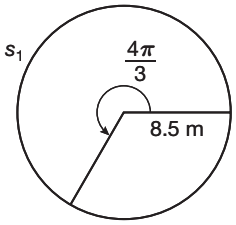
b. (Ex 1) Convert $-\frac{4\pi}{3}$ radians to degrees.

c. (Ex 2) Use the unit circle to find the exact value of cos 315°.

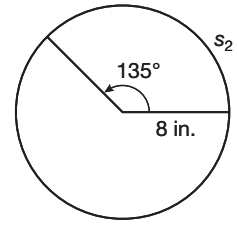
d. (Ex 2) Use the unit circle to find the exact value of tan $\frac{3\pi}{4}$.

e. (Ex 3) Use a reference angle to find the sine, cosine, and tangent of 210°. Find exact values.

f. (Ex 4) Find the length of arc s_1. Approximate to the nearest tenth.



g. (Ex 4) Find the length of arc s_2. Approximate to the nearest tenth.



h. (Ex 5) The minute hand of a certain clock is 15 centimeters long. How far does the tip of the minute hand travel in 25 minutes, to the nearest centimeter?

Practice Distributed and Integrated

Simplify.

1. (48) $\dfrac{\frac{1}{x}+\frac{4}{y}}{3+\frac{1}{xy}}$

2. (48) $\dfrac{\frac{4}{x}-3}{\frac{7}{x}+2}$

3. (48) $\dfrac{x+\frac{1}{x^2}}{x^2-\frac{2}{x^2}}$

Determine if the following probability experiments are binomial experiments. If not, explain why.

4. (Inv 5) Rolling a number cube 18 times and recording the results

5. (Inv 5) Flipping 2 coins 20 times to test if they match half of the time

Simplify.

6. (40) $5\sqrt{45}-2\sqrt{75}+2\sqrt{108}$

7. (40) $3\sqrt{12}\left(4\sqrt{3}-3\sqrt{3}\right)$

Expand.

8. (19) $(x+5)^3$

9. (19) $(x+4)^3$

Solve.

10. (35) $36x^2 - 36 = 0$

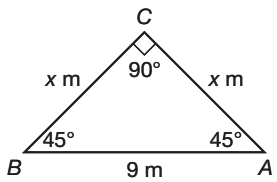
11. (35) $24x = -11x^2 - x^3$

12. (50) **Error Analysis** Julian is finding the inverse of the function $y = 2x^3$. Which step is incorrect?

Step One: $y = 2x^3 \longrightarrow$ Step Two: $x = 2y^3 \longrightarrow$ Step Three: $\frac{x}{2} = y^3 \longrightarrow$ Step Four: $\pm\sqrt{\frac{x}{2}} = y$

***13.** (62) Write the expression in terms of i and then add: $\left(15 - \sqrt{-4}\right) + \left(10 + \sqrt{-1}\right)$.

14. (44) **Geometry** In the triangle to the right, find x.



***15.** (63) **Planet Rotation** Jupiter is the largest planet. Its radius is approximately 44,365 miles. Jupiter rotates faster than Earth. It takes approximately 9.8 hours for Jupiter to make one complete rotation. If an object is fixed on Jupiter's equator, how far does it travel in 1 hour due to Jupiter's rotation?

16. (57) **Multiple Choice** Which of the following functions could be used to model exponential growth?

A $y = \frac{1}{e^x}$ **B** $y = 2e^{-x}$ **C** $y = \frac{2}{e^{2x}}$ **D** $y = \frac{2.4}{e^{-2x}}$

17. (46) **Surveying** A surveyor stands on the roof of a building that is 180 feet above the ground and spots the top of a taller building at an angle of elevation of 34°. The two buildings are 480 feet apart. Find the height of the taller building. Round to the nearest foot.

***18.** (54) **Graphing Calculator** Describe the feasible region for the set of inequalities $x \geq 0$, $y \geq 0$, $y \leq -1.5x + 9.5$.

***19.** (60) **Formulate** A fair coin is flipped. Write the equation that can be used to solve for the probability that three flips result in heads, then four flips, and seven flips. Write a formula using exponents that can be used to find the probability that the same independent event will occur n times.

20. (48) **Multiple Choice** Which of the following expressions is equivalent to $\frac{\frac{1}{x} + \frac{1}{y}}{1 - \frac{1}{x}}$?

A $\frac{1 - x}{x + y}$ **B** $\frac{x^2 - xy - x - y}{x^2y}$ **C** $\frac{x + y}{xy - y}$ **D** $\frac{2(1 - x)}{x + y}$

***21.** (62) **Write** How is adding and subtracting complex numbers similar to adding and subtracting polynomials?

***22.** (45) **Model** Sketch a graph of five points whose correlation coefficient is about 0.1.

23. (56) **Multi-Step** The formula $\theta = \frac{180°(n-2)}{n}$ gives the measure of each interior angle of an n-sided regular polygon.

a. Use the formula to find the measure of an interior angle of a regular decagon.

b. Find the reference angle for this angle.

24. (63) **Estimate** Use what you know about the unit circle and special angles to estimate $\cos(0.3\pi)$. Do not use a calculator. Explain your method.

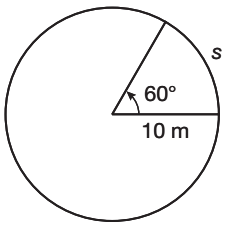
25. (58) **Jupiter** Gravity on the planet Jupiter is almost three times stronger than on Earth. If an object is thrown into the air, with an initial velocity of 100 feet per second, from the surface of Jupiter, the height, h, of the object in feet after t seconds can be approximated by the equation $h = -42t^2 + 100t$. Find the time at which the object will first reach 50 feet. Round your answer to the tenths place.

26. (53) **Analyze** Let $f(x) = ax + b$ and $g(x) = cx + d$. Under which conditions will the composite functions $f(g(x))$ and $g(f(x))$ be equal?

27. (61) **Construction** The volume of a typical American-made brick in cubic millimeters can be represented by the polynomial $x^3 - 20x^2 - 4500x + 126{,}000$ where the length is given by $x + 70$. Fully factor the expression for the volume of a brick.

28. (59) **Coordinate Geometry** Sketch a graph of the equation $y = \sqrt[3]{x}$, where y is the side length of a cube and x is the volume of the cube. Use the graph to estimate the volume of a cube with side lengths of 2.4 in. Use the graph to estimate the side length of a cube with a volume of 6.9 in^3.

***29.** (63) **Error Analysis** A student incorrectly found s, the length of the indicated arc, as shown below.



$s = r\theta$

$= 10 \cdot 60$

$= 600$ m

What is the error? Find the correct arc length.

***30.** (55) **Pass codes** A computer generates temporary pass codes for users. Each pass code is exactly 3 letters long with no repeating letters. What is the probability of getting a pass code with 3 vowels (A, E, I, O, or U)?

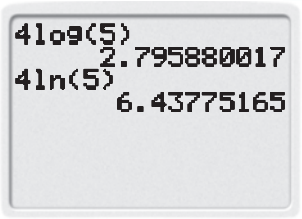
LAB
10

Using the Log Keys

Graphing Calculator Lab ***(Use with Lesson # 64, 72, 81, 87, 93, 102, and 110)***

Evaluating Logarithmic Expressions

1. Calculate the common logarithmic expression 4 log (5).
   a. To enter the value **4 log (5)**, press 4 LOG 5).
   b. To calculate this value, press ENTER.

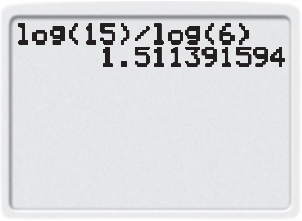


2. Calculate the natural logarithmic expression 4 ln (5).
   a. To enter the value **4 ln (5)**, press 4 LN 5).
   b. To calculate this value, press ENTER.

Hint

Use the Change of Base Law when calculating a log that does not have a base of 10 or *e*.

3. Use the Change of Base Law to calculate the logarithmic expression $\log_6 (15)$.
   a. To enter the value **$\log_6$ (15)**, press LOG 1 5) ÷ LOG 6).
   b. Press ENTER to calculate this value.

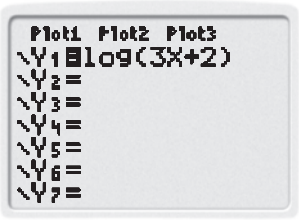


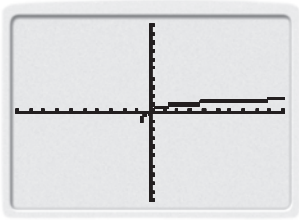
Graphing Logarithmic Functions

Graphing Calculator Tip

When graphing logarithmic expressions, use the standard zoom window. Adjust the window to view specific parts of the graph by zooming in or out.

1. Graph the common logarithmic function log $(3x + 2)$.
   a. Press Y= to access the screen to enter the logarithmic function.
   b. To enter the function **log $(3x + 2)$**, press LOG 3 X,T,θ,*n* + 2).
   c. Press ZOOM, and then press ▼ to select the **ZStandard** window.
   d. Then press ENTER to view the graph.





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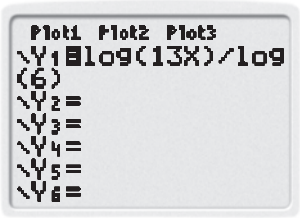
2. Graph the logarithmic function $\log_6 (13x)$ whose base is not natural or common.

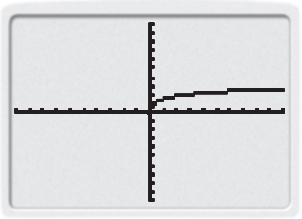
   a. Press [Y=] to access the screen to enter the logarithmic function.

   b. To enter the function $\log_6 (13x)$, press [LOG] [1] [3] [X,T,θ,n] [)] [÷] [LOG] [6] [)].

   c. Press [ZOOM], and then press [▼] to select the **ZStandard** window.

   d. Then press [ENTER] to view the graph.





Lab Practice

1. Calculate 6 log (8).

2. Calculate 3 ln (1).

3. Calculate $\log_7 (9)$.

4. Graph $y = \log (15x + 8)$.

5. Graph $y = \log_5 (x + 6)$.

LESSON
64

Using Logarithms

Warm Up

1. (3) **Vocabulary** In the expression 4^5, the base is 4 and the _____ is 5.
2. (3) What is the value of 2^3?
3. (47) What value of x makes the equation $10^x = 100$ true?

New Concepts

A **logarithm** is the exponent that is applied to a specified base to obtain a given value. Every exponential equation has a logarithmic form and vice versa.

Exponential Equation	Logarithmic Equation
$b^x = a$	$\log_b a = x$

$b > 0, b \neq 1$

The logarithmic equation $\log_b a = x$ is read "the log base b of a equals x." Notice that x is both the exponent and the logarithm in the equations above.

Example 1 Converting from Exponential to Logarithmic Form

Write each exponential equation in logarithmic form.

a. $2^5 = 32$

SOLUTION

$2^5 = 32 \Leftrightarrow \log_2 32 = 5$ The base is the same in both the exponential equation and the logarithmic equation.

The logarithmic form is $\log_2 32 = 5$.

b. $5^1 = 5$

SOLUTION

$5^1 = 5 \Leftrightarrow \log_5 5 = 1$ The exponent is the logarithm.

The logarithmic form is $\log_5 5 = 1$.

c. $8^0 = 1$

SOLUTION

$8^0 = 1 \Leftrightarrow \log_8 1 = 0$ Any nonzero base to the zero power is 1.

The logarithmic form is $\log_8 1 = 0$.

Reading Math

Read the symbol $\Leftrightarrow$ as "if and only if." When used between two equations, it means the equations are equivalent.

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Reading Math

The equations in Example 1d can be written $5^{-2} = \frac{1}{25}$ and $\log_5 \frac{1}{25} = -2$.

d. $5^{-2} = 0.04$

SOLUTION

$5^{-2} = 0.04 \iff \log_5 0.04 = -2$ An exponent (or log) can be negative.

The logarithmic form is $\log_5 0.04 = -2$.

Example 2 Converting from Logarithmic to Exponential Form

Write each logarithmic equation in exponential form.

a. $\log_{10} 1000 = 3$

SOLUTION

$\log_{10} 1000 = 3 \iff 10^3 = 1000$ The base is the same in both the logarithmic equation and the exponential equation.

The exponential form is $10^3 = 1000$.

b. $\log_7 7 = 1$

SOLUTION

$\log_7 7 = 1 \iff 7^1 = 7$ The logarithm is the exponent.

The exponential form is $7^1 = 7$.

c. $\log_3 81 = x$

SOLUTION

$\log_3 81 = x \iff 3^x = 81$ The logarithm (and the exponent) can be a variable.

The exponential form is $3^x = 81$.

A logarithm with base 10 is called a **common logarithm.** If no base is written for a logarithm, the base is assumed to be 10. For example, $\log 5 = \log_{10} 5$.

Example 3 Application: Acidity of Rainwater

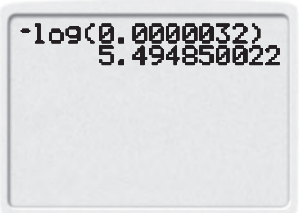
Because of the phenomenon of acid rain, the acidity of rainwater is important to environmental scientists. The acidity of a liquid is measured in pH, given by the function $\text{pH} = -\log[\text{H}^+]$, where $[\text{H}^+]$ represents the concentration of hydrogen ions in moles per liter. In 1999, the hydrogen ion concentration of rainwater in western Nevada was found to be approximately 0.0000032 moles per liter. What was the pH of the rainwater?

SOLUTION

$\text{pH} = -\log[\text{H}^+]$

$\text{pH} = -\log(0.0000032)$ Substitute the known value in the function.

Use a calculator to find the value of the logarithm in base 10. Press the LOG key.



The rainwater had a pH of about 5.5.

Hint

The display 3.16227766E − 6 means $3.16227766 \times 10^{-6}$, which is approximately 3.2×10^{-6}, or 0.0000032.

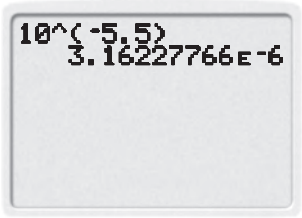
Check

$-\log (0.0000032) \approx 5.5$ Write the result found in the Solution.

$\log (0.0000032) \approx -5.5$ Multiply both sides by –1.

$10^{-5.5} \approx 0.0000032$ Write the logarithmic equation in exponential form.

Use a calculator to verify that the value of $10^{-5.5}$ is approximately 0.0000032.



Lesson Practice

Write each exponential equation in logarithmic form.

(Ex 1)

a. $3^2 = 9$ **b.** $4^1 = 4$

c. $9^0 = 1$ **d.** $8^{-1} = 0.125$

Write each logarithmic equation in exponential form.

(Ex 2)

e. $\log_{10} 100 = 2$ **f.** $\log_8 8 = 1$ **g.** $\log_5 125 = x$

(Ex 3)

h. The acidity of a liquid is measured in pH, given by the function $\text{pH} = -\log[\text{H}^+]$, where $[\text{H}^+]$ represents the concentration of hydrogen ions in moles per liter. In 1999, the hydrogen ion concentration of rainwater in northern Maine was found to be approximately 0.0000200 moles per liter. What was the pH of the rainwater, to the nearest tenth?

Practice Distributed and Integrated

Use synthetic substitution.

1. (51) Find $f(6)$ for $f(x) = x^4 - 4x^3 + 2x^2 + 4x - 18$.

2. (51) Find $f(-7)$ for $f(x) = x^4 + 5x^3 - 12x^2 - 4x + 8$.

3. (26) Find the equation of the line that passes through $(-2, 5)$ and $(-6, -3)$.

4. (41) The sides of a triangle measure 7 in., 6 in., and 8 in. Is the triangle a right triangle?

Find the zeros of each quadratic function.

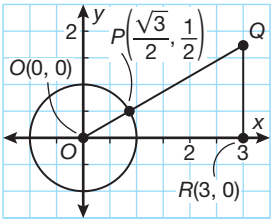
5. (35) $f(x) = 2x^2 + x - 15$ **6.** (35) $f(x) = -32x^2 - 28x$ **7.** (35) $f(x) = x^2 - 9$

Determine if (0, −3) is a solution of the inequalities.

8. (39) $5x + 2y < -4$ **9.** (39) $y - 2x > 6$ **10.** (39) $16y - 2 \geq x$

11. (52) **Formulate** A parasail is attached to a boat with a rope 300 feet long. The angle of elevation from the boat to the parasail is 48 degrees. Write a formula that would help you to estimate the parasail's height above the boat.

*12. (63) **Coordinate Geometry** Find the area of triangle OQR. Give the answer in exact form and to the nearest tenth of a square unit.



13. (50) Find an equation for the inverse of $y = 5x - 8$. Identify the domain and range of the inverse function.

*14. (64) **Write** Explain why $\log_0 9$ and $\log_1 9$ do not exist.

15. (53) **Population** Currently the U.S. population is growing at a rate of approximately 0.6 percent per year. According to the 2000 census, the population was about 281 million. Use composite functions to predict the U.S. population in the year 2015, and justify your prediction.

*16. (59) **Multi-Step** Use $\left(\frac{9}{7}\right)^{\frac{1}{4}}\left(\frac{9}{7}\right)^{\frac{1}{4}}$.

a. Which property of rational exponents should be applied first to simplify the expression? Explain.

b. Apply the property from part a.

c. Which property of rational exponents should be applied next?

d. Apply the property from part c.

e. If possible, finish simplifying the expression from part d.

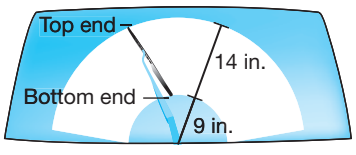
*17. (61) **Transportation** The approximate volume of a moving truck, in cubic feet, that can move four to five household rooms can be given by $x^3 - 15x^2 + 56x + 6x^2 - 90x + 336$. Factor this expression by grouping. Then find the dimensions when $x = 15$.

18. (57) **Geometry** The figures below show how to create Sierpinski's Triangle. Start with a solid black equilateral triangle (Iteration 0); at every iteration, join the midpoints of the sides of each black triangle to get a smaller triangle, and color the interior of that triangle white.

Formulate an exponential function for the number of black triangles $N(x)$ in the x^{th} iteration.

19. (31) Find the slope of the line that passes through the points $A\left(\frac{3}{x}, \frac{1}{3}\right)$ and $B\left(\frac{1}{5}, \frac{5}{x}\right)$, where x is any non-zero real number.

*20. (63) **Windshield Wipers** A rear windshield wiper moves through an angle of 135° on each swipe. To the nearest inch, how much greater is the length of the arc traced by the top end of the wiper blade than the length of the arc traced by the bottom end of the wiper blade?



21. (44) **Multiple Choice** Which of the following expressions is equivalent to $\frac{3}{5-\sqrt{2}}$?

A $\frac{3-15\sqrt{2}}{21}$ **B** $\frac{15+3\sqrt{2}}{23}$ **C** $\frac{15-2\sqrt{3}}{21}$ **D** $\frac{21-3\sqrt{2}}{15}$

22. (Inv 6) **Multi-Step** What number needs to be added to both sides of the equation to make $x^2 + 12x + 42 = 0$ a perfect square? Solve the equation.

***23.** (64) **Rainwater** The acidity of a liquid is measured in pH, given by the function $pH = -\log[H^+]$, where $[H^+]$ represents the concentration of hydrogen ions in moles per liter. In 1999, the hydrogen ion concentration of rainwater in the Chesapeake Bay region of Maryland was found to be approximately 0.0000316 moles per liter. What was the pH of the rainwater, to the nearest tenth? Show how to check your answer by writing a logarithmic equation and its equivalent exponential equation. (Hint: You will need to use the $\approx$ symbol instead of the $=$ symbol.)

***24.** (60) **Write** Two number cubes are tossed once. Explain why the events, one cube shows a number less than 5 and the sum of the two cubes is a multiple of 2, are dependent.

25. (55) **Justify** Give an example of an event that has a probability of 0. Explain why the probability is 0.

***26.** (62) **Graphing Calculator** Set the mode on your graphing calculator to $a + bi$ by pressing the Mode key and using the arrow keys. Then find $\sqrt{-1296} \div \sqrt{-324}$.

27. (56) **Error Analysis** Rizwan tried to find the value of $\cos\theta$, where θ is an angle in standard position with the point $Q(-6, 8)$ on its terminal side. His work is shown below. What was Rizwan's error?

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{(-6)^2 + 8^2}$

$r = \sqrt{100}$

$r = 10$

$\cos\theta = \frac{x}{r}$

$\cos\theta = \frac{6}{10} = 0.6$

***28.** (64) **Multiple Choice** Which equation is equivalent to $\log_2 16 = x$?

A $2^x = 16$ **B** $x^2 = 16$ **C** $16^2 = x$ **D** $2^{16} = x$

29. (45) **Analyze** Compare and contrast correlation coefficient values of -0.45 and 0.45.

***30.** (58) **Falling Objects** The height, in feet, of an object that is falling or is projected into the air can be described by $h = -16t^2 + v_0t + h_0$, where h is the height in feet after t seconds, v_0 is the initial velocity of the object in feet per second, and h_0 is the initial height of the object in feet per second. A coin is tossed from the top of a 100 foot tall building, with an initial velocity of 92 feet per second. Write the equation that models the height of the coin. Find the time when the coin will reach the ground. Round your answer to the tenths place.

LESSON 65

Using the Quadratic Formula

Warm Up

1. (62) **Vocabulary** The square root of a negative number is an _____ number.

2. (58) Solve by completing the square $x^2 - 26x = 13$.

3. (62) Simplify $3\sqrt{-25}$.

4. (2) Evaluate $b^2 - 4ac$ when $a = 2$, $b = 7$, and $c = 5$.

New Concepts

In general, a quadratic equation in standard form is $ax^2 + bx + c = 0$, with $a \neq 0$. The general equation can be solved for x by completing the square.

$$ax^2 + bx = -c$$ Subtract c from each side.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$ Divide both sides by a.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$ Complete the square.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$ Factor the left side.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$ Make common denominators on the right.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$ Combine the fractions on the right.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$ Apply the Square Root Property.

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$ Isolate x.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ Simplify the radical.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Combine the fractions.

The last line is the quadratic formula.

Math Reasoning

Analyze

Explain how $\frac{-c}{a} + \frac{b^2}{4a^2}$ becomes $-\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$.

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Online Connection
www.SaxonMathResources.com

The quadratic formula can be used to find the solutions of any quadratic equation.

Math Language

The **solutions** of a quadratic equation are the **zeros** of the related function.

Example 1 Solving Quadratic Equations with Real Zeros

Solve each equation.

a. $5x^2 + 34x - 7 = 0$

SOLUTION Use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-34 \pm \sqrt{34^2 - 4(5)(-7)}}{2(5)}$$ Substitute 5 for a, 34 for b, and -7 for c.

$$= \frac{-34 \pm \sqrt{1296}}{10}$$ Simplify the radicand and denominator.

$$= \frac{-34 \pm 36}{10} = \frac{1}{5} \text{ and } -7$$ Evaluate the square root and simplify.

The solutions are $\frac{1}{5}$ and -7.

b. $9x^2 + 6x + 1 = 0$

SOLUTION Use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$x = \frac{-6 \pm \sqrt{6^2 - 4(9)(1)}}{2(9)}$$ Substitute 9 for a, 6 for b, and 1 for c.

$$= \frac{-6 \pm \sqrt{0}}{18}$$ Simplify.

$$= \frac{-6 \pm 0}{18} = -\frac{6}{18} = -\frac{1}{3}$$

The solution is $-\frac{1}{3}$.

Hint

The standard form of a complex number is $a + bi$.

Example 2 Solving Quadratic Equations with Complex Zeros

a. Solve the equation $3x^2 + 5x + 4 = 0$. Write the solutions as complex numbers in standard form.

SOLUTION Use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{5^2 - 4(3)(4)}}{2(3)}$$ Substitute 3 for a, 5 for b, and 4 for c.

$$= \frac{-5 \pm \sqrt{-23}}{6}$$ Simplify the radicand and denominator.

$$= \frac{-5 \pm i\sqrt{23}}{6}$$ Write the negative root as an imaginary number.

$$= -\frac{5}{6} \pm \frac{\sqrt{23}}{6}i$$ Write the solutions in standard form.

The solutions are $-\frac{5}{6} + \frac{\sqrt{23}}{6}i$ and $-\frac{5}{6} - \frac{\sqrt{23}}{6}i$.

b. Solve the equation $x^2 - \frac{1}{2}x + \frac{1}{8} = 0$. Write the solutions as complex numbers in standard form.

SOLUTION To make the calculations simpler, clear the fractions before using the quadratic formula.

$8x^2 - 4x + 1 = 0$ — Multiply both sides of the equation by 8.

Use the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(8)(1)}}{2(8)}$ — Substitute 8 for a, -4 for b, and 1 for c.

$= \frac{4 \pm \sqrt{-16}}{16}$ — Simplify the radicand and denominator.

$= \frac{4 \pm 4i}{16}$ — Write the negative root as an imaginary number.

$= \frac{1 \pm i}{4}$ — Factor 4 out of the numerator and denominator.

$= \frac{1}{4} \pm \frac{1}{4}i$ — Write the solutions in standard form.

The solutions are $\frac{1}{4} + \frac{1}{4}i$ and $\frac{1}{4} - \frac{1}{4}i$.

Because the solutions of a quadratic equation are the zeros, or x-intercepts of the equation, a graphing calculator can be used to find the solutions of a quadratic equation when the solutions are real.

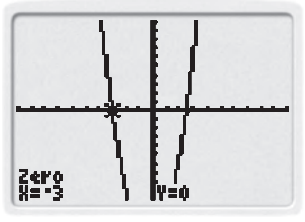
Example 3 Solving Quadratic Equations using a Graphing Calculator

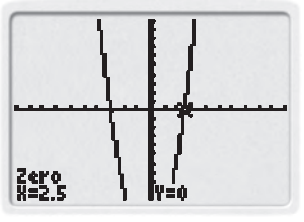
Solve $2x^2 + x - 15 = 0$ by using a graphing calculator.

SOLUTION Graph the related function, $y = 2x^2 + x - 15$.

If needed, adjust the window so that the x-intercepts of the graph are displayed. Access the **Calc** function (2nd + Trace) and choose **zero.**

When asked for left and right bounds, choose points that are on either side of one of the x-intercepts. Repeat for the second intercept.





The solutions are -3 and 2.5.

Math Reasoning

Verify Graph $y = 3x^2 + 5x + 4$. How does the graph confirm that there are no real solutions to $3x^2 + 5x + 4 = 0$?

Example 4 Application: Travel

Train A leaves the station at 1 p.m. and travels due north at 55 miles per hour. Train B leaves the same station at 4 p.m. and travels due east at 35 miles per hour. About how long will it take for the trains to be 500 miles apart?

SOLUTION

Write each distance as *rate* × *time*.
Train B is 3 hours behind Train A.
Train A: $d = rt = 55t$
Train B: $d = rt = 35(t - 3)$

$$a^2 + b^2 = c^2$$

$$(55t)^2 + (35(t-3))^2 = 500^2 \quad \text{Use the Pythagorean Theorem.}$$

$$3025t^2 + (35t - 105)^2 = 250{,}000 \quad \text{Simplify.}$$

$$3025t^2 + 1225t^2 - 7350t + 11{,}025 = 250{,}000 \quad \text{Square the binomial.}$$

$$4250t^2 - 7350t - 238{,}975 = 0 \quad \text{Combine like terms.}$$

Now use the quadratic formula.

$$t = \frac{-(-7350) \pm \sqrt{(-7350)^2 - 4(4250)(-238{,}975)}}{2(4250)}$$

$$= \frac{7350 \pm \sqrt{4{,}116{,}597{,}500}}{8500}$$

$t \approx 8.41$ and $t \approx -6.68$

Disregard the negative time. It will take about 8.41 hours for the trains to be 500 miles apart.

Check Find the distance each train traveled after 8.41 hours.

Train A: $55(8.41) = 462.55$ miles, Train B: $35(8.41 - 3) = 189.35$ miles

Use the Pythagorean Theorem: $\sqrt{462.55^2 + 189.35^2} = 499.8 \approx 500$.

Lesson Practice

Solve each equation.

(Ex 1)

a. $3x^2 - 7x + 2 = 0$

b. $9x^2 + 12x + 4 = 0$

Solve each equation. Write the solutions as complex numbers in standard form.

(Ex 2)

c. $x^2 + 5x = -9$

(Ex 2)

d. $-x^2 + \frac{1}{3}x - \frac{1}{9} = 0$

(Ex 3)

e. Solve $8x^2 + 7x - 100 = 0$ by using a graphing calculator.

(Ex 4)

f. A hiker leaves a campground at 8 a.m. and heads due south at a rate of 4 miles per hour. Another hiker leaves the same campground at 9 a.m. and heads due east at a rate of 2 miles per hour. Approximately how long will it take for the hikers to be 15 miles apart?

Practice Distributed and Integrated

1. (57) Secret ingredient X has a half-life of 23 hours. What portion of 360 kg will remain after 7 hours?

Use long division.

2. (38) Divide $(3x^3 + 12x^2 - 15x + 15)$ by $(3x - 9)$

3. (38) Divide $(2x^3 + 14x^2 - 4x - 48)$ by $(2x + 4)$

Solve.

***4.** (58) $x^2 = 7 + 3x$

***5.** (58) $x^2 = -x + 1$

Simplify.

6. (40) $2\sqrt{3}(5\sqrt{3} - 2\sqrt{6})$

7. (40) $4\sqrt{63} - 3\sqrt{28}$

8. (40) $3\sqrt{2} \cdot 2\sqrt{6} \cdot 3\sqrt{6}$

9. (55) Jordon spins a spinner with six equal-sized sections numbered 1–6. In one spin, what is the likelihood that the spinner will stop on a 1 or a 5?

***10.** (65) **Error Analysis** Find and correct the error a student made in solving $4x^2 - 37x + 9 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-37 \pm \sqrt{(-37)^2 - 4(4)(9)}}{8}$$

$$= \frac{-37 \pm \sqrt{1369 - 144}}{8}$$

$$= \frac{-37 \pm \sqrt{1225}}{8} = \frac{-37 \pm 35}{8} = -\frac{1}{4} \text{ and } -9$$

11. (49) List the coefficients in the 6^{th} row of Pascal's Triangle.

12. (55) **Probability** A history test has 20 questions, 7 of which are about the Civil War. Students are asked to answer 5 of the 20 questions. What is the probability that a student randomly selecting questions will select 5 Civil War questions?

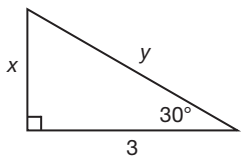
***13.** (63) **Write** Describe how to determine $\sin \frac{3\pi}{2}$ by using the unit circle.

14. (36) Find an equation of a line perpendicular to $y = \frac{1}{3}x - 4$.

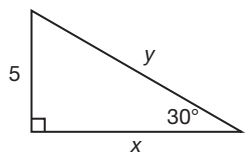
15. (46) **Analyze** In what type of right triangles will the sine and cosine values of the acute angles be equal?

Find x and y.

16. (52)



17. (52)



*18. (64) **Multi-Step** Complete parts a–d to graph a logarithmic function and use the graph to estimate a logarithm.

a. Write the equation $y = \log_2 \frac{1}{4}$ in exponential form. What value of y makes both equations true?

b. Writing exponential equations if needed, complete the table of values for the function $y = \log_2 x$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \log_2 x$						

c. Plot the points and sketch the graph of the function.

d. Use your graph to estimate the value of $\log_2 6$. Check your estimate by evaluating the appropriate power of 2 on a calculator.

19. (58) **Multiple Choice** Choose the letter that best represents the value of x in the equation $x^2 + 2x + 7 = 8$.

A $x = -1 \pm \sqrt{2}$ **B** $x = -2, 0$ **C** $x = 2, 0$ **D** $x = 1 \pm \sqrt{2}$

20. (39) **Measurement** Given that 1 strip of red ribbon = 3 inches and 1 strip of blue ribbon = 4 inches, write and graph the inequality that represents that the total number of inches must be no less than 60.

21. (Inv 1) **Multi-Step** A car is sitting outside in a driveway. Consider the following statement, "If it starts raining, the car will get wet."

a. Is the converse of the statement true? Explain.

b. Is the contrapositive true? Explain.

22. (44) **Geometry** The base of a triangle is $5\sqrt{2}$ feet, and its height is $\frac{1}{\sqrt{3}}$ feet. Find the area of the triangle.

*23. (62) **Physics** A fireworks technician wants a firework to explode 70 feet below the top of the Gateway Arch in St. Louis. The Gateway Arch has a height of 630 feet. The technician has figured the height of the firework to be modeled by $-16t^2 + 160t$, where t is the time in seconds after it is launched. The solutions of $-16t^2 + 160t = 560$ give the number of seconds it will take for the firework to reach the desired height. Solve the equation. Will the firework reach the desired height? If so, when?

24. (53) **Justify** Let $f(x) = 3x - 7$ and $g(x) = \frac{1}{3}x + \frac{7}{3}$.

a. Justify the claim that $f(g(x)) = x$.

b. Does $g(f(x)) = x$ also? Explain.

25. (61) **Error Analysis** Find and correct the error a student made in factoring the expression below.

$$9x^4 - 63x^3 - 8x + 56$$
$$9x^3(x - 7) - 8(x - 7)$$
$$(x - 7)(9x^3 - 8)$$
$$(x - 7)(3x - 2)(6x^2 + 6x + 4)$$

*26. (65) **Physical Science** The North Falls waterfalls in Silver Falls State Park in Oregon are 136 feet high. The equation $-16t^2 + 136 = 0$ gives the number of seconds it takes for a drop of water at the top of the falls to reach the pool of water below. Solve the equation by using the quadratic formula. Round to the nearest hundredth of a second.

27. (56) **Formulate** Let θ be an angle in standard position and let n be any integer. You can use the expression $\theta + n(360°)$ to indicate all the angles coterminal with θ. Write an expression for all the angles coterminal with 15°, and find 3 different ones.

*28. (65) **Graphing Calculator** Solve $5x^2 - 2x - 16 = 0$ by using a graphing calculator.

29. (59) **Velocity of an Electron** The velocity at which an electron orbits the nucleus of an atom is known as the Fermi-Thomas velocity and can be found using the equation $\frac{Z^{\frac{2}{3}}}{137}c$, where Z is the number of protons in the nucleus and c is the speed of light. What is the Fermi-Thomas velocity for platinum, which has 78 protons, in terms of c? Round to the nearest thousandths place.

*30. (64) **Analyze** Write the following equations in exponential form. What property of exponents is illustrated?

$$\log_2 1 = 0, \log_3 1 = 0, \log_{15} 1 = 0$$

LESSON
66

Solving Polynomial Equations

Warm Up

1. (65) **Vocabulary** The formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is called the _____.
2. (23) Factor $27cd^3 - 12cd$.
3. (35) Solve $x(x - 4) = 0$.
4. (61) True or False: $x + 7$ is a factor of $x^3 + 8x^2 + x - 42$.
5. (65) Solve $x^2 + x + 1 = 0$ by using the quadratic formula.

New Concepts

Sometimes a polynomial equation can be solved just by factoring and using the Zero Product Property.

Example 1 Using Factoring to Solve Polynomial Equations

Solve each equation.

a. $2x^4 - 10x^3 - 72x^2 = 0$

SOLUTION Since one side is already equal to 0, begin by factoring the polynomial.

$2x^2(x^2 - 5x - 36) = 0$	The GCF is $2x^2$.
$2x^2(x + 4)(x - 9) = 0$	Factor the trinomial.
$2x^2 = 0$ or $x + 4 = 0$ or $x - 9 = 0$	Use the Zero Product Property.
$x = 0$ or $x = -4$ or $x = 9$	Solve each equation.

The solutions are 0, −4, and 9.

b. $4x^3 + x = 4x^2$

SOLUTION First write the equation in standard form.

$4x^3 - 4x^2 + x = 0$	Subtract $4x^2$ from each side.
$x(4x^2 - 4x + 1) = 0$	The GCF is x.
$x(2x - 1)(2x - 1) = 0$	Factor the perfect square trinomial.
$x = 0$ or $2x - 1 = 0$ or $2x - 1 = 0$	Use the Zero Product Property.
$x = 0$ or $x = \frac{1}{2}$ or $x = \frac{1}{2}$	Solve each equation.

The solutions are 0 and $\frac{1}{2}$.

Hint

Recall that the **Zero Product Property** states that if $ab = 0$, then $a = 0$ or $b = 0$.

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Notice in Example 1b that the solution, or root, of $\frac{1}{2}$ appears twice. This is called a double root. It has **multiplicity** 2.

Multiplicity of Roots

The multiplicity of a root r of a polynomial equation is the number of times $x - r$ is a factor of the polynomial.

Example 2 Identifying Multiplicity

Find the roots of each equation. Give the multiplicity of each root.

a. $x^5 + 8x^4 + 16x^3 = 0$

SOLUTION Factor the polynomial to use the Zero Product Property.

$x^3(x^2 + 8x + 16) = 0$ The GCF is x^3.

$x^3(x + 4)(x + 4) = 0$ Factor the perfect square trinomial.

$x^3 = 0$ or $x + 4 = 0$ or $x + 4 = 0$ Use the Zero Product Property.

$x = 0$ or $x = -4$ or $x = -4$ Solve each equation.

The roots are 0 and -4. Because $x^3 = x \cdot x \cdot x$, x is a factor 3 times, so the root of 0 has a multiplicity 3. Because $x + 4$ is a factor 2 times, the root of -4 has multiplicity 2.

b. $x^4 = 2x^2 - 1$

SOLUTION First write the equation in standard form: $x^4 - 2x^2 + 1 = 0$.

$(x^2 - 1)(x^2 - 1) = 0$ Factor the perfect square trinomial.

$(x + 1)(x - 1)(x + 1)(x - 1) = 0$ Factor the differences of squares.

$x + 1 = 0$ or $x - 1 = 0$ or $x + 1 = 0$ or $x - 1 = 0$

$x = -1$ or $x = 1$ or $x = -1$ or $x = 1$

The roots are -1 and 1. Because both $(x + 1)$ and $(x - 1)$ occur twice, each root has multiplicity 2.

A polynomial equation like $x^3 - 2x^2 - 11x + 12 = 0$, where the degree of the polynomial is greater than 2, can be factored by using synthetic division if one of the factors is a linear binomial and is already known. When none of the factors are already given, begin factoring the polynomial by using the Rational Root Theorem.

Math Reasoning

Analyze What can you do to use the Rational Root Theorem to find the possible rational roots of $\frac{1}{5}x^3 + 2x^2 - x + 2 = 0$?

The Rational Root Theorem

If a polynomial $P(x)$ has integer coefficients, then every rational root of $P(x) = 0$ can be written in the form $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient of $P(x)$.

The Irrational Root Theorem states that irrational roots come in pairs.

The Irrational Root Theorem

If a polynomial $P(x)$ has rational coefficients, and $a + b\sqrt{c}$ is a root of $P(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $P(x) = 0$.

Hint

Remember that $x - r$ is a factor if $P(r) = 0$.

Example 3 Identifying All Real Roots of a Polynomial Equation

a. Identify all the real roots of $x^3 - 2x^2 - 11x + 12 = 0$.

SOLUTION By the Rational Root Theorem the possible rational roots are: $\frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. Use synthetic division to test roots, r, until a factor, $x - r$, is found.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & -12 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

The first try works. By the Factor Theorem $P(1) = 0$, so $x - 1$ is a factor.

The quotient is also a factor, so the polynomial can be written as $(x - 1)(x^2 - x - 12)$, which factors into $(x - 1)(x - 4)(x + 3)$.

The equation can be written as $(x - 1)(x + 3)(x - 4) = 0$.

Using the Zero Product Property, the roots are 1, -3, and 4.

b. Identify all the real roots of $3x^4 - 5x^3 - 29x^2 + 3x + 4 = 0$.

SOLUTION By the Rational Root Theorem the possible rational roots are: $\frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 3} = \pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 4, \pm\frac{4}{3}$. Test roots. Begin with integers.

Organize the work in a synthetic substitution table.

The remainder is 0 for $P(4)$, so $x - 4$ is a factor.

$3x^3 + 7x^2 - x - 1$ is the other factor.

Possible root	3	−5	−29	3	4
1	3	−2	−31	−28	−24
2	3	1	−27	−51	−98
4	3	7	−1	−1	0

The possible rational roots for $3x^3 + 7x^2 - x - 1 = 0$ are: $\frac{\pm 1}{\pm 1, \pm 3} = \pm 1, \pm\frac{1}{3}$.

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & 7 & -1 & -1 \\ & & -1 & -2 & 1 \\ \hline & 3 & 6 & -3 & 0 \end{array}$$

$x + \frac{1}{3}$ is a factor. $3x^2 + 6x - 3$ is the other factor. The trinomial factors into $3(x^2 + 2x - 1)$. Solve $x^2 + 2x - 1 = 0$ by using the quadratic formula.

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

The polynomial equation is

$$(x - 4)\left(x + \frac{1}{3}\right)\left(x - \left(-1 + \sqrt{2}\right)\right)\left(x - \left(-1 - \sqrt{2}\right)\right) = 0.$$

The real roots are 4, $-\frac{1}{3}$, $-1 + \sqrt{2}$, and $-1 - \sqrt{2}$.

The x-intercepts of a polynomial function are real roots of the related equation. Therefore, a graph can be used to help find the roots of a polynomial equation.

Example 4 Using a Calculator to Solve Polynomial Equations

Solve $x^3 + 5x^2 - 9x - 10 = 0$ by using a graphing calculator.

SOLUTION Graph the related function, $y = x^3 + 5x^2 - 9x - 10$. Use the **Calc** command to find the x-intercepts.

Math Reasoning

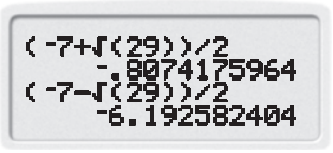
Verify Show that the Rational Root Theorem gives 2 as a possible root.

Since 2 is a rational root, use synthetic division with $r = 2$.

$$\begin{array}{r|cccc} 2 & 1 & 5 & -9 & -10 \\ & & 2 & 14 & 10 \\ \hline & 1 & 7 & 5 & 0 \end{array}$$

Solve $x^2 + 7x + 5 = 0$: $x = \dfrac{-7 \pm \sqrt{7^2 - 4(1)(5)}}{2(1)} = \dfrac{-7 \pm \sqrt{29}}{2}$.

The roots are 2, $\dfrac{-7 + \sqrt{29}}{2}$, and $\dfrac{-7 - \sqrt{29}}{2}$. Notice that the x-intercepts given by the graphing calculator above are estimates of the exact values of the irrational roots.



Example 5 Application: Packaging

A box is to be designed so that the height is 6 inches greater than the length and the width is 3 inches less than the length. What must the length be if the volume has to be 40 cubic inches?

SOLUTION

1. Understand The volume of a box, or rectangular prism, is the product of the length, width, and height: $V = lwh$.

2. Plan Write expressions for each dimension. If x represents the length, then $x + 6$ represents the height and $x - 3$ represents the width.

$$40 = x(x - 3)(x + 6)$$

Hint

Use the Rational Root Theorem to determine all the possible rational roots.

3. Solve $40 = x^3 + 3x^2 - 18x$

$$0 = x^3 + 3x^2 - 18x - 40$$

$$\begin{array}{r|cccc} 4 & 1 & 3 & -18 & -40 \\ & & 4 & 28 & 40 \\ \hline & 1 & 7 & 10 & 0 \end{array}$$

The polynomial factors into $(x - 4)(x^2 + 7x + 10)$, which factors into $(x - 4)(x + 2)(x + 5)$. The roots are 4, -2, and -5. Since the length must be positive, disregard the negative roots.

The length is 4 inches.

4. Check If the length is 4 inches, the height is 10 inches, and the width is 1 inch. Find the volume: $V = lwh = 4(1)(10) = 40$.

Lesson Practice

Solve each equation.

(Ex 1)

a. $x^4 - 3x^3 - 28x^2 = 0$

b. $18x^3 - 18x = 0$

Find the roots of each equation. Give the multiplicity of each root.

(Ex 2)

c. $2x^3 - 32x^2 + 128x = 0$

d. $x^6 + 12x^5 + 27x^4 = 0$

e. (Ex 3) Identify all the real roots of $x^3 - 5x^2 - x + 5 = 0$.

f. (Ex 3) Identify all the real roots of $x^3 + 4x^2 - 5x - 14 = 0$.

g. (Ex 4) Solve $x^3 + 2x^2 - 45x - 18 = 0$ by using a graphing calculator.

h. (Ex 5) A box is to be designed so that the height is 3 inches greater than the length and the width is 2 inches less than the length. What must the length be if the volume has to be 18 cubic inches?

Practice Distributed and Integrated

Simplify each expression.

1. (59) $\dfrac{81^{\frac{2}{4}}}{81^{\frac{3}{4}}}$

2. (37) $\dfrac{p - 4px}{p}$

3. (37) $\dfrac{4a}{a+4} + \dfrac{a+2}{2a}$

4. (37) $\dfrac{x}{x+2} + \dfrac{3+x}{x^2+4x+4}$

5. (59) $(4^6)^{\frac{1}{3}}$

6. (59) $343^{\frac{1}{3}} \cdot 343^{\frac{1}{3}}$

***7.** (60) There are 7 cherries and 3 grapes in a bag. What is the probability of selecting 2 grapes at random without replacement?

8. (36) Find the equation of the line that passes through $(2, -3)$ and is parallel to $y = -\frac{3}{8}x + 2$.

***9.** (66) **Packaging** A new food product is to be sold in cylindrical cans. The designer wants the height of the can to be 6 inches greater than the radius. The manager wants the can to have a volume of about 125 in^3.

a. Write a polynomial equation in standard form that can be used to find the radius. Leave in terms of π. Hint: The formula for the volume of a cylinder is $V = \pi r^2 h$.

b. Find the approximate radius that should be used for the can by using a graphing calculator. Round the radius to the nearest tenth.

10. (50) Find an equation for the inverse of $y = 7x^2 + 21$ for $x \geq 0$. Identify the domain and range of the inverse function.

11. (22) **Multiple Choice** $f(x) = x - 5$ is an example of a:

A continuous function **B** discrete function

C discontinuous function **D** both B and C

12. (58) **Write** Explain why it is necessary to add the constant term $\left(\frac{b}{2}\right)^2$ to both sides of the equation when using the method of completing the square to solve a quadratic equation.

***13.** (Inv 5) **Graphing Calculator** Calculate the binomial probability, to the nearest hundredth of a percent, of 8 successes out of 50 trials, if the probability of success for each trial is 0.17.

14. (56) **Geometry** Consider the circle with radius 5 units, centered at the origin. A 270° angle in standard position cuts the circle into a larger and a smaller sector. Find the area of the larger sector.

***15.** (64) **Error Analysis** A student converted the logarithmic equation $\log_2 64 = x$ to exponential form by writing $x^2 = 64$. What is the error? Write the correct exponential equation.

***16.** (65) **Multi-Step** A 24 in. by 36 in. photo is framed and the total area of the photo and the frame is 1200 in^2.

a. Write a quadratic expression in standard form to represent the situation.

b. Find the width of the frame to the nearest hundredth of an inch.

17. (57) **Formulate** Write the equation of an exponential function whose graph passes through the points (0, 5) and (−2, 125). Determine whether the function is increasing or decreasing.

***18.** (65) **Analyze** If using the quadratic formula to solve $16 = -3x^2$, what would be the values of a, b, and c?

***19.** (63) **Tires** The diameter of a certain tire is 24 inches. Through how many degrees does the tire rotate when the automobile travels 5 feet? Give your answer to the nearest degree.

20. (55) **Population** In 2006, the U.S. Census Bureau estimated there were 36,457,549 people living in California and 299,398,484 living in the United States.

a. Estimate the probability that a person randomly selected in the United States lives in California.

b. Estimate the odds in favor that a person randomly selected in the United States lives in California.

21. (62) **Probability** A student taking a test has enough time to solve 2 out of the 5 equations shown.

$$x^2 + 1 = 0, \quad x^2 + 4 = 0, \quad x^2 - 25 = 0, \quad x^2 - 16 = 0, \quad x^2 + 9 = 0$$

If she chooses equations randomly, what is the probability that both equations have imaginary solutions?

Use trigonometric ratios to solve for each of the variables.

22. (52) $\cos 30° = \frac{\sqrt{3}}{2}$ and $\cos 30° = \frac{y}{12}$

23. (52) $\sin 30° = \frac{1}{2}$ and $\sin 30° = \frac{x}{12}$

24. **Multi-Step** A student creates a chart classifying his books by subject and cover. *(60)*

a. Are the events math textbook or paperback textbook mutually exclusive or inclusive? Explain why.

b. Find the probability that four books, selected randomly with replacement, are math or paperback. Round to the thousandths place.

	Cover	
Subject	**Hardcover**	**Paperback**
Math	7	3
Sciences	23	14
Social Sciences	6	16
History	1	4

25. **Music** Frets are small metal bars placed across the neck of a guitar so that the guitar can produce notes of a specific scale. To find the distance a fret should be placed from the bridge, multiply the length of the string by $2^{-\frac{n}{12}}$, where n is the number of notes higher than the string's root note. Where should a fret be placed on the low E string to produce a note that is 5 notes higher, if the length of the string is 64 cm, rounded to the nearest hundredths place? *(59)*

***26.** **Generalize** Tell how to factor by grouping. *(61)*

27. **Probability** Find the probability of getting at least 1 tail if a coin is tossed 4 times. *(49)*

***28.** **Multiple Choice** Which is not a possible rational root of $2x^3 + 2x^2 - 21x - 27 = 0$? *(66)*

A -3 **B** -1 **C** $\frac{2}{3}$ **D** $\frac{9}{2}$

29. Let $f(x) = 8x - 6$ and $g(x) = 4x^2 + 1$. Evaluate the composition $(f \circ g)(-2)$ using two different methods. *(53)*

***30.** **Write** Explain why $\log_0 5$ and $\log_1 5$ do not exist. *(64)*

LESSON 67

Finding Inverse Trigonometric Functions

Warm Up

1. (63) **Vocabulary** A circle with a radius of 1 unit is called a _____.
2. (63) What is sin 30°?
3. (63) What is cos $\frac{\pi}{4}$?

New Concepts

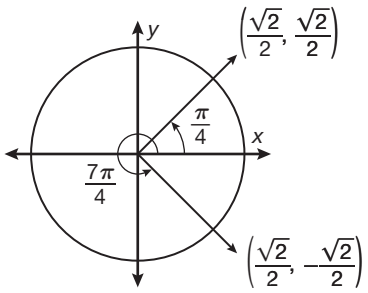
Evaluation of trigonometric functions for given angle measures was explained in Lesson 46. Angle measures that have given trigonometric function values can be found by using *inverse trigonometric relations*.

If a function value is:	Then one of the inverse relation values is:
$\sin \theta = a$	$\sin^{-1} a = \theta$
$\cos \theta = a$	$\cos^{-1} a = \theta$
$\tan \theta = a$	$\tan^{-1} a = \theta$

Reading Math

The symbol $\sin^{-1}$ represents the inverse sine relation, NOT a reciprocal. $\sin^{-1} a \neq \frac{1}{\sin a}$.

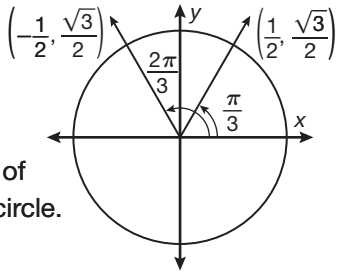
The inverses of the trigonometric functions are not functions themselves because there are many values of θ for a particular value of a. For example, consider $\cos^{-1} \frac{\sqrt{2}}{2}$. Angles with measures of $\frac{\pi}{4}$ and $\frac{7\pi}{4}$ radians have the cosine value $\frac{\sqrt{2}}{2}$. So do all angles that are coterminal with these angles.



Unit Circle

Example 1 Finding Trigonometric Inverses

Describe all the values of $\sin^{-1} \frac{\sqrt{3}}{2}$.



Step 1: Find the values of θ between 0 and 2π radians for which $\sin \theta$ is equal to $\frac{\sqrt{3}}{2}$.

$\frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}, \frac{\sqrt{3}}{2} = \sin \frac{2\pi}{3}$ Use *y*-coordinates of points on the unit circle.

Step 2: Describe the measures of all angles that are coterminal with $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

All the values of $\sin^{-1} \frac{\sqrt{3}}{2}$ are given by $\frac{\pi}{3} + (2\pi)n$ and $\frac{2\pi}{3} + (2\pi)n$, where n is an integer.

Add integer multiples of 2π radians.

Hint

Substitute positive and negative integers for n in the expressions $\frac{\pi}{3} + 2\pi n$ and $\frac{2\pi}{3} + 2\pi n$. Evaluate the sine of each result.

To define inverse trigonometric functions, it is necessary to restrict the domains of the trigonometric functions. The trigonometric functions that a calculator uses have restricted domains and will give only one value in the range that you will need to generalize to find all the possible values.

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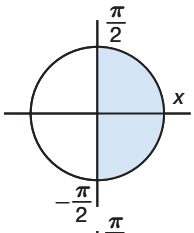
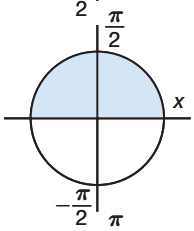
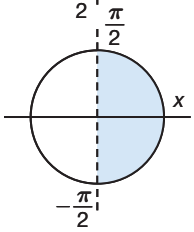
$\text{Sin}\,\theta = \sin\theta$ for $\left\{\theta \middle| -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$ — θ can terminate on the y-axis, the positive x-axis, or in Quadrant I or IV.



$\text{Cos}\,\theta = \cos\theta$ for $\{\theta \mid 0 \le \theta \le \pi\}$ — θ can terminate on the x-axis, the positive y-axis, or in Quadrant I or II.



$\text{Tan}\,\theta = \tan\theta$ for $\left\{\theta \middle| -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$ — θ can terminate on the positive x-axis, or in Quadrant I or IV.



$\text{Sin}\,\theta$, $\text{Cos}\,\theta$, and $\text{Tan}\,\theta$ can be used to define the inverse trigonometric functions. For each value of a in the domain of any inverse trigonometric function, there is one and only one value of θ in the range.

Inverse Trigonometric Functions

Words	Symbol	Domain	Range
The **inverse sine function** is $\text{Sin}^{-1}a = \theta$, where $\text{Sin}\,\theta = a$.	$\text{Sin}^{-1}a$	$\{a \mid -1 \le a \le 1\}$	$\left\{\theta \middle| -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$ $\{\theta \mid -90° \le \theta \le 90°\}$
The **inverse cosine function** is $\text{Cos}^{-1}a = \theta$, where $\text{Cos}\,\theta = a$.	$\text{Cos}^{-1}a$	$\{a \mid -1 \le a \le 1\}$	$\{\theta \mid 0 \le \theta \le \pi\}$ $\{\theta \mid 0° \le \theta \le 180°\}$
The **inverse tangent function** is $\text{Tan}^{-1}a = \theta$, where $\text{Tan}\,\theta = a$.	$\text{Tan}^{-1}a$	$\{a \mid -\infty < a < \infty\}$	$\left\{\theta \middle| -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$ $\{\theta \mid -90° < \theta < 90°\}$

Reading Math

The inverse trigonometric functions are also called the arcsine, arccosine, and arctangent functions.

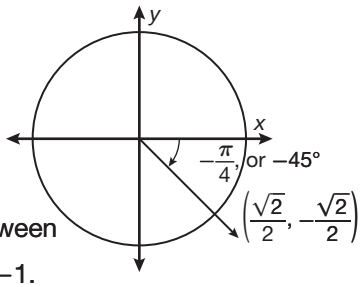
Example 2 Evaluating Inverse Trigonometric Functions

Find each inverse trigonometric function value that exists. Give answer in both radians and degrees.

a. $\text{Tan}^{-1}(-1)$

SOLUTION

Use the unit circle.



$-1 = \text{Tan}\left(-\frac{\pi}{4}\right)$ — $-\frac{\pi}{4}$ is the angle measure between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is -1.

$\text{Tan}^{-1}(-1) = \left(-\frac{\pi}{4}\right)$, or $\text{Tan}^{-1}(-1) = -45°$

Hint

$\text{Tan}\,\theta = \frac{y}{x}$

$\text{Tan}\left(-\frac{\pi}{4}\right) = -\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$

b. $\text{Cos}^{-1}\,2$

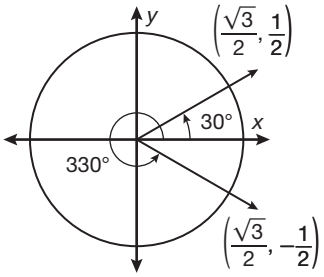
SOLUTION

The domain of the inverse cosine function is $\{a \mid -1 \le a \le 1\}$. Because 2 is not in this domain, $\text{Cos}^{-1}\,2$ does not exist.

Example 3 Solving Trigonometric Equations

a. Solve $2 \cos \theta = \sqrt{3}$, for $0° \le \theta \le 360°$.

SOLUTION



$2 \cos \theta = \sqrt{3}$ — $\cos \theta$ is unknown.

$\frac{2 \cos \theta}{2} = \frac{\sqrt{3}}{2}$ — Divide each side by 2 to isolate $\cos \theta$.

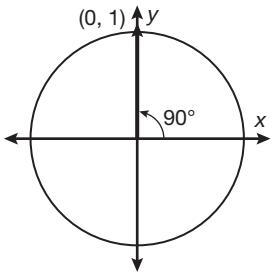
$\cos \theta = \frac{\sqrt{3}}{2}$

$\theta = 30°$ or $330°$ — Use the unit circle.

The solutions are 30° and 330°.

b. Solve $5 \sin \theta = 5$, for $0° \le \theta \le 360°$.

SOLUTION



$5 \sin \theta = 5$ — $\sin \theta$ is unknown.

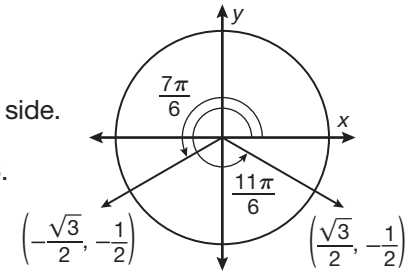
$\sin \theta = 1$ — Divide each side by 5.

$\theta = 90°$ — Use the unit circle.

The solution is 90°.

c. Solve $6 \sin \theta + 3 = 0$, where θ is any real number of radians.

SOLUTION



$6 \sin \theta + 3 = 0$ — $\sin \theta$ is unknown.

$6 \sin \theta = -3$ — Subtract 3 from each side.

$\sin \theta = -\frac{1}{2}$ — Divide each side by 6.

$\theta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$ or any coterminal angle

The solutions are $\frac{7\pi}{6} + 2\pi n$ and $\frac{11\pi}{6} + 2\pi n$, where n is an integer.

Caution

The inverse cosine key on a calculator will only give the solution in the domain of the inverse cosine function.

$\text{Cos}^{-1} \frac{\sqrt{3}}{2} = 30°$

Example 4 Solving Trigonometric Equations Using a Calculator

Solve each equation to the nearest tenth. Use the given restrictions.

a. $10 \text{ Cos } \theta = 6$, for $0° \le \theta \le 180°$

SOLUTION

$10 \text{ Cos } \theta = 6$ — Cos θ is unknown.

$\text{Cos } \theta = 0.6$ — Divide each side by 10.

$\theta = \text{Cos}^{-1}\ (0.6)$ — Use the inverse cosine key on your calculator.

$\theta \approx 53.1°$

The solution is 53.1°.

Caution

If the answer on your calculator screen is 0.927295218 when you enter $\cos^{-1}$ (0.6), your calculator is set to radian mode instead of degree mode.

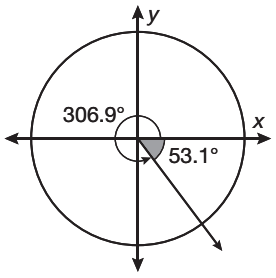
b. $10 \cos \theta = 6$, for $270° < \theta < 360°$

SOLUTION

Because of the restrictions, the terminal side of θ is in Quadrant IV. Find the angle that terminates in Quadrant IV that has the same cosine value as 53.1°.

$\theta \approx 360° - 53.1°$

$\theta \approx 306.9°$



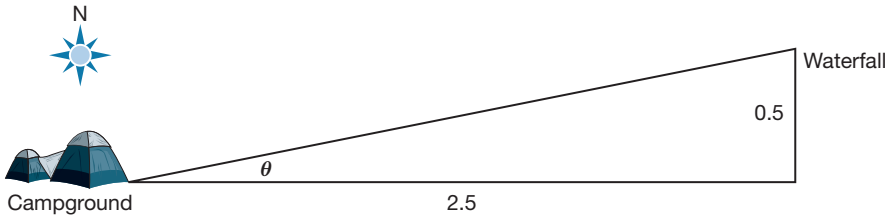
θ has a reference angle of 53.1°, and $270° < \theta < 360°$

Example 5 Application: Navigation

A pair of hikers plan to walk from a campground to a waterfall. The waterfall is 2.5 miles east and 0.5 mile north of the campground. To the nearest degree, in what direction should the hikers head?

Step 1: Draw a diagram.

The hikers' direction should be based on θ, the measure of an acute angle of a right triangle.



Step 2: Find the value of θ.

$\text{Tan } \theta = \dfrac{\text{opposite}}{\text{adjacent}}$ Use the tangent ratio.

$\text{Tan } \theta = \dfrac{0.5}{2.5} = 0.2$ Substitute 0.5 for opposite and 2.5 for adjacent. Then simplify.

$\theta = \text{Tan}^{-1} 0.2$

$\theta \approx 11°$

The hikers should head 11° north of east.

Lesson Practice

(Ex 1) **a.** Describe all the values of $\cos^{-1}\frac{\sqrt{3}}{2}$.

(Ex 2) **b.** Evaluate $\text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ in both radians and degrees if it exists

(Ex 2) **c.** Evaluate $\text{Sin}^{-1}\left(-\frac{3}{2}\right)$ in both radians and degrees if it exists.

(Ex 3) **d.** Solve $2 \sin \theta = -1$, for $0° \leq \theta \leq 360°$.

(Ex 3) **e.** Solve $-\cos \theta = -1$, for $0° \leq \theta \leq 360°$.

(Ex 3) **f.** Solve $10 \cos \theta - 5 = 0$, where θ is any real number of radians.

g. (Ex 4) Solve the equation to the nearest tenth. Use the given restrictions.
$5 \operatorname{Sin} \theta = 2$, for $-90° \leq \theta \leq 90°$

h. (Ex 4) Solve the equation to the nearest tenth. Use the given restrictions.
$5 \operatorname{Sin} \theta = 2$, for $90° < \theta < 180°$

i. (Ex 5) A group of hikers plan to walk from a parking lot to an overlook. The overlook is 1.6 miles west and 0.6 mile north of the parking lot. To the nearest degree, in what direction should the hikers head?

Practice Distributed and Integrated

1. (56) Given $\theta = -1787°$, find the measure of the reference angle.

Convert each measure from degrees to radians.

2. (63) $60°$ **3.** (63) $125°$ **4.** (63) $-240°$

5. (51) Use synthetic substitution to find $f(-3)$ for $f(x) = 7x^4 - 3x^3 + x^2 + 11$.

6. (58) Complete the square and factor the resulting perfect square trinomial: $x^2 + 6x$.

Simplify.

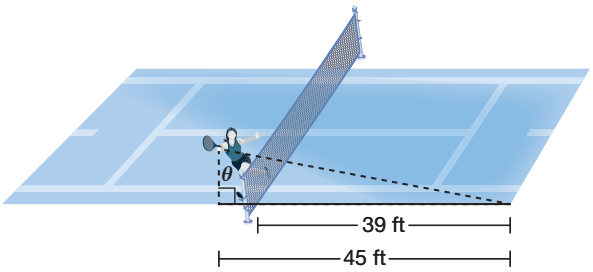
7. (48) $\dfrac{\frac{a}{b}}{\frac{a+b}{b}}$

8. (48) $\dfrac{\frac{s}{a+b} + \frac{x}{b}}{\frac{a}{s+x}}$

***9.** (67) **Error Analysis** A student incorrectly solved the equation $\sin\theta = -0.55$, for $90° \leq \theta \leq 270°$, as shown below. What error did the student make? Find the correct solution.

$$\sin \theta = -0.55$$
$$\sin^{-1}(-0.55) \approx -33.4°$$
$$\theta \approx 180° - 33.4°$$
$$\theta \approx 146.6°$$

***10.** (67) **Tennis** A tennis player 6 feet from the net hits an overhead shot, hitting the ball at a height of 8.5 feet. Assume that the path of the ball is a straight line. To the nearest degree, what is the maximum angle θ at which the ball can be hit and land within the court?



***11.** (66) **Graphing Calculator** **a.** Graph $y = x^4 + 2x^3 - 12x^2 + 14x - 5$ on a graphing calculator. What are the x-intercepts?

b. Find the multiplicity of each root of $x^4 + 2x^3 - 12x^2 + 14x - 5 = 0$ by factoring.

12. (54) Minimize $P = 4x + 6y$ for the constraints $0 \le x \le 4$, $y \ge 1$, $y \ge -x + 4$

***13.** (65) **Multiple Choice** Which is a solution of $5 + 2x^2 = -3x$?

A 1 **B** $\frac{5}{2}$ **C** $\frac{3}{4} + \frac{\sqrt{31}}{4}$ **D** $-\frac{3}{4} - \frac{\sqrt{31}}{4}i$

***14.** (Inv 5) **Contact Lenses** The contact lens prescription for Asia's left eye is stronger than the prescription for her right eye. If Asia accidentally mixed up her contact lenses, what is the probability that she will put the correct lens in either eye on the first try? Explain.

***15.** (63) **Multi-Step** Complete parts a–d to determine the measure of the angle formed by the hands of a clock when the time is 3:24. Find all measures in radians.

a. Find x, the angle formed at 3:00.

b. Find $(x + y + z)$, the angle through which the minute hand moves in 24 minutes.

c. Find y, the angle through which the hour hand moves in 24 minutes.

d. Find z, the angle formed by the minute hand and hour hand when the time is 3:24.

16. (62) **Model** Make a Venn Diagram to illustrate how the complex numbers, real numbers, and imaginary numbers are related.

17. (61) **Finance** The value of a stock over the past five years where x is the number of years is modeled by $x^3 - 2x^2 + 5x - 10$. Factor this expression.

18. (54) **Geometry** The Community Center wants to build a playground area and enclose it within a wooden, split rail fence. The cost of fencing is \$20 per finished foot. The cost for painting the fence is \$12 per finished foot. The Center cannot spend more than \$2000 on fencing. The playground cannot be any longer than 30 ft and needs to be at least 15 ft wide. What is the maximum area the Center can get for the size of the playground? How much will this cost?

19. (60) **Error Analysis** A cup contains 15 pink beads, 24 purple beads, and 18 gold beads. Three beads are individually selected with replacement. The work below is used to find the probability that one of each color bead is selected in the order pink, purple, gold. Is the work correct? If not, explain the error and find the correct solution. If so, explain each step of the work.

K = event of pink bead drawn

U = event of purple bead drawn

G = event of gold bead drawn

$$P(K,U,G) = P(K) \cdot P(U) \cdot P(G)$$
$$= \frac{5}{19} \cdot \frac{8}{19} \cdot \frac{6}{19} \approx 0.035$$

$$P(K) = \frac{15}{57} = \frac{5}{19} \quad P(U) = \frac{24}{57} = \frac{8}{19} \quad P(G) = \frac{18}{57} = \frac{6}{19}$$

20. (59) **Multiple Choice** Choose the letter that best represents $\sqrt[3]{\frac{216x^{12}}{4}}$ simplified.

A $\frac{3x^4\sqrt[3]{4}}{2}$ **B** $3x^4\sqrt[3]{2}$ **C** $6x^4\sqrt[3]{4}$ **D** $\frac{3x^4\sqrt[3]{2}}{2}$

*21. (66) **Analyze** When solving $x^3 + 7x^2 + 7x - 6 = 0$, why is it not necessary to use synthetic substitution after one of the roots is found?

22. (58) **Write** A ball is tossed with an initial velocity of 74 feet per second from the top of a building that is 50 feet tall. The height, h, of the ball in feet after t seconds can be approximated by the equation $h = -16t^2 + 74t + 50$. The ball will be 20 feet from the ground when $t = -0.375$ second and $t = 5$ seconds. Explain why you would not use the solution $t = -0.375$ second.

*23. (67) **Coordinate Geometry** Find $m\angle AOB$ to the nearest tenth of a degree, using these points: $A(8, 3)$, $O(0, 0)$, and $B(8, 0)$.

24. (55) **Verify** Explain why the probability of choosing first, second, and third place winners for a photo when randomly choosing from the top 10 athletes is $\frac{1}{{}_{10}C_3}$.

*25. (66) **Storage** The volume of a can of dog food is 450 cubic inches. The height of the can is 5 inches greater than its radius. Write a polynomial in standard form, using 3.14 for π, that can be used to find the radius. (Round the constant to the nearest integer.) Then find the approximate radius by using a graphing calculator.

26. (56) **Generalize** Suppose θ is an angle in standard position and $Q(x, y)$ is a point on the terminal side of the angle. Explain how the signs (positive or negative) of $\sin \theta$, $\cos \theta$, and $\tan \theta$ depend on which quadrant Q lies in.

27. (60) **School** A teacher has 3 red crayons, 7 yellow crayons, 5 green crayons, and 2 blue crayons. A yellow crayon and then a blue crayon are chosen at random without replacement. Are these events dependent or mutually exclusive?

28. (57) The number of rabbits increases exponentially. At first there were 400. Three years later there were 1600. How many rabbits would there be after 10 years?

29. (54) Maximize $P = 5x + 2y$ for the constraints
$$\begin{aligned} y &\geq 0 \\ x &\geq 0 \\ y &\leq -x + 10 \\ y &\leq 2x + 1 \end{aligned}$$

*30. (65) **Physical Science** The South Falls waterfalls in Silver Falls State Park in Oregon are 177 feet high. The equation $-16t^2 + 177 = 0$ gives the number of seconds it takes for a drop of water at the top of the falls to reach the pool of water below. Solve the equation by using the quadratic formula. Round to the nearest hundredth of a second.

LESSON
68

Finding Conditional Probability

Warm Up

1. (55) **Vocabulary** If the occurrence of one event affects the probability of the second event, the two events are _____.

2. (55) As a percent, what is the probability of getting heads on all three flips of a coin?

3. (55) A bag contains 8 red marbles, 16 white marbles, and 4 black marbles. What is the probability of randomly selecting a white or black marble?

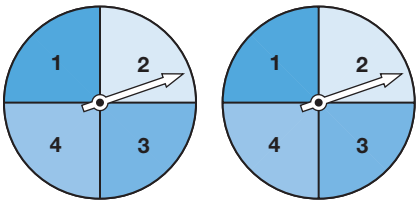
New Concepts

The probability of two independent events is the product of their individual probabilities. However, finding the probability of two dependent events requires finding a *conditional probability*. Recall that $P(B|A)$ is a conditional probability. It is read as "the probability of B given that A has occurred."

Dependent Events
If A and B are dependent events, then $P(A \text{ and } B) = P(A) \cdot P(B|A)$.

Example 1 Finding the Probability of Dependent Events

a. Suppose the two spinners shown are each spun once. Find the probability that the first spinner lands on 2 and the sum of the spinners is 6.



Math Reasoning

Analyze Name another way of finding the probability of the events in Example 1a.

SOLUTION Assign event A to be the first spinner landing on 2 and event B to be the sum of the spinners is 6. Determine if the events are dependent. It helps to list all the outcomes in the sample space. The first numbers are from the first spinner, and the second numbers are from the second spinner. If event A *does not* occur, $P(B) = \frac{3}{16}$ because out of 16 possible outcomes, 3 have a sum of 6. These pairs are boxed. If event A *does* occur, $P(B) = \frac{1}{4}$ because out of 4 possible pairs only one has a sum of 6.

1, 1	2, 1	3, 1	4, 1
1, 2	2, 2	3, 2	**4, 2**
1, 3	2, 3	**3, 3**	4, 3
1, 4	**2, 4**	3, 4	4, 4

1, 1	2, 1	3, 1	4, 1
1, 2	2, 2	3, 2	4, 2
1, 3	2, 3	3, 3	4, 3
1, 4	**2, 4**	3, 4	4, 4

Since the occurrence of event A affects the probability of event B, the events are dependent.

$$P(A \text{ and } B) = P(A) \cdot P(B|A) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

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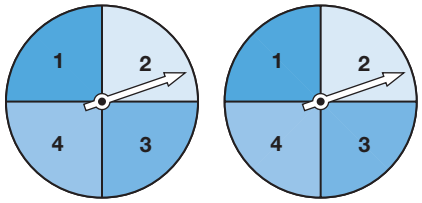
b. Suppose the two spinners shown are each spun once. Find the probability that the first spinner lands on an odd number and the sum of the spinners is greater than 4.



SOLUTION

Confirm that the events are dependent.

If event A *does not* occur, $P(B) = \frac{10}{16} = \frac{5}{8}$ because out of 16 possible outcomes, 10 have a sum greater than 4.

If event A *does* occur, $P(B) = \frac{4}{8} = \frac{1}{2}$ because out of 8 possible pairs (those in the first and third columns), 4 have a sum greater than 4.

1, 1	2, 1	3, 1	**4, 1**
1, 2	2, 2	**3, 2**	**4, 2**
1, 3	**2, 3**	**3, 3**	**4, 3**
1, 4	**2, 4**	**3, 4**	**4, 4**

1, 1	2, 1	3, 1	4, 1
1, 2	2, 2	**3, 2**	4, 2
1, 3	2, 3	**3, 3**	4, 3
1, 4	2, 4	**3, 4**	4, 4

Since the occurrence of event A affects the probability of event B, the events are dependent.

$$P(A \text{ and } B) = P(A) \cdot P(B|A) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Example 2 Using a Table to Find Conditional Probability

The table shows the results of a survey that asked students to state the year and season of their birthday.

	1992	1993	1994
Winter	10	16	14
Spring	6	9	11
Summer	12	11	8
Fall	14	9	12

Hint

First circle or highlight the row(s) or column(s) with the given condition(s), that is, event *A*.

a. Find $P(\text{Winter}|1993)$.

SOLUTION $P(\text{Winter}|1993)$ is the probability of a birthday occurring in the winter, given that it occurred in 1993. Since the condition is that it must be in the year 1993, focus on the 1993 column.

$$\frac{\text{winter}}{\text{all seasons}} : \frac{16}{16 + 9 + 11 + 9} = \frac{16}{45} \approx 35.6\%$$

b. Find $P(1992|\text{Spring or Summer})$.

SOLUTION $P(1992|\text{Spring or Summer})$ is the probability of a birthday occurring in 1992, given that it occurred in the spring or summer. Since the condition is that it must be in the spring or summer, focus on those rows.

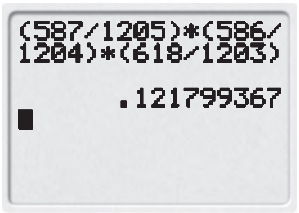
$$\frac{1992}{\text{spring or summer all years}} : \frac{6 + 12}{6 + 9 + 11 + 12 + 11 + 8} = \frac{18}{57} = \frac{6}{19} \approx 31.6\%$$

When items are selected and not replaced, the events are dependent. For each successive selection, the total number of outcomes, which are the values in the denominators, decreases by one.

Example 3 Using a Calculator to Find Conditional Probability

A raffle is being held at a school function. A container has the names of the 587 juniors and 618 seniors who are participating on slips of paper. Prizes are being given to the first, second, and third names pulled from the container. Once a student wins a prize, his or her name is not returned to the container. What is the probability that the first two prizes go to juniors and the third prize goes to a senior?

SOLUTION After the first junior wins, there is one less junior name in the container, which is also one less name overall. After the second junior wins, there is one less name overall again. There are $587 + 618 = 1205$ names in all.



$$P(\text{junior, junior, senior}) = \frac{587}{1205} \cdot \frac{586}{1204} \cdot \frac{618}{1203} \approx 0.122$$

The probability is about 12.2%.

Math Reasoning

Analyze Calculate P(junior, senior, junior). How does it compare to Example 3?

Example 4 Application: Quality Control

At a mailing center, it was found that on Tuesday, 44 out of 662 envelopes were stamped upside down, on Wednesday, 16 out of 251 envelopes were stamped upside down, and on Thursday, 3 out of 598 envelopes were stamped upside down. Find the probability that an envelope was stamped on Wednesday given that it was stamped right-side up.

SOLUTION

	Tues.	Wed.	Thurs.
Right-side Up	618	235	595
Upside Down	44	16	3

1. **Understand** Make a table to organize the information. Subtract to find the number of envelopes stamped right-side up.

2. **Plan** The condition is that the envelope was stamped right-side up. Find the sum of the envelopes stamped right-side up; this will be the denominator.

3. **Solve** Use a calculator.

$$\frac{235}{618 + 235 + 595} = \frac{235}{1448} \approx 0.162$$

The probability is about 16.2%.

4. **Check** Use estimation to check for reasonableness. From the given information, about 1500 envelopes were stamped and about 250 of those were stamped on Wednesday. The majority of those stamped on Wednesday were stamped right-side up, and $\frac{250}{1500} = \frac{1}{6} = 16\frac{2}{3}\%$.

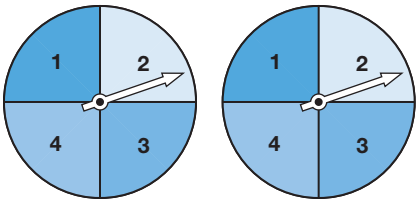
Math Reasoning

Estimate Estimate the probability that an envelope was stamped on Thursday given that it was stamped upside down.

Lesson Practice

Suppose the two spinners shown are each spun once.

(Ex 1)



a. Find the probability that the first spinner lands on 4 and the product of the spinners is less than 10.

b. Find the probability that the first spinner lands on a number less than 4 and the sum of the numbers is less than 6.

The table shows the results of a survey that asked students from three grade levels to pick their favorite subject from those listed.

(Ex 2)

	10th	11th	12th
Math	34	56	11
History	45	19	26
Science	52	45	66
English	9	12	58

c. Find $P(\text{Math}|12)$.

d. Find $P(10\text{th}|\text{History})$.

(Ex 3) **e.** A raffle is being held at a mall. A box has the names of the 237 men and 318 women who are participating on slips of paper. Gift certificates are being given to the first five names pulled from the box. Once a person wins a certificate, his or her name is not returned to the box. What is the probability that the first three certificates go to men and the last two certificates go to women?

(Ex 4) **f.** At a factory, it was found that during the morning shift, 32 out of the 2666 boxes of nails that were filled had unacceptable weights, and during the evening shift, 11 out of the 1870 boxes of nails that were filled had unacceptable weights. Find the probability that a box was filled in the evening, given that its weight was found acceptable.

Practice Distributed and Integrated

(62) **1.** Solve $s^2 - 22 = -112$. Write the solutions in terms of i.

Factor completely.

(23) **2.** $-3pax + pax^2 + 2pa$

(23) **3.** $30 + 3x^2 - 21x$

(58) **4.** **Multi-Step** Use $x^2 - 62x + 961 = 26$.

a. Explain why the left side of the equation is a perfect square trinomial.

b. Factor $x^2 - 62x + 961$.

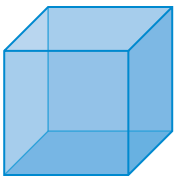
c. Solve the equation for x.

*5. (66) **Error Analysis** Find and correct the error a student made in determining the roots and their multiplicity in $x^3 - 4x^2 + 4x = 0$.

$$x^3 - 4x^2 + 4x = 0$$
$$x(x^2 - 4x + 4) = 0$$
$$x(x-2)(x-2) = 0 \qquad \text{The only root is 2 and its multiplicity is 2.}$$

6. (55) **Analyze** What does it mean for an event to have the first number in the odds in favor ratio greater than the second number?

7. (49) **Geometry** Write an expanded expression for the volume of the cube shown.



*8. (68) **Housing** The table shows estimated housing occupancy for three states for the year 2006.

Find the probability that a randomly selected house from one of these states is in Nevada. Then find the probability that a randomly selected house from one of these states is in Nevada given that it is vacant. Round each percent to the nearest tenth.

	Occupied Houses	Vacant Houses
Mississippi	1,075,521	165,918
Nevada	936,828	128,439
New York	7,088,376	819,138

Simplify.

9. (59) $-64^{-\frac{2}{3}}$

10. (59) $\dfrac{-3}{-9^{-\frac{3}{2}}}$

11. (59) $(-8)^{\frac{1}{3}}$

12. (63) Find the arc length of a sector with a radius of 4 in. and central angle $\theta = \frac{\pi}{6}$.

*13. (68) **Probability** A bag contains 30 red marbles and 10 white marbles. If three marbles are randomly chosen without replacement, what is the probability of selecting all red or all white marbles?

14. (46) **Surveying** A surveyor is standing on top of a hill that is 600 meters from a bridge. Her angle of elevation to the top of the bridge is 14° and her angle of depression to the bottom of the bridge is 26°. To the nearest tenth of a meter, what is the height of the bridge?

*15. (68) **Generalize** Insert $=$ or $\neq$ to make each statement true.

If A and B are independent events, then $P(B|A) \square P(B)$.

If A and B are dependent events, then $P(B|A) \square P(B)$.

16. (60) **Statistics** How do probability and statistics compare and contrast?

17. (56) **Multiple Choice** Which angle does *not* have a reference angle of 30°?

A $-30°$ **B** $120°$ **C** $570°$ **D** $-150°$

*18. (67) **Estimate** Use what you know about the unit circle and special angles to estimate $\text{Cos}^{-1}(0.48)$ in degrees. Do not use a calculator. Explain your method.

19. (63) **Write** Describe how to determine cos 150° by using its reference angle.

Simplify.

20. (44) $3\sqrt{\frac{5}{2}} - 2\sqrt{\frac{2}{5}}$

21. (44) $3\sqrt{18} + 2\sqrt{50} - \sqrt{98}$

22. (Inv 5) Calculate the binomial probability $P(12, 200, 0.09)$ to the nearest hundredth of a percent using the formula $P(n, N, p) = \frac{N!}{n!(N-n)!}p^n(1-p)^{N-n}$.

*23. (54) **Graphing Calculator** Describe the feasible region for the set of inequalities $x \geq 0$, $x \leq 4$, $y \geq 0$, $2y + x \leq 25$.

24. (62) **Error Analysis** Find and correct the error a student made in subtracting complex numbers.

$$(7 - 4i) - (2 + 3i)$$
$$5 - i$$

*25. (67) **Multiple Choice** Which describes all the values of $\sin^{-1}\frac{1}{2}$?

A $\frac{5\pi}{6}$ **B** $\frac{\pi}{6}$

C $\frac{2\pi}{3}$ **D** $\frac{\pi}{3}$

26. (59) **Estimate** Sketch a graph of the equation $y = \frac{\sqrt[3]{6\pi^2 x}}{2\pi}$, where y is the radius of a sphere and x is the volume of the sphere. Use the graph to estimate the volume of a sphere with a radius of 2 cm. Use the graph to estimate the radius of a sphere with a volume of 14 cm^3.

27. (61) **Volume** The volume of a California king size mattress is $12x^3 - 36x^2 - 432x - 384$ cubic inches. Factor the polynomial given that $x - 8$ is a factor. Then find the volume given that $x = 20$.

*28. (64) **Plant Nutrition** The pH of a soil is given by the function $\text{pH} = -\log [\text{H}^+]$, where $[\text{H}^+]$ represents the concentration of hydrogen ions in moles per liter. Most plants require soil with a hydrogen ion concentration between 0.0001 and 0.00000001 moles per liter. What pH values correspond to these hydrogen ion concentration values?

*29. (65) **Travel** A driver leaves a parking lot and heads due south at a rate of 40 miles per hour. An hour later, another driver leaves the same parking lot and heads due west at a rate of 60 miles per hour. Approximately how long will it take for the cars to be 300 miles apart?

*30. (68) **Multi-Step** Two bags each contain papers labeled with numbers 1 through 5. One paper is selected from each.

a. Explain why selecting a 5 from the first bag and the sum of the numbers is greater than 5 are dependent events.

b. Find the probability of both events.

LESSON 69

Simplifying Complex Expressions

Warm Up

1. (62) **Vocabulary** A number that can be written in the form $a + bi$ where a and b are real numbers is a _____.

2. (19) Multiply $(3x - 5)(4x + 2)$.

3. (40) Simplify $\frac{5}{\sqrt{5}}$.

4. (62) Add $(3 + 2i) + (6 - 8i)$.

New Concepts

Complex numbers $(a + bi)$ can be graphed on a complex plane, where the horizontal axis represents the real part, a, and the vertical axis represents the imaginary part, bi.

Math Language

The **imaginary axis** is sometimes referred to as the ***i*-axis.**

Example 1 Graphing Complex Numbers

Graph each number on the complex plane.

a. $3 + 4i$ **b.** $-2 - 3i$

c. $-4i$ **d.** 1

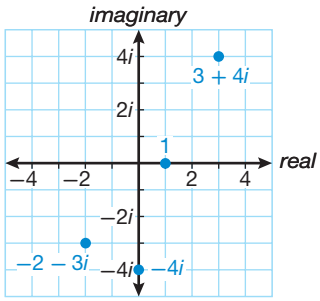
SOLUTION

a. Move 3 units to the right and 4 units up.

b. Move 2 units to the left and 3 units down.

c. Move 0 units right or left and 4 units down.

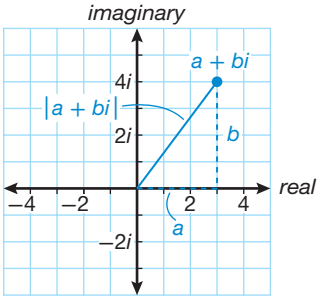
d. Move 1 unit to the right and 0 units up or down.



The **absolute value of a complex number** is its distance from the origin. Unless the point is on the x- or i-axis, the distance is the length of a line segment that forms the hypotenuse of a right triangle and can be found using the Pythagorean Theorem.

Absolute Value of a Complex Number

$$|a + bi| = \sqrt{a^2 + b^2}$$



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Example 2 Finding the Absolute Value of Complex Numbers

Find the absolute value of each number.

a. $3 + 4i$

SOLUTION $|a + bi| = \sqrt{a^2 + b^2}$

$$|3 + 4i| = \sqrt{3^2 + 4^2}$$
$$= \sqrt{9 + 16} = \sqrt{25} = 5$$

b. $-2 - 3i$

SOLUTION $|a + bi| = \sqrt{a^2 + b^2}$

$$|-2 - 3i| = \sqrt{(-2)^2 + (-3)^2}$$
$$= \sqrt{4 + 9} = \sqrt{13}$$

c. $-5i$

SOLUTION $|a + bi| = \sqrt{a^2 + b^2}$

$$|-5i| = \sqrt{0^2 + (-5)^2}$$
$$= \sqrt{25} = 5$$

Math Reasoning

Verify Use the formula for the absolute value of a complex number to show that $|-1|$ is 1.

When multiplying complex numbers, remember that $i^2 = -1$.

Example 3 Multiplying Complex Numbers

Multiply. Write answers in the form $a + bi$.

a. $-4i(7 + 3i)$

SOLUTION

$-4i(7 + 3i)$	
$-28i - 12i^2$	Distributive Property
$-28i - 12(-1)$	$i^2 = -1$
$-28i + 12$	Simplify.
$12 - 28i$	Write in standard form.

b. $(5 + 4i)(6 - 2i)$

SOLUTION

$(5 + 4i)(6 - 2i)$	
$30 - 10i + 24i - 8i^2$	Use the FOIL method.
$30 + 14i - 8i^2$	Simplify.
$30 + 14i - 8(-1)$	$i^2 = -1$
$30 + 14i + 8$	Simplify.
$38 + 14i$	Write in standard form.

Study the powers of i below.

Powers of i	
$i^1 = i$	$i^5 = i \cdot i^4 = i(1) = i$
$i^2 = -1$	$i^6 = i \cdot i^5 = i(i) = i^2 = -1$
$i^3 = i \cdot i^2 = i(-1) = -i$	$i^7 = i \cdot i^6 = i(-1) = -i$
$i^4 = i \cdot i^3 = i(-i) = -i^2 = -(-1) = 1$	$i^8 = i \cdot i^7 = i(-i) = -i^2 = -(-1) = 1$

Notice that the four simplified values of i, -1, $-i$, and 1 repeat such that $i = i^5$, $i^2 = i^6$, $i^3 = i^7$, and $i^4 = i^8$. Any power of i can be found by dividing the exponent of i by four and using the remainder.

When the remainder is …	The expression is equivalent to …
0	$i^4 = 1$
1	$i^1 = i$
2	$i^2 = -1$
3	$i^3 = -i$

Example 4 Evaluating Powers of i

Simplify each expression.

a. i^{30}

SOLUTION $30 \div 4 = 7$ R 2, so $i^{30} = i^2 = -1$.

b. $5i^{12}$

SOLUTION $12 \div 4 = 3$ R 0, so $i^{12} = i^4 = 1$.
$5i^{12} = 5(1) = 5$

c. $-2i^{13}$

SOLUTION $13 \div 4 = 3$ R 1, so $i^{13} = i^1 = i$.
$-2i^{13} = -2(i) = -2i$

Caution

A simplified expression does not have a radical in the denominator. Because $i = \sqrt{-1}$, an expression with i in the denominator is not simplified.

To eliminate a radical in a denominator, multiply both the numerator and denominator of the fraction by the conjugate of the denominator. The **complex conjugate** of $a + bi$ is $a - bi$.

Example 5 Dividing Complex Numbers

a. Divide 3 by $6 + i\sqrt{5}$. Write the answer in the form $a + bi$.

SOLUTION $\dfrac{3}{6 + i\sqrt{5}}$

$\dfrac{3}{6 + i\sqrt{5}}\left(\dfrac{6 - i\sqrt{5}}{6 - i\sqrt{5}}\right)$ Multiply the numerator and the denominator by the complex conjugate of the denominator.

$\dfrac{18 - 3i\sqrt{5}}{36 - 5i^2}$ Distribute and simplify.

$\dfrac{18 - 3i\sqrt{5}}{41} = \dfrac{18}{41} - \dfrac{3\sqrt{5}}{41}i$

b. Divide $4 + i$ by $6i$. Write the answer in the form $a + bi$.

SOLUTION $\dfrac{4+i}{6i}$

$\dfrac{4+i}{6i}\left(\dfrac{-6i}{-6i}\right)$ Multiply the numerator and the denominator by the complex conjugate of the denominator.

$\dfrac{-24i-6i^2}{-36i^2}$ Distribute.

$\dfrac{-24i-6(-1)}{-36(-1)}$ Simplify.

$\dfrac{-24i+6}{36} = -\dfrac{24}{36}i + \dfrac{6}{36} = -\dfrac{2}{3}i + \dfrac{1}{6} = \dfrac{1}{6} - \dfrac{2}{3}i$

Math Reasoning

Analyze Why is $-6i$ the conjugate of $6i$?

Lesson Practice

Graph each number on the complex plane.

(Ex 1)

a. $1 - 3i$ **b.** $-2 + 4i$ **c.** -3 **d.** $2i$

Find the absolute value of each number.

(Ex 2)

e. $7 - 2i$ **f.** $-5 + i$ **g.** $10i$

Multiply. Write answers in the form $a + bi$.

(Ex 3)

h. $8i(-9 - 5i)$ **i.** $(2 - 3i)(5 + 9i)$

Simplify each expression.

(Ex 4)

j. i^{18} **k.** $-3i^{11}$ **l.** $\dfrac{1}{2}i^{21}$

m. (Ex 5) Divide $3 + 3i$ by $3i$. Write the answer in the form $a + bi$.

n. (Ex 5) Divide -4 by $2 + i\sqrt{3}$. Write the answer in the form $a + bi$.

Practice Distributed and Integrated

Solve using the quadratic formula.

1. (65) $f(x) = x^2 + 8x - 3$

2. (65) $f(x) = 3x^2 - 10x + 4$

3. (55) **Probability** A bag holds 53 yellow marbles, 17 red marbles, and 30 green marbles. What is the probability of not choosing a red marble?

4. (57) A savings account earns interest at an annual rate of 3.8%, compounded quarterly. If the account begins with a principal amount of \$2,700, what will its value be after 14 years?

Solve for x.

5. (59) $x^{\frac{1}{7}} = 5$

6. (59) $x^{\frac{2}{3}} = 36$

*7. (69) **Multiple Choice** In which quadrant of the complex plane is $5 - i$ located?

A Quadrant I **B** Quadrant II **C** Quadrant III **D** Quadrant IV

8. (61) **Write** Explain how to use synthetic division to determine if $x + 6$ is a factor of $x^3 + 3x^2 - 28x - 60$ and then perform the division.

Simplify.

9. (62) $\frac{1}{4}\sqrt{-256}$.

10. (62) $\frac{1}{2}\sqrt{-2500}$.

11. (56) Given $\theta = -920°$, find the measure of the reference angle.

*12. (69) **Generalize** Show that the product of any two complex conjugates, $a + bi$ and $a - bi$, is a real number.

*13. (67) Evaluate the inverse trigonometric function $\text{Sin}^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. Give your answer in both radians and degrees.

14. (Inv 4) Why can't the matrix below be used as an encoding matrix for a cipher?

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 13 \end{bmatrix}$$

15. (56) **Amusement Parks** You get on a ferris wheel ride, and the ferris wheel turns through 1370°. How much further must the wheel turn before you are at the bottom again?

*16. (68) **Error Analysis** Find and correct the error a student made in finding $P(\text{AM}|\text{Male})$.

$$P(\text{AM}|\text{Male}) = \frac{16}{20} = \frac{4}{5} = 80\%$$

	Male	Female
AM	16	4
PM	8	12

*17. (66) **Graphing Calculator** **a.** Graph $y = x^3 + 6x^2 - 32$ on a graphing calculator. What are the x-intercepts?

b. Find the multiplicity of each root of $x^3 + 6x^2 - 32 = 0$ by factoring.

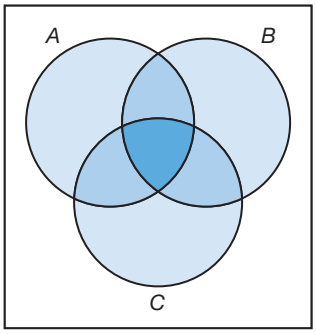
*18. (68) **Formulate** Give a rule for finding $P(B|A)$. *Hint:* Consider the rule for $P(A \text{ and } B)$ for dependent events.

19. (64) **Multiple Choice** Which equation is equivalent to $10^x = 0.1$?

A $\log_{0.1}10 = x$ **B** $\log_{0.1}x = 10$

C $\log_{10}0.1 = x$ **D** $\log_{10}x = 0.1$

20. (60) **Formulate** Write a formula for $P(A \text{ or } B \text{ or } C)$. Use the diagram to the right to help you.



21. (67) **Multi-Step** Let $f(\theta) = \text{Sin}\theta$ and $g(\theta) = \text{Sin}^{-1}\theta$, where θ is in radians.

a. Evaluate $f\left(g\left(\frac{1}{2}\right)\right)$. Show your steps.

b. Evaluate $g\left(f\left(\frac{\pi}{6}\right)\right)$. Show your steps.

c. Based on part a, what is the value of $f(g(x))$? Based on part b, what is the value of $g(f(x))$?

22. (58) **Error Analysis** Examine the work below. Find and explain the error. Then find the correct solution.

$$x^2 - 44x + 484 = 121$$
$$(x - 22)^2 = 121$$
$$x - 22 = 11$$
$$x = 33$$

***23.** (65) **Probability** A student graphs $y = 9x^2 - 29x + 22$ and uses a graphing calculator to find the value of the zeros one at a time. What is the probability that the first zero the student finds is negative? Why?

***24.** (69) **Multi-Step** Simplify $i^{22} + i^{15}$. Then graph the number on the complex plane.

25. (63) **Planet Rotation** Venus is only slightly smaller than Earth; its radius is approximately 3757 miles. But it rotates much more slowly; it takes 243 Earth days for Venus to make one complete rotation. If an object is fixed on Venus's equator, how far does it travel in 1 hour due to Venus's rotation?

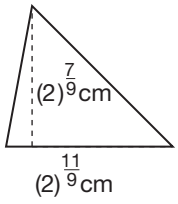
26. (64) **Analyze** Describe the value(s) of b^x when $b = 1$ and when $b \neq 1$.

***27.** (69) Find the absolute value of $|3 - 4i|$.

***28.** (69) Find the absolute value of $|6 + 2i|$.

29. (62) **Sports** A football player playing in a domed stadium kicks a football straight up into the air. The height of the football after t seconds can be modeled by $-16t^2 + 80t$. The dome has a height of about 288 feet. The solutions of $-16t^2 + 80t = 288$ give the number of seconds it will take for the football to reach the top of the dome. Solve the equation. Will the football reach the top of the dome? If so, when?

30. (59) **Geometry** Find the area of the triangle to the right.



LESSON 70

Solving Radical Equations

Warm Up

1. **Vocabulary** In the expression $\sqrt[3]{x}$, 3 is referred to as the _____. (40)
2. Simplify $\sqrt{64}$. (40)
3. Simplify $\sqrt[3]{-125}$. (40)
4. Solve $x^2 + 5x - 30 = 6$. (35)
5. Write $(x + 6)^{\frac{1}{2}}$ in radical form. (40)

New Concepts

A **radical equation** is an equation that contains at least one radical whose radicand contains a variable.

Radical Equations	Not Radical Equations
$\sqrt{x+3} + 1 = 4$	$x - \sqrt{5} = 8$
$\sqrt[3]{2x} + 5 = \sqrt[3]{x-1}$	$\sqrt[4]{25} = x - 3$

Math Reasoning

Analyze Does cubing each side of $x = 5$ result in extraneous solutions? Explain.

To solve a radical equation, first isolate the radical on one side of the equation. Then raise both sides of the equation to the power of the index of the radical. This will clear the radical.

It is critical that the solutions of a radical equation be checked because some of the solutions obtained may be extraneous solutions. An extraneous solution is a solution of a derived equation that is not a solution of the original equation.

For example, the equation $x = 5$ has one solution, 5. But when both sides of the equation are squared, the resulting equation $x^2 = 25$ has two solutions, -5 and 5. The solution -5 is extraneous because it is not a solution of the original equation.

Example 1 Solving Equations Containing One Square Root

Solve each equation.

a. $\sqrt{x-5} = 14$

SOLUTION The radical is already isolated on the left side.

$$\sqrt{x-5} = 14$$

$$\left(\sqrt{x-5}\right)^2 = 14^2 \qquad \text{Square both sides of the equation.}$$

$$x - 5 = 196 \qquad \text{Simplify.}$$

$$x = 201 \qquad \text{Add 5 to both sides.}$$

Check $\sqrt{x-5} = 14$

$$\sqrt{201-5} \stackrel{?}{=} 14$$

$$\sqrt{196} \stackrel{?}{=} 14$$

$$14 = 14 \quad \checkmark$$

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b. $\sqrt{6x+7}+3=9$

SOLUTION First isolate the radical on the left.

$$\sqrt{6x+7}+3=9$$

$\sqrt{6x+7}=6$	Subtract 3 from both sides of the equation.
$\left(\sqrt{6x+7}\right)^2=6^2$	Square both sides.
$6x+7=36$	Simplify.
$6x=29$	Subtract 7 from both sides.
$x=\frac{29}{6}$	Divide both sides by 6.

Check $\sqrt{6x+7}+3=9$

$$\sqrt{6\left(\frac{29}{6}\right)+7}+3\stackrel{?}{=}9$$

$$\sqrt{36}+3\stackrel{?}{=}9$$

$$9=9 \quad \checkmark$$

Example 2 Solving Equations Containing One Cube Root

Solve each equation.

a. $\sqrt[3]{4x+4}-6=0$

SOLUTION First isolate the radical.

$$\sqrt[3]{4x+4}-6=0$$

$\sqrt[3]{4x+4}=6$	Add 6 to each side.
$\left(\sqrt[3]{4x+4}\right)^3=6^3$	Cube both sides to clear the cube root.
$4x+4=216$	Simplify.
$4x=212$	Subtract 4 from each side.
$x=53$	Divide both sides by 4.

Check $\sqrt[3]{4x+4}-6=0$

$$\sqrt[3]{4(53)+4}-6\stackrel{?}{=}0$$

$$\sqrt[3]{216}-6\stackrel{?}{=}0$$

$$6-6\stackrel{?}{=}0$$

$$0=0 \quad \checkmark$$

b. $-2\sqrt[3]{x-8}=14$

SOLUTION Isolate the radical by dividing.

$$-2\sqrt[3]{x-8}=14$$

$\sqrt[3]{x-8}=-7$	Divide both sides by −2.
$\left(\sqrt[3]{x-8}\right)^3=(-7)^3$	Cube both sides.
$x-8=-343$	Simplify.
$x=-335$	Add 8 to each side.

Math Reasoning

Verify Check that -335 is a solution of $-2\sqrt[3]{x-8}=14$.

Math Language

Indices is the plural of index.

Sometimes an equation has two radicals. If their indices are the same and each radical is on a different side of the equation, it is simplest to raise each side of the equation to the power of the index.

If the indices on the radicals differ, the solution can be estimated by using a graphing calculator. Graphing calculators are also very useful in checking solutions.

Example 3 Solving Equations Containing Two Radicals

Solve each equation. Check the solutions graphically.

a. $\sqrt{8x - 15} = \sqrt{4x + 5}$

SOLUTION $\sqrt{8x - 15} = \sqrt{4x + 5}$

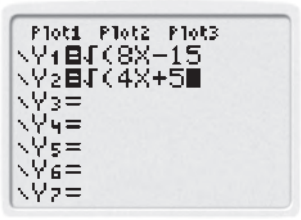
$$\left(\sqrt{8x - 15}\right)^2 = \left(\sqrt{4x + 5}\right)^2 \quad \text{Square both sides.}$$

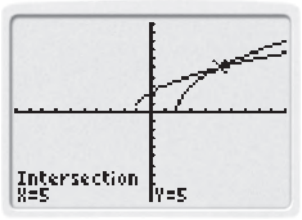
$$8x - 15 = 4x + 5 \quad \text{Simplify.}$$

$$4x = 20 \quad \text{Isolate } x.$$

$$x = 5$$

Check Graph Y1 $= \sqrt{8x - 15}$ and Y2 $= \sqrt{4x + 5}$. The intersection point is the solution. Use the **intersect** command located in the **Calculate** (2nd Trace) menu. The curves intersect at $x = 5$.





b. $\sqrt[3]{3x} = 2\sqrt[3]{x - 1}$

SOLUTION $\sqrt[3]{3x} = 2\sqrt[3]{x - 1}$

$$\left(\sqrt[3]{3x}\right)^3 = \left(2\sqrt[3]{x - 1}\right)^3 \quad \text{Cube both sides.}$$

$$3x = 8(x - 1) \quad \text{Simplify.}$$

$$3x = 8x - 8 \quad \text{Distribute 8.}$$

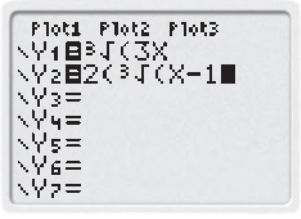
$$-5x = -8 \quad \text{Isolate } x.$$

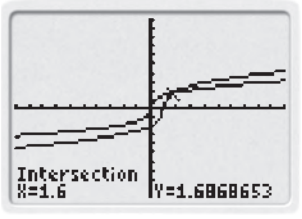
$$x = \frac{8}{5} = 1.6$$

Graphing Calculator Tips

To type the cube root symbol, press **Math** and choose it from the items in the Math list.

Check Graph Y1 $= \sqrt[3]{3x}$ and Y2 $= 2\sqrt[3]{x - 1}$. The curves intersect at $x = 1.6$.





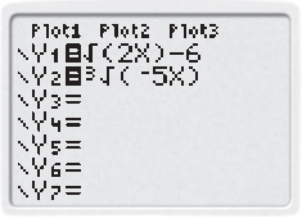
c. $\sqrt{2x} - \sqrt[3]{-5x} - 6 = 0$

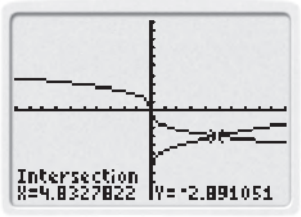
SOLUTION Move one of the radicals to the right side. Then use a graphing calculator.

$$\sqrt{2x} - \sqrt[3]{-5x} - 6 = 0$$

$$\sqrt{2x} - 6 = \sqrt[3]{-5x} \qquad \text{Add } \sqrt[3]{-5x} \text{ to each side.}$$

Graph Y1 $= \sqrt{2x} - 6$ and Y2 $= \sqrt[3]{-5x}$. The curves intersect at $x \approx 4.83$.





Example 4 Solving Equations with Extraneous Solutions

Solve each equation.

Math Reasoning

Analyze How can you see in the first step of 4a that the equation has no solutions?

a. $\sqrt{x} - 4 = -8$

SOLUTION $\sqrt{x} - 4 = -8$

$$\sqrt{x} = -4 \qquad \text{Add 4 to each side to isolate the radical.}$$

$$(\sqrt{x})^2 = (-4)^2 \qquad \text{Square both sides.}$$

$$x = 16 \qquad \text{Simplify.}$$

Check $\sqrt{x} - 4 = -8$

$$\sqrt{16} - 4 \stackrel{?}{=} -8$$

$$4 - 4 \stackrel{?}{=} -8$$

$$0 \neq -8$$

The solution of 16 is extraneous. The equation has no solutions.

b. $x + 1 = \sqrt{2x + 10}$

SOLUTION $x + 1 = \sqrt{2x + 10}$ The radical is isolated on the right.

$$(x + 1)^2 = \left(\sqrt{2x + 10}\right)^2 \qquad \text{Square both sides.}$$

$$x^2 + 2x + 1 = 2x + 10 \qquad \text{Simplify.}$$

$$x^2 - 9 = 0 \qquad \text{Solve the quadratic equation.}$$

$$x^2 = 9$$

$$x = \pm 3$$

Check $x + 1 = \sqrt{2x + 10}$

$$3 + 1 \stackrel{?}{=} \sqrt{2(3) + 10}$$

$$4 \stackrel{?}{=} \sqrt{16}$$

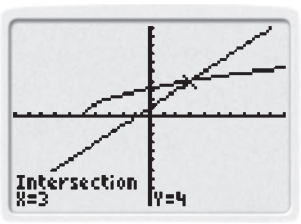
$$4 = 4 \quad \checkmark$$

$x + 1 = \sqrt{2x + 10}$

$$-3 + 1 \stackrel{?}{=} \sqrt{2(-3) + 10}$$

$$-2 \stackrel{?}{=} \sqrt{4}$$

$$-2 \neq 2$$



The solution of -3 is extraneous. The only solution is 3.

Example 5 Application: Physics

The period of a pendulum, T, is the time it takes for the pendulum to make one back-and-forth swing, and is given by $T = 2\pi\sqrt{\frac{L}{9.8}}$, where L is the length of the pendulum in meters. Find the length of a pendulum whose period is 3 seconds.

SOLUTION Substitute 3 for T in the formula and solve for L.

$$3 = 2\pi\sqrt{\frac{L}{9.8}}$$

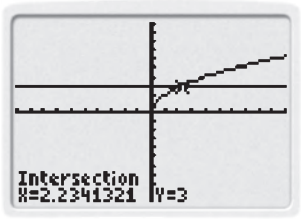
$$\frac{3}{2\pi} = \sqrt{\frac{L}{9.8}} \qquad \text{Isolate the variable.}$$

$$\left(\frac{3}{2\pi}\right)^2 = \left(\sqrt{\frac{L}{9.8}}\right)^2 \qquad \text{Square both sides.}$$

$$\frac{9}{4\pi^2} = \frac{L}{9.8} \qquad \text{Simplify.}$$

$$9.8\left(\frac{9}{4\pi^2}\right) = L \qquad \text{Isolate } L.$$

$$2.23 \approx L$$



The length of the pendulum is about 2.23 meters.

Lesson Practice

Solve each equation.

a. (Ex 1) $\sqrt{2x+3} = 21$

b. (Ex 1) $11 = \sqrt{x-7} - 2$

c. (Ex 2) $\sqrt[3]{2x} + 1 = 0$

d. (Ex 2) $4\sqrt[3]{3x-4} = 16$

e. (Ex 3) $\sqrt{9x} = \sqrt{5x+2}$

f. (Ex 3) $-\sqrt[3]{2x} = \sqrt[3]{4x+7}$

g. (Ex 3) $\sqrt{3x} + \sqrt[3]{x-4} - 5 = 0$

h. (Ex 4) $\sqrt{-1-4x} = \sqrt{x-11}$

i. (Ex 4) $x = \sqrt{6-5x}$

j. (Ex 5) The period of a pendulum, T, is the time it takes for the pendulum to make one back-and-forth swing, and is given by $T = 2\pi\sqrt{\frac{L}{9.8}}$, where L is the length of the pendulum in meters. Find the length of a pendulum whose period is 5 seconds.

Practice Distributed and Integrated

Solve the equations. Write the solutions in the form $a + bi$.

1. (62) $3p^2 + 7 = -9p^2 + 4$

2. (62) $2x^2 + 31 = 9$

***3.** (66) Find the roots and give the multiplicity of each root: $x^3 - 7x^2 + 11x - 5 = 0$

4. (53) Let $f(x) = 3x + 1$ and $g(x) = x - 2$. Find the composite function $f(g(x))$.

5. (60) Of the 220 people who came into the Italian deli on Friday, 104 bought pizza and 82 used a credit card. Half of the people who bought pizza used a credit card. What is the probability that a customer bought pizza or used a credit card?

Solve for x by completing the square.

6. (58) $x^2 = 9x - 7$

7. (58) $-5x - 6 = -x^2$

***8.** (70) **Lighthouses** The number of miles m a person can see on a clear day is approximated by $m = 1.2116\sqrt{h}$, where h is the person's height in feet above sea level. If a person standing at the top of the Barnegat Lighthouse in New Jersey on a clear day can see a distance of 16 miles, what is the person's height above sea level?

9. (55) **Error Analysis** The odds in favor of an event are 9 to 2. To the nearest percent, what is the probability of the event?

10. (61) **Multiple Choice** What is $P(-3)$ if $x + 3$ is a factor of $2x^3 + 2x^2 - 10x + 6$?

A -3 **B** 0 **C** 3 **D** 6

11. (59) **Velocity of an Electron** The velocity at which an electron orbits the nucleus of an atom is known as the Fermi-Thomas velocity and can be found using the expression $\frac{Z^{\frac{2}{3}}}{137}c$, where Z is the number of protons in the nucleus and c is the speed of light. What is the Fermi-Thomas velocity for cobalt, which has 27 protons, in terms of c? Round to the nearest thousandths place.

***12.** (70) **Write** What is an extraneous solution? When can it occur?

13. (63) **Estimate** Use what you know about the unit circle and special angles to estimate $\tan 43°$. Do not use a calculator. Explain your method.

***14.** (69) **Graphing Calculator** With your calculator in **a + bi** mode, find the following.

a. $(64 - 82i)(26 + 31i)$ **b.** $-3.7i(2.25 + 4.5i)$

15. (62) **Electrical Engineering** A circuit has a current of 5 amps, and a second circuit has a current of $(14 - 9i)$ amps. Find the sum of the currents.

16. (54) **Temporary Employees** A department store is planning to hire up to 24 temporary employees for a tent sale. Experienced workers will be paid \$20 per hour and inexperienced workers will be paid \$15 per hour. The company can pay up to \$400 per hour for the temporary employees. Experienced workers produce 1.5 times more profit than inexperienced workers. How many of each type of worker should be hired to maximize the company's profits?

***17.** (69) **Multi-Step** The product of a number and its multiplicative inverse is 1.

a. State the multiplicative inverse of $2 + 3i$.

b. Simplify the multiplicative inverse.

c. Verify Show that your answer in part b is the multiplicative inverse by multiplying it by $2 + 3i$.

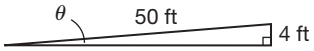
Factor each expression completely.

18. (61) $8c^3 + 343$

19. (61) $-5z^3 + 320$

20. (66) **Generalize** How does the Rational Root Theorem make factoring polynomials easier? What must be true about a polynomial before it used?

***21.** (67) **Wheelchair Accessibility** A contractor designed the wheelchair ramp shown in the diagram. The ramp must satisfy a building code that specifies 4.7° as the maximum angle of elevation, labeled θ in the diagram. Does the design satisfy the building code? Justify your answer.



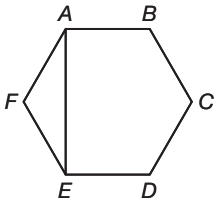
22. (50) Find an equation for the inverse of $y = 5x^3 - 20$. Determine whether each relation is a function.

23. (65) **Verify** A student used factoring to find that the solutions of $4x^2 - 11x - 20 = 0$ are -1.25 and 4. Show how to verify these solutions with the Quadratic Formula.

24. (54) Maximize $P = 4.5x + 2.4y$ for the constraints $\begin{array}{l} y \geq 0 \\ x \geq 0 \\ y \geq x \\ y \leq -2x + 9 \end{array}$

25. (64) **Error Analysis** A student converted the exponential equation $4^{-2} = \frac{1}{16}$ to logarithmic form by writing $\log_4(-2) = \frac{1}{16}$. What is the error? Write the correct logarithmic equation.

26. (52) **Geometry** Each side of the regular hexagon shown is 7 cm. Find the length of the diagonal $\overline{AE}$. Round your answer to the nearest tenth. (The measure of one interior angle of an n-gon is $\frac{(n-2)180^\circ}{n}$.)



27. (60) **Write** Examine the formula for the probability of two inclusive events. Explain why $P(A$ and $B)$ is subtracted from the formula.

$$P(A \textbf{ or } B) = P(A) + P(B) - P(A \textbf{ and } B)$$

28. (53) **Multi-Step** A clothing store has a sale in which all the shirts are 20% off. On the last day of the sale, all shirts have an additional 25% off. Find a function describing how much a shirt with price p will cost on the last day. Solve the function.

***29.** (70) **Probability** A number cube is rolled twice. What is the probability that the number facing up on each roll shows a solution of $\sqrt{2x - 8} = x - 4$?

***30.** (68) **Multiple Choice** A bag of grapes contains 9 green grapes and 15 red grapes. If 4 grapes are chosen at random, without replacement, what is the approximate probability that the first two grapes are green and the last two grapes are red?

A 4.6% **B** 5.9% **C** 7.1% **D** 13%

INVESTIGATION

7

Collecting Data

Exercise Suppose you want to find out what students at your school do for exercise, and then make a graph to display the results.

1. You construct this survey question: "What do you do for exercise?" Think about some possible responses. How many different responses do you think you would get if you asked 100 students?
2. Is "dance" a response you thought of for question 1? Is "skateboarding" or "martial arts" a response you thought of?
3. Is it possible that some students do more than one type of exercise?

The answers to survey questions are data items, so collecting the answers is a form of collecting data. When constructing a survey question, it is often important to limit the number of possible responses so that you will be able to organize, display, analyze, and interpret the data you collect. Provide enough response choices so that there is a choice for everyone, and make sure the choices don't overlap. That is, construct them so that there is clearly only one response. In the table below, the question about exercise is rephrased and answer choices are supplied, along with numbers of responses.

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Math Reasoning

Analyze Refer to the table. Which choice is needed so that everyone can choose at *least one* response? What word(s) in the question allows everyone to be able to choose *exactly one* response?

"Which of the following best describes your main type of exercise?"

Response	Frequency
A. Competitive sports (team and individual)	24
B. Run, jog, or walk (not competitive)	16
C. Swim or bicycle (not competitive)	5
D. Strength training (weights and exercise equipment)	15
E. Other	20

A bar graph and a circle graph are both appropriate ways to display this data. Copy and complete the bar graph and circle graph below.

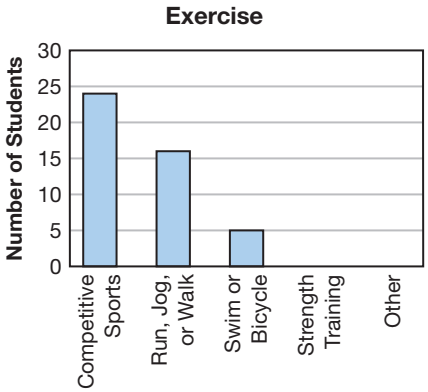
Hint

24 of 80 responses were *Competitive*. To get the central angle measure for the *Competitve* sector in the circle graph, multiply 360° by the appropriate decimal.

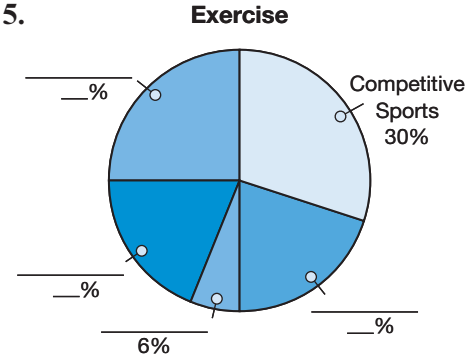
$$\frac{24}{80} = 0.30$$

$$0.30 \times 360° = 108°$$

4.



5.



Analyze and Write A survey question is *biased* if there is a reasonable chance that it could influence a response because of the way it is constructed. Determine whether each question below is biased. Explain your answer. If the question is biased, rewrite it so that it is not biased.

6. Do you agree with most high school students that parents give them too little freedom to make their own decisions?

7. Do you support the unfair policy of requiring community service as a high school requirement?

8. Do you think energy conservation should be a high priority for state and national elected officials?

9. Write a biased survey question. Then rewrite it to remove the bias.

A *population* is a group being studied. Populations are often too large to survey, so a subset of the population called a *sample* is surveyed. If a sample is not a fair representation of the population, it is biased. If a sample is a fair representation of the population, it is unbiased. When properly chosen in an appropriate situation, any of the following can be an unbiased sample.

- *Random sample* – Each member of the population has an equal chance of being selected.
- *Systematic sample* – Members are chosen based on a rule.
- *Stratified sample* – The population is divided into non-overlapping subgroups, and members are chosen at random from the subgroups.
- *Self-selected sample* – Members volunteer to be in the sample.

Analyze and Write Identify the type of sample and determine whether it is biased. Explain your answer.

10. A manager asks every tenth customer who enters her store whether they like the recent remodeling of the store.

11. A candidate mails a survey about an upcoming election along with a request for campaign contributions, enclosing a return envelope for responses.

12. For a survey about school policies, a student randomly chooses 25 students from each grade. (There are approximately 200 students in each grade.)

13. There are 645 employees at a company. The president assigns the integers 1 through 645, one to each employee, and then randomly chooses 50 integers and surveys the corresponding employees.

14. Every voter in a town is registered as a Democrat, a Republican, or an Independent. For a survey of registered voters, 50 are randomly chosen from each group.

15. **Multi-Step** Complete parts a–d.

a. Construct a survey question with 4 or 5 response choices.

b. Identify a population. Describe 3 or 4 possible samples of different types. Identify each type and state whether it is biased or unbiased.

Math Reasoning

Analyze A bar graph and circle graph both show relative sizes of categories. Which type better shows how the size of each category compares to the whole?

c. Construct a frequency table. Show a reasonable distribution of responses.

d. Construct a bar graph and circle graph to show the data.

It is useful to describe some data sets with *measures of center* such as *mode, median,* and *mean,* and with *measures of spread* such as *variance* and *standard deviation.*

A teacher chose 2 students at random from each of the 10 sophomore homeroom classes in a school and asked "To the nearest half hour, what is the average number of hours per school night that you spend on homework?" The table below shows the data distribution.

Hours	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0 or more
Frequency	0	1	4	5	5	3	1	1	0

Hint

To find the median, order the 20 data values from least to greatest, and then find the average of the 10th and 11th values.

16. Identify the population and the type of sample. Determine whether the sample is biased.

17. Find the mode(s) and median.

18. Supply the missing terms in the numerator, and then evaluate the expression to find the mean: $\overline{x} = \frac{1(0.5) + 4(1.0) + 5(1.5) + \ldots + 1(3.5)}{20}$.

Reading Math

The symbol for standard deviation is σ, the Greek lower case letter *sigma*.

19. The formula for variance is $\sigma^2 = \frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \ldots + 1(x_n - \overline{x})^2}{n}$, where $x_1, x_2, \ldots, x_n$ are the data values and $\overline{x}$ is the mean. Supply the missing terms in the numerator, and then evaluate the expression to find the variance: $\sigma^2 = \frac{1(0.5 - \overline{x})^2 + 4(1.0 - \overline{x})^2 + 5(1.5 - \overline{x})^2 + \ldots + 1(3.5 - \overline{x})^2}{20}$

20. The standard deviation is the square root of the variance: $\sigma = \sqrt{\sigma^2}$. Find the standard deviation to the nearest thousandth.

Investigation Practice

a. **Model** Students were asked the survey question, "On average, how many hours per week do you email or use the Internet? Round to the nearest hour." The frequency table below shows the results. Construct a histogram and a circle graph to show the data. (A histogram is a bar graph in which adjacent bars touch.)

Response	0–2	3–5	6–8	9–11	12 or more
Frequency	4	12	22	7	5

b. **Analyze** Suppose the instruction "Round to the nearest hour" were not included in the survey question in part a. Why might it be difficult for a student to choose a response?

c. **Analyze** Suppose the response choices were 0–3, 3–6, 6–9, 9–12, and 12 or more. Why might it be difficult to interpret a histogram showing the results?

Determine whether each question below is biased. Explain your answer. If the question is biased, rewrite it so that it is not biased.

d. How many people live in your household?

e. Knowing that teenagers tend to be irresponsible, should people younger than 18 be allowed to drive without an adult in the vehicle?

f. Knowing that the current sales tax rate is 5%, do you support a new rate of 6%, beginning next year?

g. There are some scientists who believe that recent climate changes are part of a natural long-term climate cycle rather than global warming caused by humans. Do you believe that global warming is a serious problem?

Identify the type of sample and determine whether it is biased. Explain your answer.

h. A utility company mails a survey to all its customers asking them to rate customer service, enclosing a return envelope for responses.

i. A developer wants to build a store. Town council members want to know if town residents approve. They survey all residents that live within a half mile of the proposed site.

j. A school principal selects 2 students at random from each class in the school to survey about students' study habits.

k. The junior class president wants opinions on student parking rules. She uses a list of all 255 juniors, and assigns each junior a different integer from 1 through 255. She uses a random integer generator on a calculator until she gets 20 different integers from 1 through 255. She chooses the 20 corresponding students to survey.

Six students were chosen at random from each grade in a high school and asked "At what age did you have your first job?" The table below shows the data distribution.

Age	13	14	15	16	17	18	19
Frequency	1	4	8	7	3	0	1

l. Identify the population and the type of sample. Determine whether the sample is biased.

m. Find the mode, median, and mean.

n. Find the variance.

o. Find the standard deviation.

p. Construct your own survey question. Identify the population and the type of sample. Determine whether the sample is biased. Find the mode(s), median, and mean. Find the variance and standard deviation.

LESSON 71

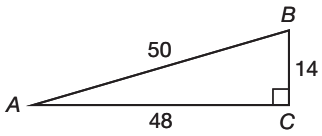
Using the Law of Sines

Warm Up

1. (46) **Vocabulary** Given an acute angle of a right triangle, the ratio of the opposite side and the hypotenuse is the _____ **(*sinelcosine*)** of the angle.

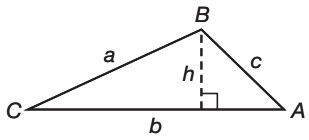
2. (46) Find the value of the sine of $\angle A$.

3. (67) Estimate $m\angle x$ if $\sin x \approx 0.438$.



New Concepts

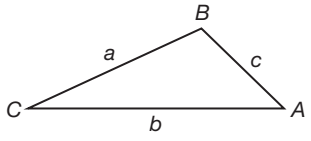
In $\triangle ABC$ at right, $\sin A = \frac{h}{c}$. To solve this equation for h, multiply both sides by c: $c \sin A = h$.



The area of $\triangle ABC$ is given by $\frac{1}{2}bh$. By substitution, it can be written as $\frac{1}{2}b(c \sin A)$. This means that the area of a triangle can be found when the height is not known, but the lengths of any two sides and their included angle are known.

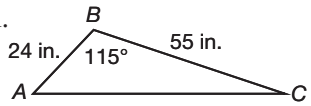
Math Language

The **included angle** between two sides is the angle formed by the intersection of those sides.

Area of a Triangle	
$\text{Area} = \frac{1}{2}bc \sin A$ $\text{Area} = \frac{1}{2}ac \sin B$ $\text{Area} = \frac{1}{2}ab \sin C$	

Example 1 Finding the Area of a Triangle with Unknown Height

Find the area of $\triangle ABC$. Round to the nearest tenth.



SOLUTION Because a and c are the known sides, use

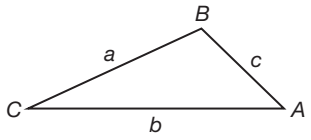
$$\text{Area} = \frac{1}{2}ac \sin B.$$

$$\text{Area} = \frac{1}{2}(55)(24) \sin 115° \qquad \text{Substitute known values.}$$

$$\text{Area} \approx 598.1631394 \qquad \text{Simplify the right side.}$$

The area of the triangle is about 598.2 square inches.

Based on which measurements are known, any of the three formulas can be used to find the area of a given triangle, so $\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$. Simplifying these equations results in the **Law of Sines.**

Law of Sines	
By the Law of Sines, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	

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By the Law of Sines, any of the following equations are true: $\frac{\sin A}{a} = \frac{\sin B}{b}$, $\frac{\sin B}{b} = \frac{\sin C}{c}$, and $\frac{\sin A}{a} = \frac{\sin C}{c}$. These equations can be used to find a missing angle or side measure in a triangle.

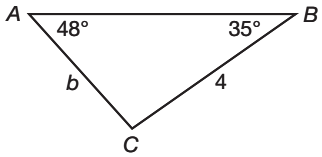
The Law of Sines can be used when …

- the measures of two angles and any side of a triangle are known (angle-angle-side (AAS) or angle-side-angle (ASA))
- the measures of two sides and an angle opposite one of those sides is known (side-side-angle (SSA)).

Sometimes, a problem asks for a triangle to be solved. This means to find all the missing angle and side measures.

Example 2 Finding the Unknown Side of a Triangle

a. Find b. Round to the nearest tenth.



SOLUTION Use $\frac{\sin A}{a} = \frac{\sin B}{b}$ because all but one of the terms in this equation are known.

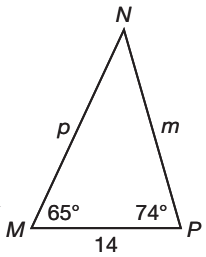
$$\frac{\sin 48°}{4} = \frac{\sin 35°}{b}$$ Substitute known values.

$$b \sin 48° = 4 \sin 35°$$ Cross multiply.

$$b = \frac{4 \sin 35°}{\sin 48°}$$ Divide both sides by sin 48°.

$$b \approx 3.1$$ Use a calculator to simplify.

b. Solve $\triangle MNP$.



SOLUTION Find $m\angle N$ by subtracting the sum of $m\angle M$ and $m\angle P$ from 180°. $m\angle N = 180° - (65° + 74°) = 41°$.

Use $\frac{\sin M}{m} = \frac{\sin N}{n}$ to find m.

$$\frac{\sin 65°}{m} = \frac{\sin 41°}{14}$$

$$m \sin 41° = 14 \sin 65°$$

$$m = \frac{14 \sin 65°}{\sin 41°}$$

$$m \approx 19.3$$

Use $\frac{\sin P}{p} = \frac{\sin N}{n}$ to find p.

$$\frac{\sin 74°}{p} = \frac{\sin 41°}{14}$$

$$p \sin 41° = 14 \sin 74°$$

$$p = \frac{14 \sin 74°}{\sin 41°}$$

$$p \approx 20.5$$

Math Reasoning

Analyze Why is it not possible to use a sine, cosine, or tangent ratio to find a missing side length in Example 2A?

Math Reasoning

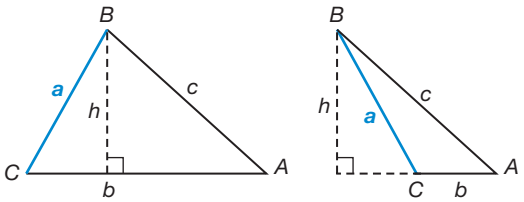
Generalize Without using variables, state what must be true for there to be two triangles given SSA.

When the given information consists of two sides and a nonincluded angle (SSA), it may be possible that the information describes two different triangles.

Possible Triangles

Given a, c, and $m\angle A$ in $\triangle ABC$, if $h < a < c$, then there are two possible triangles where:

- $h = c \sin A$
- the sum of the measures of $\angle C$ in each triangle is 180°.



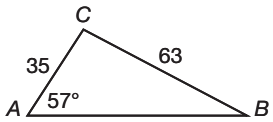
When using the Law of Sines if the side opposite the given angle is less than the other given side, the sides must be compared to the height to determine the number of possible triangles.

Example 3 Finding the Unknown Angle of a Triangle

Hint

Unless otherwise stated, do not assume that diagrams are drawn to scale.

a. Find $m\angle B$. Round to the nearest tenth.



SOLUTION Since $63 > 35$, there is only one triangle with these measures.

Use $\frac{\sin A}{a} = \frac{\sin B}{b}$.

$\frac{\sin 57°}{63} = \frac{\sin B}{35}$ Substitute the known values.

$63 \sin B = 35 \sin 57°$ Cross multiply.

$\sin B = \frac{35 \sin 57°}{63}$ Divide both sides by 63.

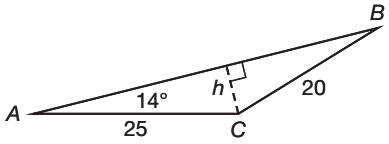
$m\angle B \approx 27.8°$ Use $\text{Sin}^{-1}x$.

b. Given $\triangle ABC$, $m\angle A = 14°$, $a = 20$, and $b = 25$, find $m\angle B$. Round to the nearest tenth.

Hint

A triangle has three heights. Use the height from the vertex of the included angle of the given sides when comparing *a*, *b*, and *h*.

SOLUTION Sketch the triangle.
The side opposite the given angle is less than the other given side. Find *h*:
$h = 25\sin 14° \approx 6$.



Because $6 < 20 < 25$, there are two possible triangles.

Use $\frac{\sin A}{a} = \frac{\sin B}{b}$.

$\frac{\sin 14°}{20} = \frac{\sin B}{25}$ Substitute the known values.

$20 \sin B = 25 \sin 14°$ Cross multiply.

$\sin B = \frac{25 \sin 14°}{20}$ Divide both sides by 20.

$m\angle B \approx 17.6°$ Use $\text{Sin}^{-1}x$.

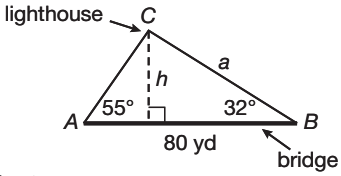
Subtract from 180° to find the other possible value of $m\angle B$:
$180° - 17.6° = 162.4°$. So $m\angle B \approx 17.6°$ or $\approx 162.4°$.

The process of determining a location or distance of an object by sighting it from two fixed points a known distance apart is called triangulation. Triangulation can be used to solve many real word problems.

Example 4 Application: Surveying

A surveyor stands on the east end of a bridge and finds that the angle formed by the bridge and the line of sight with a lighthouse is 32°. The surveyor then walks to the west end and finds the angle formed by the bridge and lighthouse to be 55°. Estimate the distance from the bridge to the lighthouse, given that the bridge has a length of 80 yards.

SOLUTION Draw a diagram of the situation. The distance from the bridge to the lighthouse is a height of the triangle.



Use the Law of Sines to find a. Then find h.

Find $m\angle C$ by subtracting from 180°. Then use $\frac{\sin A}{a} = \frac{\sin C}{c}$.

$$\frac{\sin 55°}{a} = \frac{\sin 93°}{80} \rightarrow a \sin 93° = 80 \sin 55° \rightarrow a = \frac{80 \sin 55°}{\sin 93°} \approx 65.62$$

$$\sin 32° = \frac{h}{65.62} \rightarrow h = 65.62 \sin 32° \approx 34.8$$

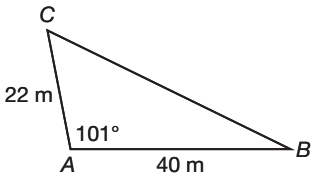
The distance is about 34.8 yards.

Check Use the tangent ratio on the two right triangles:

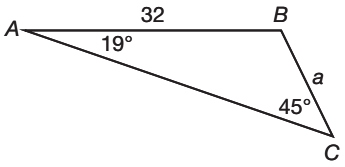
$$\tan 32° = \frac{34.8}{x} \rightarrow x \approx 56, \tan 55° = \frac{34.8}{y} \rightarrow y \approx 24, \text{ and } 56 + 24 = 80.$$

Lesson Practice

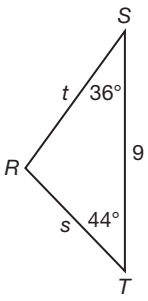
a. *(Ex 1)* Find the area of $\triangle ABC$. Round to the nearest tenth.



b. *(Ex 2)* Find a. Round to the nearest tenth.



c. *(Ex 2)* Solve $\triangle RST$.



d. *(Ex 3)* Find $m\angle A$. Round to the nearest tenth.

e. *(Ex 3)* Given $\triangle ABC$, $m\angle A = 63°$, $a = 38$, and $c = 40$, find $m\angle C$. Round to the nearest tenth.

f. *(Ex 4)* A surveyor stands on the south end of a roof and finds that the angle formed by the roof and the line of sight with a flagpole is 38°. The surveyor then walks to the north end of the roof and finds the angle formed by the roof and flagpole to be 64°. Estimate the distance from the roof to the flagpole, given that the roof has a length of 225 feet.

Practice Distributed and Integrated

Solve.

1. (66) $3x(x^2 + 7x - 25) = 0$

2. (66) $4x(2x^2 + 9x - 45) = 0$

Use synthetic division to divide.

3. (51) $5x^3 - 10x^2 + 20x - 15$ by $x - 1$

4. (51) $4x^5 - 129$ by $4x - 8$

5. (58) **Write** The square root property states: If $x^2 = a$, then $x = \pm\sqrt{a}$. Explain why x must be positive or negative.

6. (50) Find an equation for the inverse of $y = -3x$. Identify the domain and range of the inverse function.

7. (61) **Coordinate Geometry** Because $x - a$ is a factor of $P(x)$ when $P(a) = 0$, a is an x-intercept of the graph of $P(x)$. Which of the following ordered pairs must be on the graph of $P(x) = x^3 - 7x^2 + 36$?

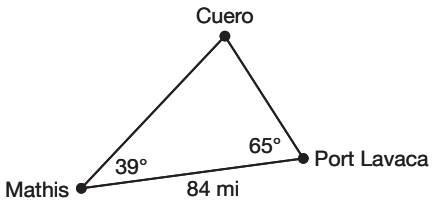
$(-6, 0)$ $(-4, 0)$ $(-2, 0)$ $(1, 0)$ $(3, 0)$ $(6, 0)$

8. (54) **Flowers** Jay is planting a maximum of 40 bulbs of lilies and tulips in her garden. She wants more tulips, x, than lilies, y. Write a system of inequalities and find the maximum number of lily bulbs Jay can plant.

9. (59) **Multiple Choice** Choose the letter that best represents $\sqrt[4]{625x^{28}}$ simplified.

A $5x^7$ **B** $25x^{14}$ **C** $25x^7$ **D** $5x^{14}$

***10.** (71) **Geography** On a map, line segments connecting three southeast Texas cities form a triangle as shown to the right. To the nearest mile, what is the distance from Mathis to Cuero?



11. (65) **Write** Why must you write a quadratic equation in standard form before using the Quadratic Formula?

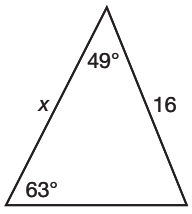
12. (52) Angle θ is an acute angle of a right triangle and $\cos\theta = \frac{7}{11}$. Find the sine of angle θ.

13. (66) **Analyze** $3 - 6\sqrt{2}$ is a root of a polynomial equation. What must be another root of the equation? Why?

***14.** (71) **Error Analysis** Explain and correct the error a student made in finding the value of x.

$$\frac{\sin 63°}{16} = \frac{\sin 49°}{x}$$

$$x = \frac{16 \sin 49°}{\sin 63°} \approx 13.55$$



*15. (68) **Quality Control** In a factory, a sample of items is checked during all three work shifts. During the first shift, 2 out of 11 items were found defective. During the second shift, 0 out of 13 items were found defective. During the third shift, 4 out of 20 items were found defective. What is the probability that an item was checked during the first shift, given that is was not defective?

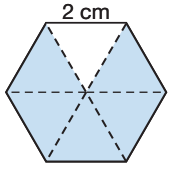
*16. (70) **Graphing Calculator** Estimate the solution of $\sqrt{3x+1} - 4 + \sqrt[3]{x} = 0$ to the nearest hundredth.

17. (57) **Formulate** The half-life of Thorium-229 is 7340 years. Formulate an exponential function representing the amount of a grams of Thorium-229 remaining after t years. Use the function to find how much of a 200 gram sample of Thorium-229 will be left after 10,000 years.

*18. (63) **Error Analysis** To find the tangent of 300°, a student used its reference angle as follows: $\tan 60° = \sqrt{3}$, therefore $\tan 300° = \sqrt{3}$. What is the error? Find the correct value of $\tan 300°$.

19. (53) Let $f(x) = 5x^2$ and $g(x) = -4x$. Find the composite function $f(g(x))$.

20. (52) **Geometry** Find the area of the shaded region of the regular hexagon.



21. (54) Maximize $P = 2x + y$ for the constraints $x \geq 0$, $y \geq 0$, $x + y \leq 6$

*22. (70) **Analyze** Explain why $x - \sqrt{5} = 2x$ is not a radical equation.

23. (38) Use long division to divide $-x^4 - 7x^3 + 6x^2 - 1$ by $x - 3$.

*24. (69) **Multi-Step** Simplify $\frac{4 - 8i + 2 + i}{7 + 3i + 2i}$. Write the answer in the form $a + bi$.

25. (Inv 7) **Multi-Step** A dietician asked 26 high school students how often they eat breakfast. The students eat breakfast a mean of 2.5 days per week and there is a standard deviation of 1.36.

a. If Stephanie eats breakfast 5 days per week, how many standard deviations above the mean is she?

b. If Darryl eats breakfast 1 day per week, how many standard deviations below the mean is he?

c. Who is closest to the sample mean?

26. (62) **Bay Bridge** A sailor in a boat beneath the San Francisco Bay Bridge shoots a flare straight up. The height of the flare after t seconds is modeled by $-16t^2 + 96t$. The bridge is 220 feet above the water and the sailor is 12 feet above the water. The solutions of $-16t^2 + 96t = 208$ give the number of seconds it will take for the flare to reach the bridge. Will the flare reach the bridge? If so, when?

*27. (71) **Multiple Choice** What is the area of $\triangle ABC$ if $a = 7$ in., $b = 9$ in., and $m\angle C = 23°$?
A about 12.3 in^2 **B** about 24.6 in^2 **C** about 29 in^2 **D** about 49.2 in^2

Simplify.

*28. (69) $\frac{2 + 5i}{i}$

*29. (69) i^7

30. (59) $49^{\frac{5}{12}} \cdot 49^{\frac{1}{12}}$

LESSON
72

Using the Properties of Logarithms

Warm Up

1. **Vocabulary** In the expression $\log_5 20$, the number 5 is the _____. (64)

2. Evaluate $(2^3)^3$. (3)

3. Simplify $2^x \cdot 2^{(x+y)}$. (3)

New Concepts

Since a logarithm is the exponent applied to a base to obtain a given value, $b^x = a$ is equivalent to $\log_b a = x$.

The definition of a logarithm can be used to find the inverse of an exponential function.

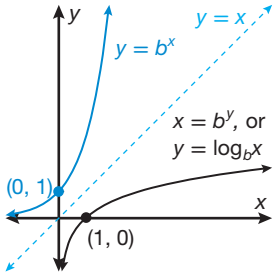
$y = b^x$ The exponential function with base b

$x = b^y$ Interchange the variables to find the inverse.

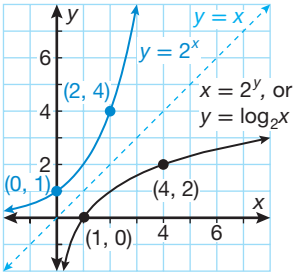
$y = \log_b x$ Write $x = b^y$ in its logarithmic form.

Inverse functions

The graphs of inverse functions are reflective images of each other with respect to line $y = x$. The parent function $y = b^x$ for $b > 1$ is shown below, along with its inverse function $y = \log_b x$.

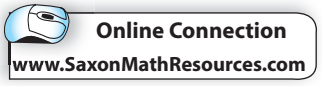


The value of b determines the exact shape of the graphs. The function graphs for $b = 2$ are shown below.



The definition of logarithm can also be used to obtain the following rules.

Inverse Properties of Logarithms
$b^{\log_b x} = x$ and $\log_b b^x = x$



These rules can be used to simplify exponential or logarithmic expressions.

Example 1 Simplifying Exponential and Logarithmic Expressions

a. Simplify $\log_5 5^{2c+3}$.

SOLUTION

$\log_5 5^{2c+3} = 2c + 3$ Inverse Property of Logarithms

b. Simplify $2^{\log_2 p}$.

SOLUTION

$2^{\log_2 p} = p$ Inverse Property of Logarithms

Most calculators calculate logarithms only in base 10 or base e. One way a logarithm with a base other than 10 or e can be evaluated is by applying the **change of base formula.**

Change of Base Formula
For $a > 0$ and $a \neq 1$ and any base b such that $b > 0$ and $b \neq 1$, $\log_b x = \frac{\log_a x}{\log_a b}$.

Example 2 Evaluating a Logarithm by Changing the Base

Evaluate $\log_4 8$.

SOLUTION

Method 1: Change to base 10.

$$\log_4 8 = \frac{\log 8}{\log 4}$$
$$\approx \frac{0.903}{0.602}$$ Use a calculator.
$$= 1.5$$ Divide.

Method 2: Change to base 2, because both 4 and 8 are powers of 2.

$$\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2}$$
$$= 1.5$$

Hint

You can also evaluate $\log_4 8$ by changing to base e. $\log_4 8 = \frac{\ln 8}{\ln 4} = 1.5$

Exploration Discovering the Properties of Logarithms

The logarithm $\log_2 32$ can be written as $\log_2 (4 \cdot 8)$.

1. Evaluate $\log_2 4$ and $\log_2 8$.
2. What is the sum of $\log_2 4$ and $\log_2 8$?
3. Since $\log_2 32$ can be written as $\log_2 2^5$ by the inverse Property of Logarithms, $\log_2 2^5 =$ _____.
4. Compare the value of $\log_2 32$ and $\log_2 4 + \log_2 8$.
5. **Generalize** If $\log_2 32 = \log_2 (4 \cdot 8) = \log_2 4 + \log_2 8$, then a make a conjecture about $\log_b (mn)$.
6. **Generalize** Use the process shaped by Problems 1–5 to make a conjecture about $\log_b \left(\frac{m}{n}\right)$. Use the statement $\log_2 4 = \log_2 \frac{64}{16} = \log_2 64 - \log_2 16$ as a model.

Product Property of Logarithms

For any positive numbers m, n, and b $(b \neq 1)$, $\log_b mn = \log_b m + \log_b n$

To prove the Product Property of Logarithms, let $b^x = m$ and $b^y = n$. Then $\log_b m = x$ and $\log_b n = y$.

$$b^x b^y = mn$$

$b^{x+y} = mn$ Product of Powers Property

$\log_b b^{x+y} = \log_b mn$ Take the logarithm of each side.

$x + y = \log_b mn$ Inverse Property of Logarithms

$\log_b m + \log_b n = \log_b mn$ Substitute $\log_b m$ for x and $\log_b n$ for y.

Example 3 Approximating the Logarithm of a Product

a. Approximate the value of $\log_3 63$. Use $\log_3 7 \approx 1.7712$.

SOLUTION

$\log_3 63 = \log_3 (7 \cdot 3^2)$ Factor 63 so that one of the factors is a power of 3.

$= \log_3 7 + \log_3 3^2$ Product Property of Logarithms

$= \log_3 7 + 2$ Inverse Property of Logarithms

$\approx 1.7712 + 2$ Substitute 1.7712 for $\log_3 7$.

≈ 3.7712 Add.

b. Approximate the value of $\log 60$. Use $\log 6 \approx 0.7782$.

SOLUTION

$\log 60 = \log (6 \cdot 10)$ Factor 60 so that one of the factors is a power of 10.

$= \log 6 + \log 10$ Product Property of Logarithms

$= 0.7782 + \log 10$ Substitute 0.7782 for log 6

$\approx 0.7782 + 1$ Inverse Property of Logarithms

≈ 1.7782 Add.

Hint

The solutions in Example 3 can be checked by approximating $3^{3.7712}$ and $10^{1.7782}$ with a calculator.

Quotient Property of Logarithms

For any positive numbers m, n, and b $(b \neq 1)$, $\log_b \frac{m}{n} = \log_b m - \log_b n$

To prove the Quotient Property of Logarithms, let $b^x = m$ and $b^y = n$. Then $\log_b m = x$ and $\log_b n = y$.

$$\frac{b^x}{b^y} = \left(\frac{m}{n}\right)$$

$b^{x-y} = \frac{m}{n}$ Quotient of Powers Poperty

$\log_b b^{x-y} = \log_b \left(\frac{m}{n}\right)$ Take the logarithm of each side.

$x - y = \log_b \left(\frac{m}{n}\right)$ Inverse Property of Logarithms

$\log_b m - \log_b n = \log_b \left(\frac{m}{n}\right)$ Substitute $\log_b m$ for x and $\log_b n$ for y.

Example 4 Approximating the Logarithm of a Quotient

a. Approximate the value of $\log_4 6$. Use $\log_4 48 \approx 2.7925$ and $\log_4 8 = 1.5$.

SOLUTION

$\log_4 6 = \log_4 \frac{48}{8}$	Express 6 as a quotient.
$= \log_4 48 - \log_4 8$	Quotient Property of Logarithms
$\approx 2.7925 - 1.5$	Substitute the given values.
≈ 1.2925	Subtract.

b. Approximate the value of $\ln \frac{e^4}{3}$. Use $\ln 3 \approx 1.0986$.

SOLUTION

$\ln \frac{e^4}{3} = \ln e^4 - \ln 3$	Quotient Property of Logarithms
$\approx 4 - \ln 3$	Inverse Property of Logarithms
$\approx 4 - 1.0986$	Substitute 1.0986 for ln 3
≈ 2.9014	Subtract.

Hint

The solution in Example 4b can be checked by approximating $e^{2.9014}$ with a calculator.

Power Property of Logarithms

For any real number p and positive numbers a and b $(b \neq 1)$,
$\log_b a^p = p \log_b a$

Example 5 Applying the Power Property of Logarithms

a. Simplify $\log_3 81^2$.

SOLUTION

$\log_3 81^2 = 2 \log_3 81$	Power Property of Logarithms
$= 2 \cdot 4$	$\log_3 81 = 4$ because $3^4 = 81$.
$= 8$	Multiply.

b. Write $\ln \left(\frac{5e}{x}\right)^2$ as a sum or difference of terms.

SOLUTION

$\ln \left(\frac{5e}{x}\right)^2 = 2 \ln \frac{5e}{x}$	Power Property of Logarithms
$= 2 (\ln 5e - \ln x)$	Quotient Property of Logarithms
$= 2 (\ln 5 + \ln e - \ln x)$	Product Property of Logarithms
$= 2 (\ln 5 + 1 - \ln x)$	Inverse Property of Logarithms
$= 2 \ln 5 + 2 - 2 \ln x$	Distribute the 2.

Example 6 Application: Geology

The Richter magnitude M of an earthquake is expressed by $M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$, where E is the energy released in ergs. In 1989 an earthquake of magnitude 7.1 struck San Francisco. Find the energy released by that earthquake.

SOLUTION

$7.1 = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$	Substitute 7.1 for *M*.
$\frac{3}{2}(7.1) = \log\left(\frac{E}{10^{11.8}}\right)$	Multiply both sides by $\frac{3}{2}$.
$10.65 = \log E - \log 10^{11.8}$	Quotient Property of Logarithms
$10.65 = \log E - 11.8$	Inverse Property of Logarithms
$22.45 = \log E$	Add 11.8 to both sides.
$10^{22.45} = E$	Write the logarithmic equation in exponential form.
$E \approx 2.82 \times 10^{22}$	Use a calculator to evaluate.

The energy released was approximately 2.82×10^{22} ergs.

Hint

The display 2.818382931E22 means $2.818382931 \times 10^{22}$, which is approximately 2.82×10^{22}.

Lesson Practice

(Ex 1) **a.** Simplify $10^{\log(3a-1)}$.

(Ex 1) **b.** Simplify $\ln e^{x+1}$.

(Ex 2) **c.** Evaluate $\log_{32} 64$.

(Ex 3) **d.** Approximate the value of $\log_2 40$. Use $\log_2 5 \approx 2.3219$.

(Ex 3) **e.** Approximate the value of $\log 1900$. Use $\log 19 \approx 1.2788$.

(Ex 4) **f.** Approximate the value of $\log_5 4$. Use $\log_5 60 \approx 2.5440$ and $\log_5 15 \approx 1.6826$.

(Ex 4) **g.** Approximate the value of $\log \frac{11}{10{,}000}$. Use $\log 11 \approx 1.0414$.

(Ex 5) **h.** Simplify $\log_5 125^4$.

(Ex 5) **i.** Rewrite $\ln\left(\frac{x+2}{2e^2}\right)$ as a sum or difference of terms.

(Ex 6) **j.** The Richter magnitude M of an earthquake is expressed by $M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$, where E is the energy released in ergs. In 2007 an earthquake of magnitude 6.6 occurred near the west coast of Japan. Find the energy released by that earthquake.

Practice Distributed and Integrated

(41) **1.** Find the distance between (6,0) and (−3, −3).

(26) **2.** Find the equation of the line that has a slope of $\frac{2}{5}$ and passes through (3, −5) in slope-intercept form.

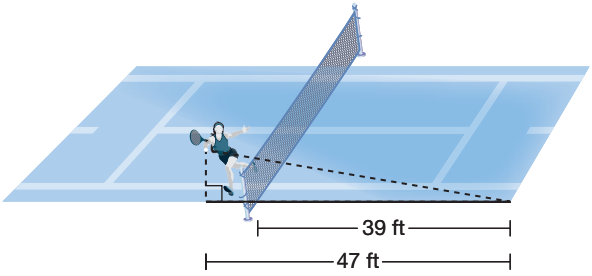
Simplify.

3. (40) $4\sqrt{3} \cdot 5\sqrt{2} \cdot 6\sqrt{12}$

4. (40) $4\sqrt{63} - 3\sqrt{28}$

***5.** **Write** (72) Explain why $\log_2 2^x = x$.

6. (67) **Tennis** A tennis player 8 feet from the net hits an overhead shot, hitting the ball at a height of 8 feet. Assume that the path of the ball is a straight line. To the nearest degree, what is the maximum angle θ at which the ball can be hit and land within the court?



7. (57) **Formulate** Formulate an expression for the amount Imani will have in her bank account at the end of ten years if she invests a principal of $2500 at an annual interest rate of 6.4%, compounded quarterly.

8. (69) Find the absolute value of $|3 + i|$.

9. (54) Describe the feasible region for the set of inequalities $0 \le x \le 15$, $y \ge 0$, $3y + 4x \le 75$.

***10.** (71) Solve $\triangle DEF$ given that $m\angle D = 65°$, $m\angle E = 47°$, and $d = 102$.

11. (63) **Multiple Choice** If an angle in standard position has a measure of $\frac{3\pi}{4}$ radians, in what point does its terminal side intersect the unit circle centered at the origin?

A $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **B** $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **C** $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **D** $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

12. (65) **Travel** A driver leaves a gas station and heads due north at a rate of 50 miles per hour. Two hours later, a truck driver leaves the same gas station and heads due east at a rate of 40 miles per hour. Approximately how long will it take for the vehicles to be 450 miles apart?

***13.** (72) **Multi-Step** Suppose that the population of a certain town increases at the rate of 6% per year and that the current population is 2210.

a. Write an exponential function that gives the population as a function of x, the number of elapsed years.

b. Write a logarithmic function that gives the number of elapsed years as a function of x, the population.

c. To the nearest tenth, how many years will it take for the population to exceed 3000?

***14.** (68) **Analyze** Explain why the probability of events that involve selection of items without replacement involve conditional probabilities.

***15.** (72) **Multiple Choice** Which is equal to $5^{\log_5 x}$?

A 1 **B** x **C** $5x$ **D** 5^x

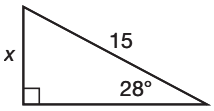
16. (63) **Clock Face** The minute hand of a certain clock is 20 inches long. How far does the tip of the minute hand travel in 45 minutes, to the nearest inch?

***17.** (Inv 3) **Graphing Calculator** Find the solution to the following system of equations using the Matrix functionality of the calculator.

$$-2x + y + 4z = -24$$
$$-5x + 7y - z = -36$$
$$8x - 3y + z = 43$$

18. (52) **Error Analysis** Kyle carefully measures two sides of a triangle and finds that the longest side has a length of 2 inches, and the second longest side has a length of $\sqrt{3}$ inches. He comes to the conclusion that the shorter leg must have length 1. What is wrong with his reasoning?

***19.** (71) **Verify** Show that the value of x can be found by either using the sine ratio or the Law of Sines.



Add or subtract. Write the answer in the form $a + bi$.

20. (62) $(6 - 3i) + (5 + 4i)$

21. (62) $(8 + 20i) - (-8 + 12i)$

***22.** (71) **Coordinate Geometry** Explain how to use the Law of Sines to find the area of a triangle whose vertices are at $(-4, 0)$, $(6, 0)$, and $(0, 6)$. Then find the area.

23. (32) Find the inverse of $\begin{bmatrix} 16 & 8 \\ 2 & 0 \end{bmatrix}$.

24. (62) **Analyze** Explain how to add a real number to a complex number. Provide an example.

***25.** (70) **Lighthouses** The number of miles m a person can see on a clear day is approximated by $m = 1.2\sqrt{h}$, where h is the person's height above sea level. Suppose a person standing at the top of the Little Sable Point Lighthouse in Michigan on a clear day can see a distance of 12.4 miles. To the nearest foot, what is their height above sea level?

***26.** (69) **Geometry** The sum of two complex numbers can be found by graphing each number on the complex plane, drawing segments from the origin to each point, completing the parallelogram that has these two segments as sides, and finding the point opposite the origin. Use this method to find $(-2 + 3i) + (5 + i)$.

27. (Inv 6) **Error Analysis** Isaiah added 196 to both sides of the equation $x^2 + 16x + 60 = 0$ so he could make the perfect square. What error did he make?

28. (Inv 7) Write the polar equation of a circle with a radius of 7.

Solve by substitution.

29. (21)
$$2y - x = 6$$
$$y - 2x = -3$$

30. (21)
$$3x + 2y = 12$$
$$8x - 2y = 10$$

LAB
11

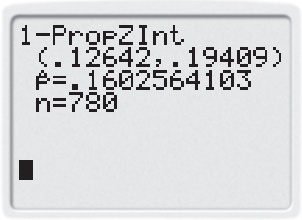
Calculating Confidence Intervals

Graphing Calculator Lab (*Use with Lesson 73, 80, 118*)

A graphing calculator can be used to find confidence intervals.

Calculating the Confidence Interval Estimate of a Population Proportion

1. Calculate the confidence interval using the **"A: PropZInt"** feature.
   - **a.** From the home screen, press the STAT button. Then press ▶ twice to reach the **TESTS** menu.
   - **b.** Press ▼ to [**A: 1-PropZInt…**] and press ENTER. The 1-PropZInt-screen will appear so that the statistic values can be entered.
   - **c.** Press 1 2 5 to enter x (successes). Then press ENTER.
   - **d.** Press 7 8 0 to enter n (sample size). Then press ENTER.
   - **e.** Press . 9 9 to enter the C-Level (confidence level). Then press ENTER to move to the **Calculate** option.
   - **f.** Press ENTER to calculate the confidence interval. The values that appear inside the parentheses form the confidence interval.

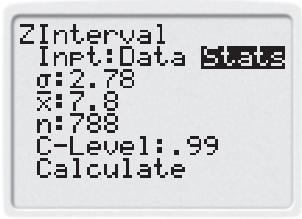


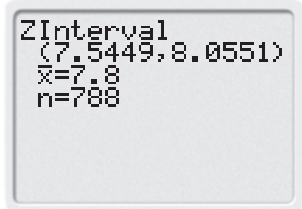
Hint

If the number of successes x is not given, then use the formula $x = n\hat{p}$ to calculate x.

Calculating the Confidence Interval Estimate of a Population Mean

2. Calculate the confidence interval when the standard deviation is known.
   - **a.** To access this feature, follow step **1a.** Press ▼ to select [**7:ZInterval…**] and press ENTER.
   - **b.** First, select the type of input. Press ▶ to highlight **Stats** and press ENTER. Then press ▼ to enter the first statistic value.
   - **c.** Enter the σ (population standard deviation). Press 2 . 7 8, and press ▼ to move to the next item.
   - **d.** Continue to enter the other values. Enter $\bar{X}$ (sample mean) as 7.8, n (sample size) as 788, and C-Level (confidence level) as .99.
   - **e.** Move the cursor to the **Calculate** option. Press ENTER to calculate the confidence interval. The values that appear inside the parentheses form the confidence interval.





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Graphing Calculator Tip

For these examples, the statistics values are given, so **Stats** is the type of input. When statistics values need to be calculated from stored data, choose the option **Data** as the type of input.

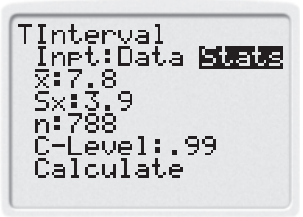
3. Calculate the confidence interval when the standard deviation is unknown.

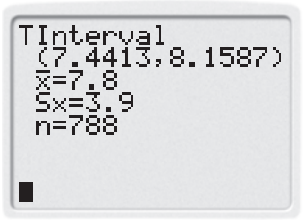
a. To access this feature, follow step **1a**. Press ▼ to select **[8: TInterval…]** and press ENTER.

b. Select the type of input. Press ▶ to highlight Stats and press ENTER.

c. Use ▼ to move to the next entry. Enter $\overline{X}$(sample mean) as 7.8, Sx (standard deviation) as 3.9, n (sample size) as 788, and C-Level (confidence level) as .99.

d. Move the cursor to the **CALCULATE** option and press ENTER to calculate the confidence interval. The values that appear inside the parentheses form the confidence interval.





Lab Practice

a. Find the 97% confidence interval estimate of a population proportion p when $x = 450$ and $n = 3655$.

b. Find the 95% confidence interval estimate of a population mean μ when $\sigma = 5744$, $\overline{X} = 38{,}576$, and $n = 43$.

c. Find the 95% confidence interval estimate of a population mean μ when $\overline{X} = 510$, $Sx = 115$, and $n = 33$.

LESSON
73

Using Sampling

Warm Up

1. (SB) **Vocabulary** Two ratios set equal to each other form a _____.

2. (7) Solve for b: $\frac{24}{b} = \frac{33}{66}$.

3. (7) Find the value of n given $n = \frac{pq}{s}$, $p = 16$, $q = 32$, and $s = 8$.

New Concepts

A **population** is a group of individuals about which information is desired. When the population is large enough to make it difficult to examine every individual in it, such as the population of all American citizens, information from a sample of the population is obtained instead.

A **sample** is part of a population. The **sample size** is the number of individuals in a sample. When the sample is representative of the population studied, results from analyzing the sample can be extended to the entire population.

Example 1 Identifying the Population and Sample

Describe the individuals in the population and in the sample.

a. A newspaper reporter surveyed 5 women from a high school basketball team to determine how they feel about the men's team getting new uniforms when they did not get new uniforms.

SOLUTION The population is all the women on that school's female basketball team. The sample is the 5 women that spoke to the reporter.

b. Every registered voter in Jefferson County received a questionnaire in the mail to be filled out and returned two weeks later.

SOLUTION The population is every registered voter in Jefferson County, and the sample is those individuals who chose to fill out and return the survey on time.

Math Reasoning

Analyze In Example 1b, why is it wrong to say that the population is every citizen in Jefferson County?

The process of choosing the sample to be examined is called **sampling.** Valid sampling is very important. If the sample does not represent the population, the results of any analyses done on the sample are useless. A representative sample will have the same characteristics as the population. For example if the population is about 30% female, a representative sample would also be about 30% female.

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In **probability sampling,** every individual in the population has a known probability of being selected and this probability is greater than 0.

Probability Samples

Type	Definition	Example
Simple Random Sample (SRS)	Every member of the population has an equal chance of being chosen.	The names of all students in a class are put in a hat and five names are chosen at random.
Stratified Sample	The population is divided into mutually exclusive groups which have similar characteristics and a SRS is done on each group.	The class is divided into boys and girls and three names are chosen at random from each group.
Systematic Sample	A member of the group is chosen at a regular interval.	The names are listed at random and every eighth name is selected.
Cluster Sample	The population is divided into groups called clusters and the clusters are randomly chosen.	A survey may be given to a randomly selected homeroom.

Math Reasoning

Analyze Why is a stratified sample not a simple random sample?

Example 2 Identifying a Probability Sample

Tell if the sampling method performed was a probability sample. If not, explain why not. If so, state the type of probability sample.

a. A student wants to know what the senior class thinks about the suggested prom theme. He surveys every other senior who steps onto his bus.

SOLUTION The sample is not a probability sample because seniors who take another bus, are absent, or walk, bike, or drive to school have no chance of being selected for the survey.

b. A teacher assigns every student in her class a number from 1 to 40 and uses a calculator to choose random integers from 1 to 40. The students corresponding to the first 5 numbers generated are designated to be group leaders.

SOLUTION The sample is a simple random sample because every student has the same chance of being chosen and every possible group of 5 students has the same chance of being chosen.

An SRS can be made by using either the random number generator on a calculator, or by using a table of random digits. A table of random digits is shown below.

62010	60482	89824	08705	57913	46815	76221
30402	04710	12255	15279	31264	24680	38689
04633	61396	93751	98362	59553	62398	66607
63157	48257	29223	66773	89022	43069	91349
53787	35824	84823	86792	74771	91784	23826

To make an SRS by using the table, first assign each individual in the population a number. All individuals should have the same number of digits in their number. If choosing a sample of 10 individuals, assign 2-digit numbers: 01, 02, 03…10. If choosing a sample of 100 individuals, assign 3-digit numbers: 001, 002, 003…100. Then start at any row in the table and read the numbers straight across.

Example 3 Choosing an SRS by using a Table of Random Digits

Twenty four names are numbered and listed below. Five must be randomly chosen for a sample. Use the Random Number Table and the instructions for using the table to determine the five names that will be picked for the sample.

01 Ann	02 Luke	03 Cassie	04 Finn	05 Jared	06 Tanela
07 Jude	08 Gunner	09 Drew	10 Halle	11 Colby	12 Anton
13 Aaron	14 Kenya	15 Ensa	16 Jarvis	17 Ming	18 Buhay
19 Alice	20 Steven	21 Pia	22 Niko	23 Leah	24 Ben

Random Number Table

046336 139693 751983 625955 362398 666071 858190 305192 911319
263157 482572 922366 773890 224306 991349 331116 368248 864530

Instructions Look at each consecutive 2-digit number. Ignore those numbers that are not one of the labeled numbers (01 through 24). Ignore numbers that repeat. Pick up on the second row wherever the first row leaves off, as if the table were written as one long continuous string of numbers.

SOLUTION Stop once five numbers from 01 through 24 have been identified. These numbers are shown in bold.

046336 **13**9693 75**19**83 625955 36**23**98 666071 858190 305192 911319
263157 482572 922366 773890 **22**4306 991349 331116 368248 864530

Make sure to look at the consecutive two-digit numbers. In the first group the numbers are 04, 63 and 36. Note: 46 is not a number.

The numbers 04, 13, 19, 23, and 22 correspond to Finn, Aaron, Alice, Leah, and Niko.

Not all sampling methods actually performed are a probability sample.

In **convenience sampling,** the surveyor simply selects whoever is convenient for them to select. For example, this may be those individuals who happen to walk by. It is not a probability sample, because people in the population who do not walk by the surveyor have no chance of being chosen.

Math Language

Voluntary response sampling is also known as **self-selected sampling**.

In **voluntary response sampling,** the individuals in the sample choose themselves. For example, a population may receive a questionnaire in the mail. The sample consists of only those people who choose to return it. Note that controversial questions tend to be answered mostly by those people who have very strong opinions about that issue.

Both convenience and voluntary response sampling can lead to **bias.** A sampling method is biased if it favors certain outcomes. The sample is then not representative of the population, and the results of surveying the individuals in the sample do not truly reflect that of the population.

Bias can result not only from poor sampling, but from poorly worded questions as well. Questions that are confusing or that lead the reader towards a certain side are biased.

Example 4 Identifying Bias

a. A survey asks students, "Do you agree that the school colors should be changed from the drab green and black we currently have to a vibrant gold and purple that will signify strength and determination?" Explain the bias in the question.

SOLUTION The question is biased because it leads students to say *yes* by using positive adjectives for only one set of the colors. It is also biased because it only gives one side of the situation. It does not discuss possible problems or issues that could arise with switching colors.

Math Reasoning

Formulate Write a question for Example 4a that will lead the students to say *no*.

b. An article in a magazine asks readers to go online and answer a question related to the article. Explain the bias in the sampling method.

SOLUTION The sample is biased against those readers without internet access because they will not be able to answer the question. It is also biased against those readers that did not read the article. Therefore, the sample will have a greater percentage of people interested in that topic and that have internet access than that of the population.

Sampling can be used to estimate the size of a population. A common method used to estimate animal populations is called the **capture-recapture method.**

In the simplest of the capture-recapture techniques, a researcher visits an area and marks, or tags, every animal they capture. They later return to the same area, after enough time has passed that the marked animals should completely mingle with any unmarked animals. Another group of this animal population is captured. The researcher uses the following proportion to then estimate the population of the animal in the area.

$$\frac{\text{number of marked animals on 2nd visit}}{\text{number of all animals captured on 2nd visit}} = \frac{\text{number of animals originally marked}}{\text{total population of animals in area}}$$

Example 5 Application: Estimating an Animal Population

Math Reasoning

Find the confidence interval estimate of the population proportion p when $x = 12$ and $n = 32$.

A researcher visits a pond and captures and marks 20 fish. On a return visit the next day, the researcher captures 32 fish, and 12 of them are marked from the previous day. Estimate the fish population in the pond.

SOLUTION Write and solve the capture-recapture proportion.

$$\frac{12}{32} = \frac{20}{n}$$

$$12n = 640$$

$$n \approx 53$$

There are about 53 fish in the pond.

Lesson Practice

Describe the individuals in the population and in the sample.

a. *(Ex 1)* A principal of a middle school talked to and questioned parents and guardians of the students about a new dress code.

b. *(Ex 1)* Receipts from a certain grocery store ask customers to go online to complete a survey about their shopping experience.

Tell if the sampling method performed was a probability sample. If not, explain why not. If so, state the type of probability sample.

c. *(Ex 2)* The school secretary takes the list of names of students who are absent that day, randomly chooses a starting point, selects every 10th student, and calls their parent or guardian to ask if they know that the student was absent.

d. *(Ex 2)* A coach wants to know about how many students are interested in joining the track team. He asks students on both the male and female soccer teams if they, or anyone they know, would like to run on the track team.

e. *(Ex 3)* Twenty four names already numbered are listed below. Five must be randomly chosen for a sample. What five names will be picked for the sample if numbers are chosen from the random number table shown below?

11 Ann	12 Luke	13 Cassie	14 Finn	15 Jared	16 Tanela
17 Jude	18 Gunner	19 Drew	20 Halle	21 Colby	22 Anton
23 Aaron	24 Kenya	25 Ensa	26 Jarvis	27 Ming	28 Buhay
29 Alice	30 Steven	31 Pia	32 Niko	33 Leah	34 Ben

046336 139693 751983 625955 362398 666071 858190 305192 911319
263157 482572 922366 773890 224306 991349 331116 368248 864530

f. *(Ex 4)* A survey asks students, "Do you disagree that the president of the school should not be allowed to run for re-election?" Explain the bias in the question.

g. *(Ex 4)* A TV show asks viewers to answer a question by texting their response on their cell phone. Explain the bias in the sampling method.

h. *(Ex 5)* A researcher visits a park and captures and tags 43 chipmunks. On a return visit two days later, the researcher captures 50 chipmunks, and 38 of them are tagged from the previous visit. Estimate the number of chipmunks in the park.

Practice Distributed and Integrated

Write the logarithmic equation in exponential form.

1. *(64)* $\log_{10} 10{,}000{,}000 = 7$

2. *(64)* $\log_6 216 = 3$

3. *(52)* **Explain** Explain why all 45° −45° −90° triangles are similar.

*4. (71) State the case applicable to the given measurements (AAS, ASA, or SSA). Then decide whether the measurements determine one triangle, two triangles, or no triangle. A = 40°; C = 75°; c = 20

Solve for the roots of the polynomial:

5. (66) $6x(x^2 + x - 20) = 0$

6. (66) $2x(x^2 + 9x - 52) = 0$

7. (66) **Probability** For $x^3 + 8x^2 + 3x - 4 = 0$, what would be the probability of randomly selecting a root on the first try from the list of possible roots given by the Rational Root Theorem?

Solve for x by completing the square:

8. (58) $-x^2 + 6x = -12$

9. (58) $-4 = -x^2 - 3x$

*10. (73) **Business** The owner of a frozen food company wants to know how many people would agree to pay slightly more for products sold in recycled packages than non-recycled packages. She hires a surveyor to stand in the aisles of randomly selected grocery stores which sell the company's product to survey buyers of frozen food products. Define the population and sample in this situation. What groups of people would not be represented in the sample and why?

11. (53) **Multi-Step** Let $f(x) = x - 6$ and $g(x) = 3x$.

a. Find the composition functions $(f \circ g)(x)$ and $(g \circ f)(x)$.

b. Evaluate the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$ for $x = -1$.

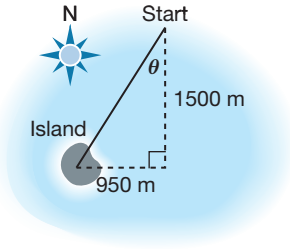
*12. (64) **Write** Explain why there is no value of x that makes the equation $\log_1 10 = x$ true.

Convert each measure from radians to degrees.

13. (63) $\frac{6\pi}{4} =$

14. (63) $\frac{2\pi}{3} =$

*15. (67) **Navigation** A camper plans to canoe to an island. The island is 1500 meters south and 950 meters west of her starting point. To the nearest degree, in what direction should she head?



*16. (73) **Multi-Step** Explain why the question below is biased. Then rewrite the question without bias. "Do you not agree that junk food should not be sold in school vending machines?"

17. (65) **Multiple Choice** Which is a solution of $3x^2 - 7x - 2 = 0$?

A $\frac{1}{3}$ **B** 2 **C** $\frac{7 + \sqrt{73}}{6}$ **D** $\frac{7}{6} - \frac{\sqrt{73}}{6}i$

18. (68) **Write** What is a conditional probability? Give an example.

*19. (73) **Estimate** A researcher using the stratified sampling method has chosen a sample of 12 female athletes and 17 non-female athletes from a population of 305 female students. About how many students in the population are athletes?

20. (59) **Error Analysis** Is the work below correct? If not, find and explain the error, and give the correct solution.

$$x^{\frac{1}{7}} = 2$$
$$\left(x^{\frac{1}{7}}\right)^{\frac{7}{1}} = 2$$
$$x^1 = 2$$
$$x = 2$$

21. (67) Solve to the nearest tenth using the given restrictions. Tan $\theta = 2.42$, for $180° \le \theta \le 360°$

***22.** (72) **Geology** The Richter magnitude M of an earthquake is expressed by $M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$, where E is the energy released in ergs. An earthquake near Peru in 2007 had a magnitude of 8.0. Find the energy released by that earthquake.

23. (58) **Multi-Step** Use $x^2 + 30x + 225 = 13$.

a. Explain why the equation is a perfect square.

b. Factor $x^2 + 30x + 225$.

c. Solve the equation for x.

***24.** (72) **Error Analysis** To rewrite $\ln 4e^3$ without an exponent, a student wrote $\ln 4e^3 = \ln 4 \cdot \ln e^3 = (\ln 4)(3) = 3 \ln 4$.

What is the error? Rewrite $\ln 4e^3$ without an exponent correctly.

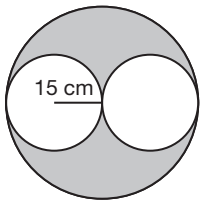
25. (57) **Population** The population of the state of Florida in various years is shown below.

Year	1900	1950	1990	2005
Population (thousands)	529	2771	12,937	17,789

Write an exponential model that approximates this data. What population is predicted by this model for the year 2010?

***26.** (71) **Graphing Calculator** Find the area of $\triangle BCD$ given $c = 3.62$, $b = 1.45$, and $m\angle D = 33.25°$. Round to the nearest hundredth.

27. (55) **Geometry** Find the probability that a point chosen randomly inside the figure is in the shaded region.



***28.** (70) **Multiple Choice** Solve $x = \sqrt{-14x - 45}$.

A no solutions **B** -9 **C** -5 **D** -9 and -5

29. (69) **Analyze** How is graphing in the complex plane similar to, and different from, graphing in the xy- coordinate plane?

30. (36) Write the equation of a line in slope intercept form that is parallel to $y = -\frac{3}{2}x + 4$ and passes through (1, 5).

LESSON
74

Finding the Discriminant

Warm Up

1. (35) **Vocabulary** Where the graph of a function intersects the x-axis is where the _____ of the related equation are found.

2. (62) Solve using the quadratic formula $2x^2 - 4x + 25 = 0$.

3. (35) Graph $y = 3x^2 + 4 - 24x$ to determine how many roots the equation $0 = 3x^2 + 4 - 24x$ has.

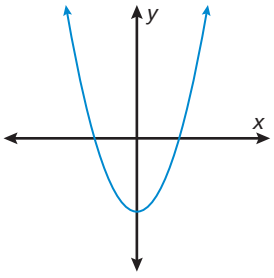
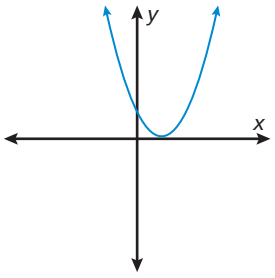
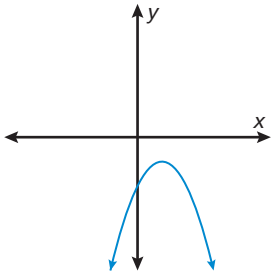
4. (27) Write $f(x) = (x - 4)^2 + 3$ in standard form.

New Concepts

The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, gives the solutions of any quadratic equation. The radicand in the formula, $b^2 - 4ac$, is called the **discriminant.** The discriminant can be used to tell if the equation has zero, one, or two solutions.

Math Language

A **solution** of an equation can also be called a **root** of the equation.

Analyzing the Discriminant		
For $ax^2 + bx + c = 0$, $(a \neq 0)$ and related function $y = ax^2 + bx + c$		
$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
Equation: 2 real roots Function: 2 x-intercepts	Equation: 1 real root with multiplicity of 2 Function: 1 x-intercept	Equation: 0 real roots 2 complex roots Function: 0 x-intercepts
		

Example 1 Describing the Roots of a Quadratic Equation

Use the discriminant to describe the roots of each equation.

a. $4x^2 + 25 = 20x$

SOLUTION Write the equation in standard form: $4x^2 - 20x + 25 = 0$.

$b^2 - 4ac = (-20)^2 - 4(4)(25) = 400 - 400 = 0$

Because the discriminant is 0, the equation has one real root.

b. $2x^2 + 7x - 99 = 0$

SOLUTION $b^2 - 4ac = (7)^2 - 4(2)(-99) = 49 + 792 = 841$

Because the discriminant is positive, the equation has 2 real roots.

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c. $3x^2 + 6x + 5 = 0$

SOLUTION $b^2 - 4ac = (6)^2 - 4(3)(5) = 36 - 60 = -24$

Because the discriminant is negative, the equation has 2 complex roots.

Example 2 Using a Positive Discriminant

A player standing on a 10-foot hill kicks a ball straight up at a velocity of 55 feet per second. He hopes it goes above the roof of a nearby building which is 50 feet above the ground. The function that gives the height of the ball after x seconds is given by $y = -16x^2 + 55x + 10$. Use the discriminant to explain why the ball *will* reach the desired height.

Math Reasoning

Analyze In Example 2, how do you know the ball's maximum height will be greater than 50 feet?

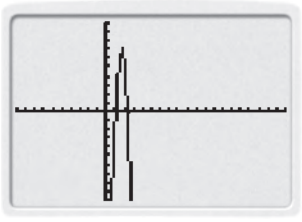
SOLUTION Substitute 50 for y and write the equation in standard form.

$50 = -16x^2 + 55x + 10 \longrightarrow 0 = -16x^2 + 55x - 40$

Find the discriminant:

$b^2 - 4ac = (55)^2 - 4(-16)(-40) = 3025 - 2560 = 465$

Since the discriminant is positive, there are two real solutions, indicating that the ball has a height of 50 feet twice. The solutions give the number of seconds at which the height is 50 feet.



Example 3 Using a Zero Discriminant

A homeowner has 120 feet of fencing material to enclose a rectangular dog pen. She wants the dog to have 900 square feet of space inside the pen. The equation $w(60 - w) = 900$ gives the width of a pen that meets these requirements. Use the discriminant to explain why these requirements *can* be met.

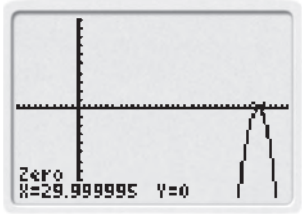
SOLUTION Write the equation as a quadratic in standard form.

$w(60 - w) = 900 \longrightarrow 60w - w^2 = 900 \longrightarrow -w^2 + 60w - 900 = 0$

Find the discriminant:

$b^2 - 4ac = (60)^2 - 4(-1)(-900) = 3600 - 3600 = 0$

Since the discriminant is 0, there is one real solution, indicating that there exists one width which meets the requirements. The solution gives this width.



Example 4 Using a Negative Discriminant

Math Reasoning

Analyze In Example 4, why is the length equal to $25 - w$ given that the width is w?

A frame maker has 50 inches of framing material with which to make a picture frame. He wants to frame a picture that has an area of 160 square inches. The equation $w(25 - w) = 160$ gives the width of the frame that meets these requirements. Use the discriminant to explain why these requirements can *not* be met.

SOLUTION Write the equation as a quadratic in standard form.

$$w(25 - w) = 160 \longrightarrow 25w - w^2 = 160 \longrightarrow -w^2 + 25w - 160 = 0$$

Find the discriminant: $b^2 - 4ac = (25)^2 - 4(-1)(-160) = 625 - 640 = -15$

Since the discriminant is negative, there are no real solutions. The frame maker cannot make a frame for this picture with only 50 inches of framing material.

Example 5 Application: Rocketry

Two scientists are firing rockets. The first scientist is setting his off at ground level at an initial velocity of 75 feet per second. The second scientist is setting hers off from a deck 25 feet above the ground at a velocity of 70 feet per second. Will either of the rockets reach a height of 100 feet above the ground? If so, which one?

SOLUTION The function for the height, in feet, of an object shot straight up at an initial velocity of v and height h is $y = -16x^2 + vx + h$.

Write an equation for each scientist's rocket.

1st scientist: $100 = -16x^2 + 75x \longrightarrow 0 = -16x^2 + 75x - 100$

2nd scientist: $100 = -16x^2 + 70x + 25 \longrightarrow 0 = -16x^2 + 70x - 75$

Find the discriminant of each equation.

1st scientist: $b^2 - 4ac \longrightarrow (75)^2 - 4(-16)(-100) = 5625 - 6400 = -775$

2nd scientist: $b^2 - 4ac \longrightarrow (70)^2 - 4(-16)(-75) = 4900 - 4800 = 100$

Math Reasoning

Analyze How long after setting it off does the second scientist's rocket reach its maximum height? What is this height?

The discriminant for the equation for the second scientist is positive, so the second equation has two real solutions. Her rocket will reach 100 feet. The rocket of the first scientist will not reach 100 feet because the equation has no real solutions.

Lesson Practice

Use the discriminant to describe the roots of each equation.

(Ex 1)

a. $16 + x^2 = 8x$

b. $2x^2 + 12x + 23 = 0$

c. $3x^2 - x = 14$

d. (Ex 2) A tennis player hits a tennis ball straight up at a velocity of 40 feet per second. The ceiling in the gym she is in has a height of 22 feet. The function that gives the height of the ball after x seconds is given by $y = -16x^2 + 40x$. Use the discriminant to explain why the ball *will* hit the ceiling above her.

e. (Ex 3) A farmer has 280 feet of fencing material to enclose a rectangular area for his pigs and chickens. He wants the fenced-in area to be 4900 square feet. The equation $w(140 - w) = 4900$ gives the width that meets these requirements. Use the discriminant to explain why these requirements *can* be met.

f. (Ex 4) A frame maker has 35 inches of framing material with which to make a picture frame. He wants to frame a picture that has an area of 80 square inches. The equation $w(17.5 - w) = 80$ gives the width of the frame that meets these requirements. Use the discriminant to determine why these requirements can *not* be met.

g. (Ex 5) Two technicians are setting off fireworks. The first technician is setting his off at ground level at an initial velocity of 140 feet per second. The second technician is setting his off from a deck 15 feet above the ground at a velocity of 120 feet per second. Will either of the fireworks reach a height of 275 feet above the ground? If so, which one?

Practice Distributed and Integrated

Use synthetic division to divide.

1. (61) $x^4 - 2$ by $x + 1$

2. (61) $x^3 - 2$ by $x - 5$

***3.** (57) **Graphing Calculator** Use a graphing calculator to investigate the behavior of the following 4 functions for the domain $x \geq 0$: $y = e^x$, $y = x$, $y = x^2$, and $y = x^3$. Write a short paragraph comparing the growth rate of these four functions as the value of x increases.

***4.** (74) **Rock Climbing** A rock climber at the bottom of the King and Queen Seat outcropping in Rocks State Park in Maryland shoots a flare from a flare gun. The height of the flare, in meters, can be modeled by $y = -5x^2 + 32x + 1$, where x is number of seconds after the flare is set off. Use the discriminant to explain if the flare will reach the height of the outcropping, given that the outcropping has a height of 58 meters.

5. (66) **Storage** The volume of a cylindrical bucket is 1696 cubic inches. The height of the bucket is 9 inches greater than its radius. Write a polynomial in standard form, without π, that can be used to find the radius (Round the constant to the nearest integer.)

6. (56) Given $\theta = 400°$, find a positive and negative coterminal angle.

7. (61) **Multiple Choice** Which is *not* a factor of $x^3 + 9x^2 + 2x - 48$?

A $x - 4$ **B** $x - 2$ **C** $x + 3$ **D** $x + 8$

8. (68) A prize box contains 5 stickers, 13 smiley erasers, 20 pencils, and 2 binders. Find the probability that the student randomly chooses 2 prizes, a sticker first and then a pencil.

9. (57) The number of mice increased exponentially. At first there were 85 mice. Twelve years later there were 1700 mice. How many mice would there be at the end of 130 years?

***10.** (74) **Analyze** How does the discriminant of a quadratic equation indicate if the real solutions are rational or irrational?

11. (69) **Multiply** $2(3i)$. Write the result in the form $a + bi$.

12. (63) **Automobiles Tires** The diameter of a certain automobile tire, mounted on the wheel, is 25 inches. How far does the automobile go when the tire rotates 10 times? Give your answer to the nearest foot.

Solve.

13. (70) $\sqrt{9x} + 11 = 14$

14. (70) $-4\sqrt{x} - 6 = -20$

15. (65) **Analyze** When using a graphing calculator to solve a quadratic equation, for what type of solutions can exact answers be found?

16. (69) **Electrical Engineering** The flow of a current, I, the resistance to the flow, Z, and the voltage, E, are related by $E = I(Z)$. Find the resistance of a circuit (in ohms) if the voltage is $265 - 135i$ volts and the current is $26 - 30i$ amps.

***17.** (71) **Geometry** Find the perimeter of $\triangle XYZ$ given $m\angle X = 40°$, $m\angle Z = 70°$, and $y = 33$ cm.

18. (62) **Error Analysis** Which student incorrectly added the complex numbers? Explain the error.

Student A	Student B
$(6 + 2i) + (-3 - 9i)$ $8i - 12i$ $-4i$	$(6 + 2i) + (-3 - 9i)$ $3 - 7i$

***19.** (73) **Government** Suppose a politician must visit 5 counties in Maine. His human resource personnel form a simple random sample by listing each county and numbering it as shown below. If they use the line of random digits shown below, which 5 counties will the politician visit?

50: Androscoggin	51: Aroostook	52: Cumberland	53: Franklin
54: Hancock	55: Kennebec	56: Knox	57: Lincoln
58: Oxford	59: Penobscot	60: Piscataquis	61: Sagadahoc
62: Somerset	63: Waldo	64: Washington	65: York

62010	60482	89824	08705	57913	46815	76221	25650	89337	53603	73597

*20. (73) **Multiple Choice** Which sampling method would introduce the greatest possible amount of bias?

A systematic sampling **B** stratified sampling

C convenience sampling **D** simple random sampling

21. (60) Katie rolls a 1–6 number cube twice. What is the probability she will roll an odd number and then an even number?

*22. (72) **Justify** Explain why $3^{\log_3 2x} = 2x$.

Factor each expression completely.

23. (61) $3y^5 - 48y^3$

24. (61) $x^3 + x^2 + x + 1$

*25. (74) **Multi-Step** **a.** Write $16x^2 = 25$ in standard form.

b. Find the discriminant of the equation in part a.

c. Use the discriminant to describe the solutions of the equation in part a.

d. Verify Solve the equation in part a.

26. (52) **Estimate** The hypotenuse of a 45° −45° −90° triangle is between 120 and 130 cm long. Estimate the length of each leg.

*27. (70) **Generalize** Raise both sides of the equation $x = 4$ to the second, third, fourth, and fifth powers. Solve each equation. In which cases are there extraneous solutions? In general, when could raising both sides by the same power generate extraneous solutions?

28. (67) **Multi-Step** Let $f(\theta) = \text{Tan } \theta$ and $g(\theta) = \text{Tan}^{-1} \theta$, where θ is in radians.

a. Evaluate $f(g(1))$ and $g\left(f\left(\frac{\pi}{4}\right)\right)$. Show your steps.

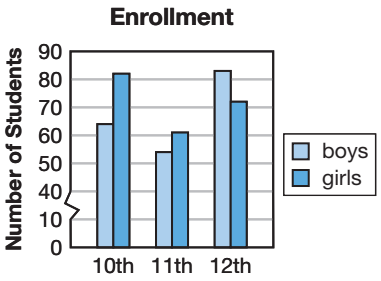
b. Based on part a, what is the value of $f(g(x))$?

c. Rename $g(\theta)$ as $f^{-1}(\theta)$. What property of inverse functions is illustrated in part a?

29. (68) **Data Analysis** Use the double-bar chart to estimate the conditional probabilities.

a. $P(\text{boy} \mid \text{10th grade})$

b. $P(\text{10th grade} \mid \text{boy})$



30. (61) **Error Analysis** What error did the student make in factoring the difference of cubes?

$$c^6 - 216$$
$$(c^2 - 6)(c^4 - 6c^2 + 36)$$

LESSON 75

Graphing Radical Functions

Warm Up

1. (50) **Vocabulary** The expression $\sqrt{x}$ is the _____ of the expression x^2.
2. (11) True or false. The expression $5x^2$ is a monomial of degree two.
3. (4) What is the domain of $y = 3x^2 + 2 - 3$?

New Concepts

A **radical function** is written in one of two forms, with a radical or with a fractional exponent.

Radical Version	Fractional Exponent Version
$f(x) = \sqrt[n]{g(x)}$	$f(x) = (g(x))^{\frac{1}{n}}$

In each case, n is an integer and $g(x)$ is a function. In the case of the radical version, the n is also referred to as the index. A radical function is the inverse of a power function.

Example 1 The Square Root Function

Math Language

A **square root function** is a function whose rule contains a variable under a square root sign.

a. Graph the square root function and its inverse. Determine the domain and range of both functions.

$y = \sqrt{x - 1}$

SOLUTION

Step 1: To find the inverse, interchange y and x and solve for y:

$x = \sqrt{y - 1}$

$x^2 = y - 1$

$y = x^2 + 1$

Since the range of the original function is $y \geq 0$, the domain of the inverse function is $x \geq 0$.

Math Reasoning

Generalize What does a comparison of the x-y table of a function and its inverse show?

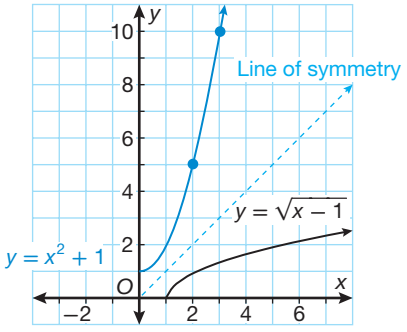
Step 2: Graph the square root function and the quadratic, its inverse, using x-y tables and the domains of each function:

$y = \sqrt{x - 1}$

x	y
1	0
2	1
5	2

$y = x^2 + 1$

x	y
0	1
1	2
2	5



The domain of the inverse, $y = x^2 + 1$, is $x \geq 0$ and the range is $y \geq 1$. For the radical function, $y = \sqrt{x - 1}$, the domain is $x \geq 1$ and the range is all real numbers greater than zero.

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Math Reasoning

Write Consider a radical function $f(x) = \sqrt[n]{g(x)}$, where n is an even number. What restrictions are there to the domain of this function?

b. Graph the square root function and its inverse.

$y = (2x - 3)^{\frac{1}{2}} + 4$

SOLUTION

Step 1: To find the inverse, interchange y and x and solve for y:

$$x = (2y - 3)^{\frac{1}{2}} + 4$$
$$x - 4 = (2y - 3)^{\frac{1}{2}}$$
$$(x - 4)^2 = 2y - 3$$
$$y = \frac{(x - 4)^2 + 3}{2} \qquad \text{Remember } x \geq 4.$$

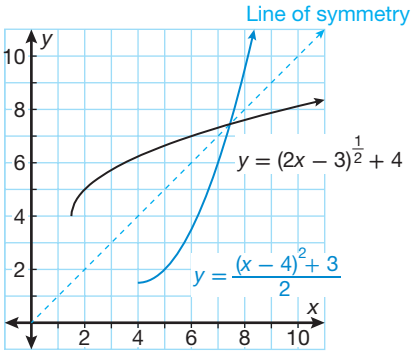
Step 2: Graph the square root function and the quadratic, its inverse:

$y = (2x - 3)^{\frac{1}{2}} + 4$

x	y
1.5	4
2	5
6	3

$y = \frac{(x - 4)^2 + 3}{2}$

x	y
4	1.5
5	2
3	6



Math Language

A **cube root function** is the function $f(x) = \sqrt[3]{x}$.

Example 2 The Cube Root Function

a. Graph the cube root function and its inverse.

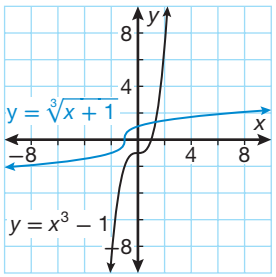
$y = \sqrt[3]{x + 1}$

SOLUTION

Step 1: To find the inverse, interchange y and x and solve for y:

$$x = \sqrt[3]{y + 1}$$
$$x^3 = y + 1$$
$$y = x^3 - 1$$

Step 2: Graph the cube root function and the cubic, its inverse:



Math Reasoning

Write Consider a radical function $f(x) = \sqrt[n]{g(x)}$, where n is an odd number. What restrictions are there to the domain of this function?

b. Graph the cube root function and its inverse.

$y = (2x + 3)^{\frac{1}{3}} - 4$

SOLUTION

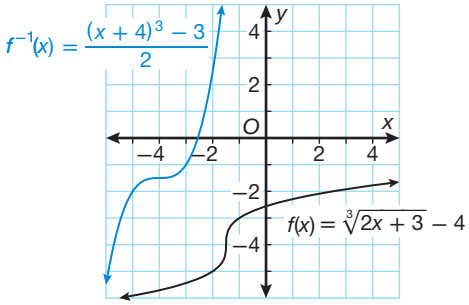
Step 1: To find the inverse, interchange y and x and solve for y:

$$x = (2y + 3)^{\frac{1}{3}} - 4$$

$$(x + 4)^3 = 2y + 3$$

$$y = \frac{(x + 4)^3 - 3}{2}$$

Step 2: Graph the cube root function and the cubic, its inverse:



Example 3 Radical Functions with Different Indexes

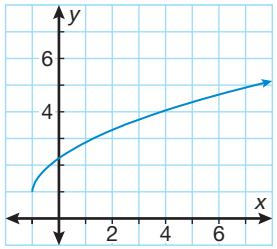
Graph the radical function using a graphing calculator. Determine the domain and range.

$f(x) = \sqrt{x + 1} + \sqrt[3]{x + 2}$

Math Reasoning

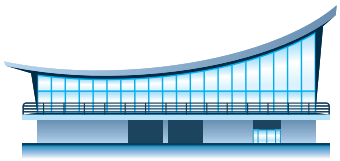
Analyze Explain why $f(x)$ does not accept x values less then -1.

SOLUTION The domain of the function is $x \geq -1$ and the range is $f(x) \geq 1$.



Example 4 Application: Architecture

An architect is constructing a building with a roof in the shape of a hyperbola. The equation shown below is for a 1:10 scale model. Graph the function and find the height of the building at $x = 1$ and $x = 15$.

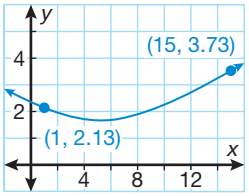


$$y = \frac{\sqrt{5^2 + (x - 5)^2}}{3}$$

SOLUTION

Height at $1 = 21.3$

Height at $15 = 37.3$



Lesson Practice

a. (Ex 1) Graph the square root function and its inverse. Determine the domain and range of both functions. $y = \sqrt{3x - 2}$

b. (Ex 1) Graph the square root function and its inverse. Determine the domain and range of both functions. $y = (4x + 9)^{\frac{1}{2}}$

c. (Ex 2) Graph the cube root function and its inverse. $y = \sqrt[3]{4x + 3}$

d. (Ex 2) Graph the cube root function and its inverse. $y = (5x - 4)^{\frac{1}{3}} + 7$

e. (Ex 3) Graph the radical function. Determine the domain and range.

$$f(x) = \sqrt{3x - 1} + \sqrt[3]{4x + 9}$$

f. (Ex 4) Graph the radical function and find where it intersects $y = 3$.

$$y = \frac{\sqrt{6^2 + (x - 7)^2}}{4}$$

Practice Distributed and Integrated

***1.** (71) Find the area of $\triangle ABC$ with the given side lengths and included angle. $b = 11$, $c = 7$, $m\angle A = 120°$

2. (66) Find the roots and give the multiplicity of each root: $2x^3 - 4x^2 + 2x = 0$

3. (57) **Medicine** If a patient ingests medicine containing a grams of a drug, the amount left in the patient's bloodstream after t hours is given by the equation $y = ae^{-0.119t}$. After how much time will exactly one-third of the drug still be circulating in the bloodstream?

***4.** (75) **Multi-Step** **a.** Find the radical function that is the inverse of $y = (x - 3)(x^2 + 3x + 9)$.

b. Identify its domain and range.

5. (69) **Error Analysis** Which student made an error in finding $|4 + 11i|$? Describe the error.

Student A	Student B
$\sqrt{4^2 + 11^2}$	$\sqrt{4^2 - 11^2}$
$\sqrt{16 + 121}$	$\sqrt{16 - 121}$
$\sqrt{137}$	$\sqrt{-105} = i\sqrt{105}$

***6.** (74) **Celebrating** On the Fourth of July, a technician on Bedloe's Island is planning to shoot confetti straight up behind the Statue of Liberty. The function $y = -4.9x^2 + 45x + 1$ gives the height of a piece of confetti, in meters, after x seconds. The statue has a height of 93 meters. Use the discriminant to explain if a piece of confetti will reach the top of the statue.

7. (70) **Sports** In basketball, the basket through which the ball is thrown consists of a metal hoop that has a 9-inch radius. Use the formula $r = \sqrt{\frac{A}{\pi}}$ to find the area of the hoop.

8. (67) **Multiple Choice** What is the solution set of the equation $2 \cos \theta + \sqrt{2} = 0$, for $0° \leq \theta \leq 360°$?

A $\{30°, 330°\}$ **B** $\{45°, 135°\}$ **C** $\{30°, 150°\}$ **D** $\{135°, 225°\}$

***9.** (75) **Predict** Consider the radical function $f(x) = (g(x))^{\frac{1}{n}}$. Suppose that the graph of $f(x)$ is a straight line for increasing values of x. What can you conclude about $g(x)$?

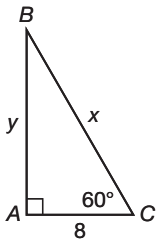
10. (63) **Write** Describe how to determine $\tan 240°$ by using its reference angle.

***11.** (68) **Probability** In a factory, a sample of items produced is checked during all three work shifts. The table shows the results of these checks for one day. What is the probability that an item was checked during the third shift given that it was found to be defective?

	1st Shift	2nd Shift	3rd Shift
Defective	3	5	18
Not Defective	119	109	132

12. (52) **Multi-Step** **a.** Given the triangle, find the lengths of side x and y.

b. Draw a perpendicular line to the hypotenuse from vertex A, calling the new point of intersection D. Find the length of AD and DC.



***13.** (71) **Write** What is meant by *triangulation*?

14. (61) **Geometry** The volume of a right prism is the product of the area of its base and its height. The height of a right prism with a volume of $6x^3 + 69x^2 + 3x - 330$ cubic units is $x + 11$ units. Find a polynomial expression for the area of the base.

15. (62) Write each expression in terms of i, and then subtract: $(-1 + \sqrt{-9}) - (-1 - \sqrt{-64})$.

16. (65) **Physics** The number of seconds it takes for a rocket fired from a deck 10 feet off the ground at a velocity of 50 feet per second to return to the ground is given by $-16t^2 + 50t + 10 = 0$. Solve the equation. Round to the nearest hundredth.

Determine whether the linear binomial is a factor of the polynomial.

17. (61) $P(x) = x^3 - 10x^2 + 19x + 30; x - 6$

18. (61) $P(x) = 3x^3 - 2x^2 - 61x - 20; x - 4$

19. (66) **Write** Tell what is meant by the *multiplicity* of a root.

Write the logarithmic equations in exponential form.

20. (64) $\log_{10} 100{,}000 = 5$

21. (64) $\log_4 1024 = 5$

*22. (75) **Error Analysis** Two students are finding the inverse of $y = \sqrt[4]{x^2 + 4}$ but get different results. Which student made the mistake?

Student A	Student B
$x = \sqrt[4]{y^2 + 4}$ $x^4 = y^2 + 4$ $y = \sqrt{x^4 - 4}$	$x = \sqrt[4]{y^2 + 4}$ $x^{\frac{1}{4}} = y^2 + 4$ $y = \sqrt{x^{\frac{1}{4}} - 4}$

23. (67) Find all the possible values of $\text{Sin}^{-1}\left(\frac{1}{2}\right)$.

24. (55) A calculator is programmed to generate random integers from 1 to 100 inclusive when the enter button is pressed. If the enter button is pressed twice, what is the probability of getting a multiple of 10 and a number greater than 60?

*25. (74) **Multiple Choice** The discriminant of a quadratic equation is -2. Which describes the solutions of the equation?

A 1 real solution **B** 1 complex solution **C** 2 real solutions **D** 2 complex solutions

Solve.

26. (72) $\log_6 \frac{5}{8}$

27. (72) $\log_6 40$

28. (62) **Multiple Choice** What are the solutions of $x^2 + 8x = -25$?

A $\pm 3i$ **B** $\pm 5i$ **C** $-4 \pm 3i$ **D** $4 \pm 5i$

*29. (73) **Graphing Calculator** To generate single digit random numbers on a calculator, press MATH and select [**randInt(**] from the PRB menu and type 0, 9 in parentheses. Then press enter.

a. Form a simple random sample of 3 individuals from the 10 listed below.

0: Greg	1: Pam	2: Tyler	3: Lakisha	4: Jon
5: Larry	6: Mark	7: Tammy	8: Li	9: Diane

b. Under what circumstances would you have to press enter more than 3 times?

*30. (72) **Justify** Explain why $\ln e^{2x-1} = 2x - 1$.

LESSON 76

Finding Polynomial Roots I

Warm Up

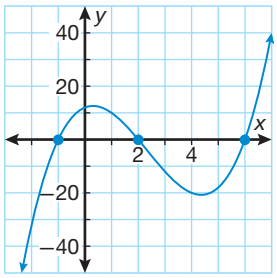
1. (23) **Vocabulary** In the expression $2x(x + 3)$, the term outside the parentheses is called a monomial _____.

2. (23) Factor the expression $2x^2 - 12x$.

3. (65) Solve using the quadratic formula. $5x^2 - 12x + 13 = 0$

New Concepts

Polynomial roots are the solutions of a polynomial equation. These roots are the zeros of the related polynomial function which are the x-intercepts of the graph of the function.

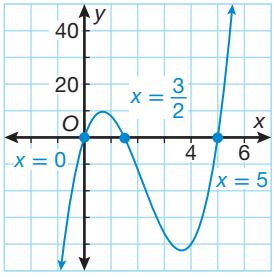


Example 1 Finding the Roots of a Factored Polynomial

Find the roots of the polynomial function.

$0 = 2x(x - 5)(2x - 3)$

SOLUTION Since the polynomial is already factored and cannot be factored further, set each factor equal to zero.



$$2x = 0 \qquad x - 5 = 0 \qquad 2x - 3 = 0$$

$$x = 0 \qquad x = 5 \qquad x = \frac{3}{2}$$

The roots are 0, 5 and $\frac{3}{2}$.

Math Reasoning

Verify If a polynomial function $f(x)$ has a zero $x = 0$, then explain why the polynomial has no constant term.

Example 2 Finding Monomial Factors to Determine Zeros

a. Find the zeros of the polynomial function.

$f(x) = 6x^3 + 10x^2 + 4x$

SOLUTION

Step 1: When there are no constant terms, there is a linear factor.

$6x^3 + 10x^2 + 4x = 0 \rightarrow 2x(3x^2 + 5x + 2) = 0 \rightarrow 2x = 0 \rightarrow x = 0$

Step 2: Now factor the quadratic using the quadratic formula.

$$x = \frac{-5 \pm \sqrt{5^2 - 4(3)(2)}}{(2)(3)} = \frac{-5 \pm \sqrt{25 - 24}}{6} = \frac{-5 \pm 1}{6}$$

The zeros are 0, -1, and $-\frac{2}{3}$.

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Math Reasoning

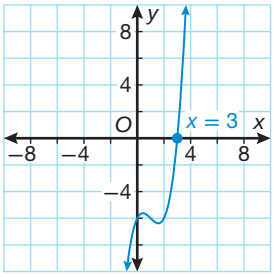
Analyze If the function $f(x)$ has a binomial factor of degree 2, does this factor always provide a real zero of the polynomial function?

b. Find the real zeros of $f(x) = x^4 - 3x^3 + 2x^2 - 6x$.

SOLUTION

Step 1: Identify the monomial factor: $x(x^3 - 3x^2 + 2x - 6) = 0$.

Step 2: Graph the cubic factor to identify additional roots.



Step 3: Divide the cubic factor by $(x - 3)$ to find the quadratic factor.

$$\begin{array}{r} x^2 + 2 \\ x - 3 \overline{)\, x^3 - 3x^2 + 2x - 6} \\ -(x^3 - 3x^2) \\ \hline 0 \qquad 2x - 6 \\ -(2x - 6) \\ \hline 0 \end{array}$$

So, $x^4 - 3x^3 + 2x^2 - 6x = x(x - 3)(x^2 + 2)$. The real zeros are $x = 0$ and $x = 3$.

Example 3 Finding Binomial Factors to Determine Roots

a. Find the real roots of $0 = (x + 3)(x^2 + 4) - (x + 3)(4x + 1)$.

SOLUTION

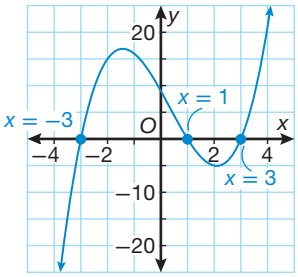
Step 1: Look for binomial factors to simplify the expression.

$$(x + 3)(x^2 + 4) - (x + 3)(4x + 1) = 0$$
$$(x + 3)(x^2 - 4x + 3) = 0$$

Step 2: Factor the quadratic expression to find the remaining roots.

$$(x + 3)(x - 3)(x - 1) = 0$$

The real roots are $x = -3$, $x = 3$, and $x = 1$.



b. Find the real roots of the polynomial equation.

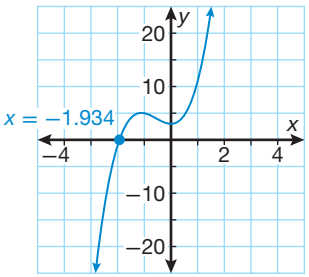
$$0 = (x - 7)(3x^3 + 4) + (x - 7)(5x^2 - 1)$$

SOLUTION

Step 1: Look for binomial factors to simplify the expression.

$$(x - 7)(3x^3 + 4) + (x - 7)(5x^2 - 1) = 0$$
$$(x - 7)(3x^3 + 5x^2 + 3) = 0$$

Step 2: Graph the cubic factor to identify additional roots.



So, $f(x) = (x - 7)(3x^3 + 4) + (x - 7)(5x^2 - 1)$ has two real roots, $x = -1.934$ and $x = 7$.

Example 4 Solving Polynomial Equations

Solve the following equation.

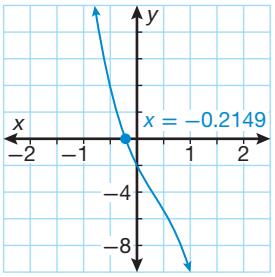
$5x^4 = 5x^3 - 8x^2 - 2x$

SOLUTION

Step 1: In this equation there is a monomial equal to a trinomial. Place all the terms on the same side of the equation and factor the linear term.

$x(-5x^3 + 5x^2 - 8x - 2) = 0$

Step 2: Graph the cubic factor to identify additional roots.



So, $x(-5x^3 + 5x^2 - 8x - 2) = 0$ has two real roots, $x = 0$ and $x \approx -0.2149$.

Reading Math

The use of the subscript 0 is commonly used in physics to denote the initial instance of time, height, velocity, etc.

Example 5 Application: Projectile Motion

The equation below can be used to find the distance from the ground at time t for a ball thrown from a height h_o at an initial speed of v_o. Suppose a ball is thrown with an initial speed of $v_o = 5$ from a height $h_o = 25$. How long will it take the ball to hit the ground?

$h = -4.9t^2 + v_o t + h_o$

SOLUTION

Finding the time means solving the following equation.

$0 = -4.9t^2 + 5t + 25$

This equation can be solved using the quadratic formula.

$$t = \frac{-5 \pm \sqrt{5^2 - 4(-4.9)(25)}}{2(-4.9)}$$

$$= \frac{-5 \pm \sqrt{515}}{-9.8}$$

$t \approx -1.80547$ or 2.82588

Ignoring the negative root, the time it will take for the ball to hit the ground is approximately 2.83 seconds.

Lesson Practice

Find the roots or zeros.

a. (Ex 1) $0 = 4x(2x - 7)(x + 9)$

b. (Ex 2) $f(x) = 7x^3 - 8x^2 + 3x$

c. (Ex 2) $f(x) = -9x^3 + 11x^2 + 4x$

d. (Ex 3) $0 = (x + 4)(x^2 - 9) - (x + 4)(4x + 1)$

e. (Ex 3) $0 = (x - 12)(4x^2 + 3) - (x - 12)(7x - 2)$

f. (Ex 4) Solve the following equation. $6x^3 = 4x^4 + 3x^2 - 7x$

g. (Ex 5) Suppose a ball is thrown with an initial speed of $v_o = 18$ from a height $h_o = 35$. To the nearest hundredth of a second, how long will it take the ball to hit the ground? Use the projectile motion equation below.

$$h = -4.9t^2 + v_o t + h_o$$

Practice Distributed and Integrated

***1.** (76) Let $f(x) = x^2 - 4$ and $g(x) = x + 1$. Find the roots of $f(g(x))$.

2. (50) Find an equation for the inverse of $y = -2x^5 - 10$.

***3.** (75) The volume of an ice cube has decreased from x^3 to $\frac{x^3}{3} - 9$, while still maintaining its cubic shape. What is the new length of each side of the cube?

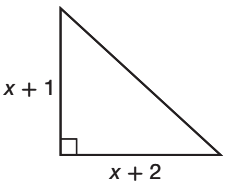
4. (63) Find the sine, cosine, and tangent of 270°.

5. (50) Find an equation for the inverse of $y = 4x^2 + 12$.

***6.** (76) **Multiple Choice** Which of the following is not a factor of $x^3 - 6x^2 + 11x - 6$?

A $(x - 4)$ **B** $(x - 3)$ **C** $(x - 2)$ **D** $(x - 1)$

***7.** (75) **Geometry** Derive the function that can be used to find the hypotenuse of the right triangle. For what value of x is the hypotenuse part of a Pythagorean triple?



***8.** (75) **Physics** The equation for finding the period of a pendulum is the radical function $T = 2\pi\sqrt{\frac{L}{9.8}}$. The period is the amount of time it takes for the pendulum to complete one back-and-forth sweep. How much longer does a pendulum need to be to double the period?

9. (57) **Biology** In the absence of predators and diseases, and with enough food, a population of brown rats can increase by 6300% per year. Suppose you start with an initial population of three breeding pairs of rats. Write an exponential function to model the population size after t years.

10. (73) **Sports** A reporter went to a baseball game and asked fans that passed through the front gate at the end of the game if they thought the town could use a larger, more improved stadium. 98% of those surveyed answered *yes.* The reporter wrote in the paper that "98% of the city's population wants a new stadium." Discuss the problems with this situation.

11. (66) **Multiple Choice** Which is a possible rational root of $2x^3 + x^2 - 72x = 36$?

A 7 **B** 9 **C** 10 **D** 72

12. (72) **Multi-Step** Two measures of sound are loudness and intensity. They are related by the function $L(I) = 10\log \frac{I}{I_0}$, where $L(I)$ is loudness in decibels, I is intensity in watts per square meter, and $I_0 \approx 10^{-12}$ watt per square meter, the intensity of a barely audible sound.

a. The loudness of conversational speech is about 60 decibels. Find the intensity of a sound with a loudness of 60 decibels. Write your answer without an exponent.

b. The threshold of discomfort to the human ear is about 120 decibels. Find the intensity of a sound with a loudness of 120 decibels.

c. When the loudness doubles from 60 dB to 120 dB, by what factor is the intensity multiplied?

13. (68) **Horses** A paddock contains 5 thoroughbreds, 6 quarter horses, 8 mustangs, and 2 ponies. Find the probability that one thoroughbred and one quarter horse will be chosen at random without replacement for the upcoming horse show.

***14.** (76) **Multi-Step** Suppose that $f(x) = (x - 5)(x^2 - 12) + (x - 5)(-3x - x)$.

a. Find the zeros of the function.

b. Based on the solution in part **a**, how many x-intercepts does the graph of the $f(x)$ have?

c. Simplify $f(x)$ and determine its degree.

15. (71) **Analyze** In $\triangle PQR$, $m\angle P = 35°$, $r = 18$, and $p = 21$. Are there one or two possible triangles that can have these measures? Explain.

16. (61) The volume of a rectangular solid with a length of $x - 8$ feet has a volume of $x^3 - 2x^2 - 40x - 64$ cubic feet. Factor the expression for the volume completely.

***17.** (74) **Graphing Calculator** Graph $y = -3.25x^2 - 0.3x + 7.5$ to determine if $-3.25x^2 - 0.3x + 7.5 = 0$ has one, two, or zero real solutions.

18. (70) **Error Analysis** Explain the error the student made below.

$$x - 2 = \sqrt{3x}$$
$$(x - 2)^2 = (\sqrt{3x})^2$$
$$x^2 - 4 = 3x$$
$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$
$$x = 4 \quad \text{or} \quad x = -1$$

19. (58) **Projectile Motion** The Leaning Tower of Pisa in Italy measures 180.4 feet from the ground on the lower side of the tower. If an object is thrown into the air, with an initial velocity of 96 feet per second, from the top of the lower side, the height, h, of the object in feet after t seconds can be described by the equation: $h = -16t^2 + 96t + 180.4$. Find the time at which the object will reach the ground. Round your answer to the tenths place.

20. (67) **Estimate** Use properties of the unit circle and special angles to estimate $\cos^{-1}(0.48)$ in degrees. Do not use a calculator. Explain your method.

21. (65) **Error Analysis** Find and correct the error a student made in solving $6x^2 + 5 - x = 0$.

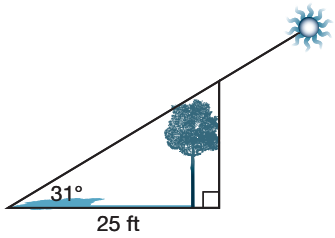
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{25 - 4(6)(-1)}}{12}$$

$$= \frac{-5 \pm \sqrt{49}}{12}$$

$$= \frac{-5 \pm 7}{12} = \frac{1}{6} \text{ and } -1$$

22. (52) A tree casts a shadow as shown in the picture. What is the height of the tree?



23. (69) **Verify** Show why $2i\sqrt{2}$ is a square root of -8.

24. (57) A patient is injected with medication. $C(t) = Ae^{-0.42t}$ gives the amount of the medication in the bloodstream after t hours. If the initial dose is 300 mg, after how many hours will it be 70 mg?

***25.** (76) **Analyze** Suppose that polynomial function $f(x)$ has n zeros and polynomial function $g(x)$ has m zeros. What must be true for $f(x) \cdot g(x)$ to have $n + m$ zeros?

26. (60) Mark rolled a 1–6 number cube four times. What is the probability of Mark rolling an odd number each time?

***27.** (76) Find the roots of the polynomial function $3x(2x - 9)(x + 5)$.

28. (55) **Probability** A photo documentary includes 36 slides in a portfolio. If the teacher randomly chooses 5 slides from the portfolio, what is the probability that the teacher picks either the first five slides or the last five slides?

Find the discriminant of the quadratic equation and give the number and type of solutions of the equation.

29. (74) $x^2 - 8x + 16 = 0$

30. (74) $s^2 + 7s + 11 = 0$

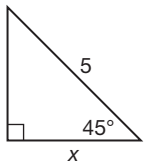
LESSON 77

Using the Law of Cosines

Warm Up

1. (46) **Vocabulary** For an acute angle in a right triangle, the ratio of the adjacent side to the hypotenuse is the _____ of the angle.

2. (46) Find the value of x to the nearest tenth.



3. (67) Estimate $m\angle A$ if $\cos A \approx 0.454$.

4. (71) Find b in $\triangle ABC$ given $m\angle A = 95°$, $m\angle B = 20°$, and $c = 14$.

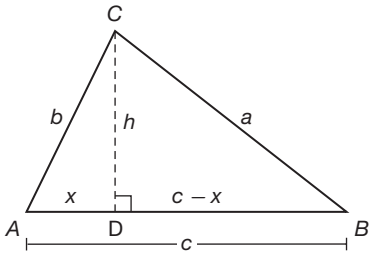
New Concepts

For a triangle that is not a right triangle, the Law of Sines can sometimes be used to find a missing measure. It can be used when given two angle measures and a side length or two side lengths and a nonincluded angle measure.

The **Law of Cosines** can be used other times, specifically when

- the measures of two sides and the included angle are known (side-angle-side (SAS)).
- the measures of all three sides of a triangle are known (side-side-side (SSS)).

To derive the Law of Cosines, let h be the altitude and $AD = x$ in $\triangle ABC$ shown at right.



Then $DB = c - x$.

In $\triangle ABC$, $b^2 = x^2 + h^2$ and $\cos A = \frac{x}{b}$ or $x = b\cos A$.

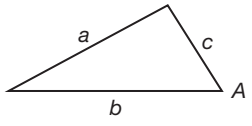
Write an equation that relates the side lengths of $\triangle DCB$.

$$a^2 = (c - x)^2 + h^2 \quad \text{Pythagorean Theorem}$$
$$= c^2 - 2cx + x^2 + h^2 \quad \text{Square the binomial.}$$
$$= c^2 - 2cx + b^2 \quad \text{Substitute } b^2 \text{ for } x^2 + h^2.$$
$$= c^2 - 2cb\cos A + b^2 \quad \text{Substitute } b\cos A \text{ for } x.$$
$$= b^2 + c^2 - 2cb\cos A \quad \text{Commutative Property of Addition}$$
$$= b^2 + c^2 - 2bc\cos A \quad \text{Commutative Property of Multiplication}$$

Math Reasoning

Generalize State all the patterns you see in the Law of Cosines.

Law of Cosines



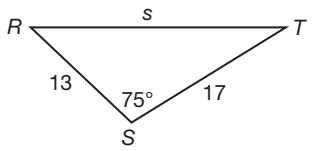
By the Law of Cosines,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Online Connection
www.SaxonMathResources.com

Example 1 Finding the Unknown Side of a Triangle

Solve $\triangle RST$.



SOLUTION Find s by using $s^2 = r^2 + t^2 - 2rt\cos S$.

$s^2 = 17^2 + 13^2 - 2(17)(13)\cos 75°$ Substitute known values.

$s^2 \approx 343.6$ Simplify the right side.

$s \approx 18.5$ Take the positive square root.

Find one of the remaining angle measures by using the Law of Sines.

Use $\frac{\sin S}{s} = \frac{\sin R}{r}$ to find $m\angle R$.

$$\frac{\sin 75°}{18.5} = \frac{\sin R}{17}$$

$$18.5\sin R = 17\sin 75°$$

$$\sin R = \frac{17\sin 75°}{18.5} \approx 0.8876$$

$$m\angle R \approx 63°$$

Find the measure of the third angle by subtracting: $180° - (75° + 63°) = 42°$.

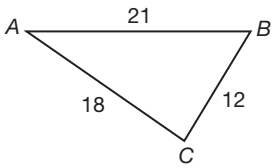
$s \approx 18.5$, $m\angle R \approx 63°$, $m\angle T \approx 42°$

Hint

Check solutions for reasonableness by remembering that the longest side is opposite the largest angle.

Example 2 Finding the Unknown Angle of a Triangle

a. Find $m\angle C$. Round to the nearest tenth.



SOLUTION Use $c^2 = a^2 + b^2 - 2ab\cos C$.

$21^2 = 12^2 + 18^2 - 2(12)(18)\cos C$ Substitute the known values.

$441 = 468 - 432\cos C$ Simplify.

$0.0625 = \cos C$ Solve for cos C.

$86° \approx m\angle C$ Use $\text{Cos}^{-1}x$.

b. In $\triangle DEF$, $d = 60$, $e = 48$, and $f = 14$, find $m\angle D$. Round to the nearest tenth.

SOLUTION Use $d^2 = e^2 + f^2 - 2ef\cos D$.

$60^2 = 48^2 + 14^2 - 2(48)(14)\cos D$ Substitute the known values.

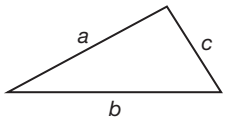
$3600 = 2500 - 1344\cos D$ Simplify.

$-0.8185 \approx \cos D$ Solve for cos D.

$145° \approx m\angle D$ Use $\text{Cos}^{-1}x$.

The area of a triangle can be determined when the three sides of the triangle are the only measures given using **Heron's Formula.** The Law of Cosines can be used to prove Heron's Formula. This proof is left for more advanced mathematics courses.

Heron's Formula

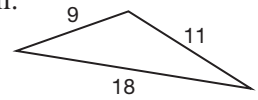


$$A = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where}$$

$$s = \frac{1}{2}(a+b+c).$$

Example 3 Using Heron's Formula to Find the Area of a Triangle

Find the area of the triangle. Round to the nearest tenth.



SOLUTION First find s.

$$s = \frac{1}{2}(a+b+c)$$

$$s = \frac{1}{2}(9+11+18) = 19$$

Then use $A = \sqrt{s(s-a)(s-b)(s-c)}$.

$$A = \sqrt{19(19-9)(19-11)(19-18)}$$

$$A = \sqrt{19(10)(8)(1)}$$

$$A = \sqrt{19(10)(8)(1)} = \sqrt{1520} \approx 39$$

The area of the triangle is about 39 square units.

Math Reasoning

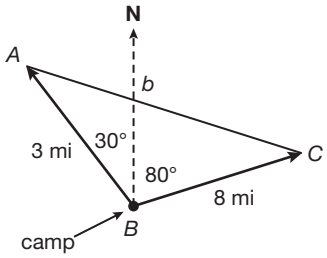
Justify Why does it not matter which values are substituted for a, b, and c?

Example 4 Application: Orienteering

Hiker A leaves camp and hikes in a straight path for 3 miles at an angle 30° west of north. Hiker B leaves the same camp and hikes in a fairly straight path for 8 miles at an angle 80° east of north. If Hiker A stays where he is and Hiker B uses her GPS device to walk straight towards him, estimate the time it will take Hiker B to reach Hiker A if Hiker B's average hiking rate is 2.75 miles per hour.

SOLUTION

Draw a diagram of the situation.



Use the Law of Cosines to find b.

Then use $t = \frac{d}{r}$ to find the time.

Use $b^2 = a^2 + c^2 - 2ac\cos B$.

$$b^2 = 8^2 + 3^2 - 2(8)(3)\cos 110°$$

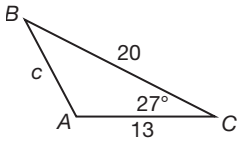
$$b^2 \approx 89.42$$

$$b \approx 9.5$$

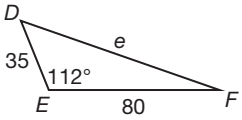
The distance is about 9.5 miles. Divide the distance by the rate to find the time: $t \approx \frac{9.5}{2.75} \approx 3.5$. It will take Hiker B about 3 hours 30 minutes to reach Hiker A.

Lesson Practice

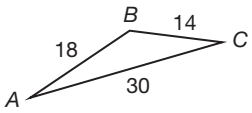
a. (Ex 1) Find c. Round to the nearest tenth.



b. (Ex 1) Solve $\triangle DEF$.

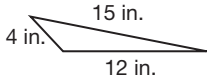


c. (Ex 2) Find $m\angle C$. Round to the nearest tenth.



d. (Ex 2) Given $\triangle FGH$, $f = 9.2$, $g = 5$, and $h = 10.1$, find $m\angle G$. Round to the nearest tenth.

e. (Ex 3) Find the area of the triangle. Round to the nearest tenth.



f. (Ex 4) A boat leaves a dock and travels in a straight path for 22 miles at an angle 75° west of north. Another boat leaves the same dock and travels in a straight path for 40 miles at an angle 60° east of north. If the second boat puts down her anchor and the first boat travels directly towards the second, estimate the time it will take the first boat to reach the second boat if the first boat travels at a constant rate of 14 miles per hour.

Practice Distributed and Integrated

***1.** (17) **Safety Regulations** A company's safety regulations require the wire rope used to suspend a 100-ton load to have a thickness that is within $\frac{3}{8}$ inch of $1\frac{3}{4}$ inches. Write and solve an absolute value equation to find the minimum safe thickness of rope.

Write the exponential equations in logarithmic form.

2. (64) $3^7 = 2187$

3. (64) $12^2 = 144$

***4.** (74) Find the discriminant of the quadratic equation $7r^2 - 5 = 2r + 9r^2$. Give the number and type of solutions of the equation.

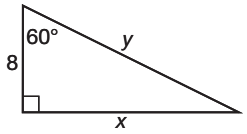
Combine into one logarithm.

5. (72) $6 \ln x + 4 \ln y$

6. (72) $\ln 40 + 2 \ln \frac{1}{2} + \ln x$

***7.** (77) **Generalize** A student said that every time the Law of Cosines is used to find an angle measure, both sides of the equation will be divided by a negative number. Is the student correct or incorrect? Explain.

8. *(52)* Find the exact values of x and y.



***9.** *(75)* **Multiple Choice** Which of the following is equivalent to $y = \sqrt[4]{\sqrt[3]{x^3 + 2x^2 + 4}}$?

A $y = (x^3 + 2x^2 + 4)^{\frac{1}{12}}$ **B** $y = (x^3 + 2x^2 + 4)^{\frac{3}{4}}$

C $y = (x^3 + 2x^2 + 4)^{\frac{4}{3}}$ **D** $y = (x^3 + 2x^2 + 4)^{12}$

***10.** *(68)* **Graphing Calculator** A school is made up of 334 freshmen and 229 sophomores. Four students are chosen at random for a newspaper interview. Use a calculator to find the probability that all four students are freshmen.

***11.** *(77)* **Coordinate Geometry** Explain how to use Heron's formula to find the area of a triangle whose vertices are at $(-7, -5)$, $(1, 7)$, and $(2, -1)$. Then find the area.

12. *(73)* **Formulate** Write a formula that estimates an animal population N, when M are marked on the first visit, and m of the n captured on the second visit are marked.

13. *(62)* **Error Analysis** Describe and correct the error in simplifying the expression.

$$(1 + 2i)(4 - i) = 4 - i + 8i - 2i^2 = 2i^2 + 7i + 4$$

***14.** *(Inv 4)* A message is put into matrix A and encoded using cipher matrix C. Will the same encoded message result from AC as CA? Why?

15. *(26)* Find the equation of the line in slope-intercept form that passes through $(5, -3)$ and has a slope of $\frac{2}{9}$.

16. *(69)* **Multiple Choice** Which expression is *not* equivalent to 1?

A i^{18} **B** i^{24} **C** i^{32} **D** i^{56}

***17.** *(77)* A triangle has side lengths of 14 feet, 15 feet, and 21 feet. Use Heron's formula to find the area of the triangle.

18. *(56)* P is a point on the terminal side of θ in standard position. Find the exact value of the six trigonometric functions for θ where $P(-5, 5)$.

19. *(66)* **Multi-Step** A rectangular prism with a square base has a height that is 8 inches longer than the length of one side of the base. Its volume is 325 cubic inches.

a. Write a polynomial equation in standard form that can be used to find the length of one side of the base.

b. List the possible rational roots.

c. Find the length of one side of the base.

*20. (76) **Error Analysis** A student is solving for the roots of $y = (x^2 - 3)(x^2 + 2) - (x^2 - 3)(6x + 5)$, what mistake did he make?

$$(x^2 - 3)(x^2 + 2) - (x^2 - 3)(6x + 5) = (x^2 - 3)(x^2 + 6x + 7)$$

Use the quadratic formula,

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)} = \frac{6 \pm \sqrt{36 - 28}}{2} = 3 \pm \sqrt{2}$$

There are a total of four roots of $f(x)$: $x = \pm\sqrt{3}$ and $3 \pm \sqrt{2}$

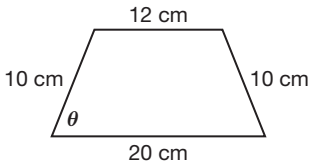
21. (65) Solve using the quadratic formula. Write the solution as complex numbers in standard form. $f(x) = x^2 + 4$

*22. (76) **Generalize** Suppose that the polynomial functions $f(x)$ and $g(x)$ are each a sum of square terms. Give an example where $f(x) \cdot g(x) = 0$ has exactly two complex solutions.

23. (70) **Verify** Explain how to check that $x = 7$ is a solution of $\sqrt{6 + x} - \sqrt{2x - 1} = 0$ by using a graphing calculator.

*24. (72) **Geology** The Richter magnitude M of an earthquake is expressed by $M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$, where E is the energy released in ergs. The 1906 San Francisco earthquake, with a magnitude between 7.7 and 8.3, was the worst earthquake disaster in U.S. history. Describe the amount of energy released by that earthquake.

25. (67) **Geometry** What is the measure of the base angle θ of the isosceles trapezoid, to the nearest tenth of a degree?



26. (61) **Sports** The area of a tennis court being used for the Special Olympics has an area that can be represented by $x^2 - 24x - 432 + 18x$ square feet. Factor this polynomial by grouping.

27. (60) There are 20 orange marbles and 13 green marbles in a sack. An orange marble is randomly selected and replaced. Then a green marble is randomly selected. Are these events dependent or mutually exclusive?

28. (74) **Generalize** What will be the value of the discriminant of an equation if the vertex of the related function is located in Quadrant II and the parabola opens upward? Why?

29. (59) **Probability** A farmer has four sheepdogs and three beagles. If he randomly chooses a dog to accompany him on a walk, what is the probability of him taking a walk with a sheepdog?

30. (58) Solve: $2x + 5 = x^2$

LESSON
78

Solving Quadratic Equations II

Warm Up

1. (65) **Vocabulary** The expression $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is part of the _____.

2. (65) Solve $5x^2 - 3x + 5 = 0$.

3. (35) Find the zero(s) of $y = x^2 + 8x + 16$ graphically.

New Concepts

Some quadratic equations can be solved by factoring the *difference of two squares* and then applying the Zero Product Property, which states that if a product is zero, then at least one of its factors is zero.

Difference of Two Squares
$a^2 - b^2 = (a + b)(a - b)$

Example 1 Solving Quadratic Equations

a. Solve $9x^2 - 25 = 0$.

SOLUTION $9x^2 - 25 = 0$

$(3x)^2 - (5)^2 = 0$ Write the left side as $a^2 - b^2$.

$(3x + 5)(3x - 5) = 0$ Factor the difference of squares.

$3x + 5 = 0$ or $3x - 5 = 0$ Zero Product Property

$3x = -5 \qquad 3x = 5$

$x = -\frac{5}{3}$ or $x = \frac{5}{3}$ Solve each equation.

The roots are $-\frac{5}{3}$ and $\frac{5}{3}$.

Check Substitute each value into the original equation.

$9x^2 - 25 = 0$

$9\left(-\frac{5}{3}\right)^2 - 25 \stackrel{?}{=} 0$

$9\left(\frac{25}{9}\right) - 25 \stackrel{?}{=} 0$

$25 - 25 \stackrel{?}{=} 0$

$0 = 0$ ✓

$9x^2 - 25 = 0$

$9\left(\frac{5}{3}\right)^2 - 25 \stackrel{?}{=} 0$

$9\left(\frac{25}{9}\right) - 25 \stackrel{?}{=} 0$

$25 - 25 \stackrel{?}{=} 0$

$0 = 0$ ✓

Math Language

To solve an equation means to find its *solutions*, or *roots*.

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b. Solve $7x^2 = 63$.

SOLUTION

$7x^2 = 63$

$7x^2 - 63 = 0$ Subtract 63 from both sides.

$7(x^2 - 9) = 0$ Factor: The GCF is 7.

$7(x + 3)(x - 3) = 0$ Factor the difference of squares.

$x + 3 = 0$ or $x - 3 = 0$ Zero Product Property

$x = -3$ or $x = 3$ Solve each equation.

The roots are –3 and 3.

Example 2 Solving a Cubic Equation with a Quadratic Factor

Solve $10x^3 - 10x = 0$.

SOLUTION

$10x^3 - 10x = 0$ Factor: The GCF is $10x$.

$10x(x^2 - 1) = 0$ Factor the difference of squares.

$10x(x + 1)(x - 1) = 0$ Zero Product Property

$10x = 0$ or $x + 1 = 0$ or $x - 1 = 0$ Solve each equation.

$x = 0$ or $x = -1$ or $x = 1$

The roots are 0, −1, and 1.

Check Substitute each value into the original equation.

$10x^3 - 10x = 0$	$10x^3 - 10x = 0$	$10x^3 - 10x = 0$
$10(0)^3 - 10(0) \stackrel{?}{=} 0$	$10(-1)^3 - 10(-1) \stackrel{?}{=} 0$	$10(1)^3 - 10(1) \stackrel{?}{=} 0$
$10(0) - 0 \stackrel{?}{=} 0$	$10(-1) + 10 \stackrel{?}{=} 0$	$10(1) - 10 \stackrel{?}{=} 0$
$0 - 0 \stackrel{?}{=} 0$	$-10 + 10 \stackrel{?}{=} 0$	$10 - 10 \stackrel{?}{=} 0$
$0 = 0$ ✓	$0 = 0$ ✓	$0 = 0$ ✓

Hint

The Zero Product Property applies to any number of factors. If $abc = 0$, then $a = 0$ or $b = 0$ or $c = 0$.

Example 3 Solving a Quartic Equation

a. Solve $x^4 - 16 = 0$.

SOLUTION

$x^4 - 16 = 0$

$(x^2)^2 - (4)^2 = 0$ Write the left side as $a^2 - b^2$.

$(x^2 + 4)(x^2 - 4) = 0$ Factor the difference of squares.

$(x^2 + 4)(x + 2)(x - 2) = 0$ Factor the difference of squares again.

$x^2 + 4 = 0$ or $x + 2 = 0$ or $x - 2 = 0$ Zero Product Property

$x^2 = -4$

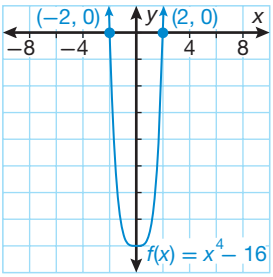
$x = \pm\sqrt{-4}$

$x = \pm 2i$ or $x = -2$ or $x = 2$ Solve each equation.

The complex roots are $2i$ and $-2i$.

The real roots are -2 and 2.

b. Verify the real roots of $x^4 - 16 = 0$ graphically.



SOLUTION Graph the related function $f(x) = x^4 - 16$. The x-intercepts are -2 and 2. They are the real roots of the equation $x^4 - 16 = 0$.

Example 4 Solving a Quadratic Equation Using Perfect Squares

Solve $(9x^2 - 12x + 4) - (x^2 + 10x + 25) = 0$.

SOLUTION

$(9x^2 - 12x + 4) - (x^2 + 10x + 25) = 0$	The trinomials are perfect squares.
$(3x - 2)^2 - (x + 5)^2 = 0$	Write each trinomial as the square of a binomial. Then factor the difference of squares.
$[(3x - 2) + (x + 5)][(3x - 2) - (x + 5)] = 0$	
$[4x + 3][2x - 7] = 0$	Simplify inside the brackets.
$4x + 3 = 0$ or $2x - 7 = 0$	Zero Product Property
$x = -\frac{3}{4}$ or $x = \frac{7}{2}$	Solve each equation.

The roots are $-\frac{3}{4}$ and $\frac{7}{2}$.

Example 5 Application: Vertical Motion

The height of a free-falling object (neglecting air resistance) is given by the function $h(t) = -4.9t^2 + v_0t + h_0$, where h_0 is the initial height in meters, v_0 is the initial velocity in meters per second, and $h(t)$ is the height in meters at time t seconds. How long does it take for an object to hit the ground after it is dropped from a height of 40 meters?

Hint

If an object is dropped, its initial velocity is zero. If projected down, its initial velocity is negative. If projected up, its initial velocity is positive.

SOLUTION

Understand When the object hits the ground, $h(t) = 0$.

Plan Write and solve an equation that is related to the height function.

Solve

$h(t) = -4.9t^2 + v_0t + h_0$	Write the height function.
$0 = -4.9t^2 + 0t + 40$	Substitute the known values.
$0 = -4.9t^2 + 40$	Simplify.
$0 = 4.9t^2 - 40$	Multiply by -1 to get a positive leading coefficient.
$0 = 49t^2 - 400$	Multiply by 10 to get the difference of two squares.

Math Reasoning

Write Explain why only one side of the equation $0 = -4.9t^2 + 40$ changes when you multiply both sides by -1.

$0 = (7t + 20)(7t - 20)$ Factor the difference of squares.

$7t + 20 = 0$ or $7t - 20 = 0$ Apply the Zero Product Property.

$t = -\frac{20}{7}$ or $t = \frac{20}{7}$ Solve each equation. Use the positive solution.

It takes the object $\frac{20}{7} \approx 2.86$ seconds to hit the ground.

Check $-4.9\left(\frac{20}{7}\right)^2 + 40 = 0$, so the height is 0 when $t = \frac{20}{7}$.

Lesson Practice

Solve.

a. *(Ex 1)* $100x^2 - 1 = 0$

b. *(Ex 1)* $6x^2 = 150$

c. *(Ex 2)* $18x^3 - 32x = 0$

d. *(Ex 3)* $16x^4 - 1 = 0$

e. *(Ex 3)* Verify the real roots of $16x^4 - 1 = 0$ graphically.

f. *(Ex 4)* Solve $(x^2 + 16x + 64) - (36x^2 - 12x + 1) = 0$.

g. *(Ex 5)* The height of a free-falling object (neglecting air resistance) is given by the function $h(t) = -4.9t^2 + v_0t + h_0$, where h_0 is the initial height in meters, v_0 is the initial velocity in meters per second, and $h(t)$ is the height in meters at time t seconds. How long does it take for an object to hit the ground after it is dropped from a height of 62.5 meters?

Practice Distributed and Integrated

***1.** *(77)* In $\triangle LMN$, $MN = 13$, $LN = 3$, and the $m\angle N = 36°$. Find LM.

2. *(56)* Given $\theta = -164°$, find a positive and negative coterminal angle.

Solve.

3. *(66)* $4x(2x^2 + 9x - 45) = 0$

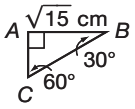
4. *(66)* $2x(x^2 - 4x + 6) = 0$

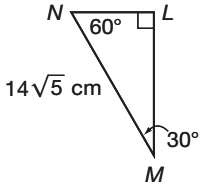
***5.** *(78)* $2x^2 + 9x + 4 = 0$

***6.** *(78)* $3x^3 - 12x = 0$

7. *(52)* **Multiple Choice** In the diagram, triangle ABC is dilated to get triangle LMN. What is the scale factor?

A $\sqrt{5}$ **B** $2\sqrt{5}$ **C** 7 **D** 14





8. *(65)* **Multi-Step** **a.** Write the equation $\frac{3}{4}x^2 + \frac{3}{8}x + 1 = 0$ without fractions.

b. Find the roots of the equation.

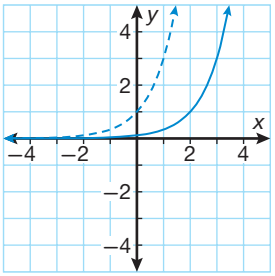
***9.** *(78)* **Analyze** Describe how to solve the equation $2.5x^2 - 0.4 = 0$ by factoring the difference of two squares.

*10. (77) **Multiple Choice** What is the area of $\triangle ABC$ if $a = 34$ m, $b = 25$ m, and $c = 48$ m?

A about 54 m^2 **B** about 404 m^2 **C** about 6147 m^2 **D** about 163,529 m^2

*11. (54) **Graphing Calculator** Describe the feasible region for the inequalities. $0 \le x \le 20$, $y \ge 10$, $6y + 8x \ge 105$.

12. (57) **Error Analysis** To graph the function $y = 3^x + 2$, Eric started out by graphing $y = 3^x$ (the dashed curve). He then translated it 2 units to the right. Explain Eric's error, and write an equation for the function he graphed.



13. (59) **Multi-Step** Use $\left(5^{\frac{1}{2}} \cdot 5^{\frac{2}{7}}\right)^{\frac{2}{11}}$

a. Which property of rational exponents should be applied first to simplify the expression? Explain.

b. Which property of rational exponents should be applied next?

c. Simplify the expression.

14. (60) **Data Analysis** The following table summarizes data obtained by the U.S. Census Bureau on computers in households. If one of the surveyed members of the household is randomly selected, find the probability that the member of the household is male or has a computer. Round to the ten thousandths place.

Gender of Household Member

Computer	Male	Female
Yes	39,475	30,461
No	20,583	22,607

Use synthetic division to divide.

15. (51) $4x^2 + 7x + 10$ by $x + 2$

16. (51) $2x^2 - 6x - 12$ by $x - 5$

17. (28) Identify any excluded values for $\frac{2x - 10}{10 - 2x}$. Then simplify the expression.

18. (62) **Analyze** Classify the number $a + bi$ when both a and b are equal to 0. Explain.

19. (67) **Write** Describe how to use the sine function to define the inverse sine function.

20. (70) **Physical Science** The number of miles m a person can see on a clear day is approximated by $m = 1.2116\sqrt{h}$, where h is the person's height above sea level. Suppose a person standing on a pier can see for 6 miles. To the nearest tenth of a foot, what is their height above sea level?

21. (69) **Predict** If every complex number with an absolute value of $\sqrt{58}$ is graphed, what geometric figure is formed and why?

22. (68) **Geometry** The angle measures of five different triangles are given below.

30°, 60°, 90° 45°, 45°, 90° 60°, 60°, 60° 20°, 20°, 140° 50°, 60°, 70°

a. If three triangles are chosen at random, without replacement, what is the probability that the first two triangles are right triangles and the last triangle is obtuse?

b. If three triangles are chosen at random, without replacement, what is the probability that all three are right triangles?

c. If three triangles are chosen at random, without replacement, what is the probability that all three are isosceles triangles?

23. (44) **Error Analysis** Alex and Robert each tried to simplify the expression $\frac{4+\sqrt{3}}{5-\sqrt{3}}$. Who is correct? Explain your reasoning.

Robert

$$\frac{4+\sqrt{3}}{5-\sqrt{3}} = \frac{4+\sqrt{3}}{5-\sqrt{3}} \cdot \frac{4-\sqrt{3}}{4-\sqrt{3}}$$
$$= \frac{4^2-\left(\sqrt{3}\right)^2}{20-5\sqrt{3}-4\sqrt{3}+3}$$
$$= \frac{13}{23-9\sqrt{3}}$$

Alex

$$\frac{4+\sqrt{3}}{5-\sqrt{3}} = \frac{4+\sqrt{3}}{5-\sqrt{3}} \cdot \frac{5+\sqrt{3}}{5+\sqrt{3}}$$
$$= \frac{4(5)+4\sqrt{3}+5\sqrt{3}+\left(\sqrt{3}\right)\left(\sqrt{3}\right)}{5^2-\left(\sqrt{3}\right)^2}$$
$$= \frac{23+9\sqrt{3}}{22}$$

***24.** (78) **Gravity on the Moon** The height of a free-falling object on the moon (neglecting air resistance) is given by the function $h(t) = -0.8t^2 + v_0t + h_0$, where h_0 is the initial height in meters, v_0 is the initial velocity in meters per second, and $h(t)$ is the height in meters at time t seconds. How long does it take for an object to hit the surface of the moon after it is dropped from a height of 20 meters? (Hint: Multiply both sides of the quadratic equation by the number that makes the coefficient of the quadratic term a perfect square.)

***25.** (77) A triangle has side lengths of 12 feet, 16 feet, and 18 feet. Use Heron's formula to find the area of the triangle.

26. (54) Maximize $P = 2x + 5y$ for the constraints
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ y &\leq 1.5x + 1 \\ y &\leq -x + 6 \end{aligned}$$

***27.** (76) **Generalize** Suppose that polynomial functions $f(x)$ and $g(x)$ are each a difference of square terms. What can you say about the roots of $f(x) \cdot g(x)$?

***28.** (74) **Fencing** A homeowner has 90 feet of fencing material to enclose a rectangular dog pen. He wants the dog to have 600 square feet of space inside the pen. The equation $w(20 - w) = 600$ gives the width of a pen that meets these requirements. Use the discriminant to explain whether or not these requirements can be met.

Use the change of base formula to evaluate the logarithm.

29. (72) $\log_7 19$

30. (72) $\log_6 \frac{24}{5}$

LESSON
79

Understanding Piecewise Functions

Warm Up

1. (4) **Vocabulary** According to the _____, a function can have only one point that intersects any vertical line.

2. (34) What are the domain and range for $y = -3x + 2$?

3. (27) For what values of the domain is $f(x) > 0$?

$f(x) = -(x + 2)^2 + 1$

4. (26) Write the equation of the line with slope 3 that passes through the point (50, 200) in slope intercept form.

New Concepts

A **piecewise function** is a function comprised of at least two functions. Each function defines a different part of the function's domain.

Example 1 Graphing a Piecewise Function

The information in the table shows the admission prices to a local museum. Graph the piecewise function showing the museum admission cost for someone x years of age.

Museum Admission Prices	
Age	**Price**
under 12	$5
12–50	$12
over 50	$10

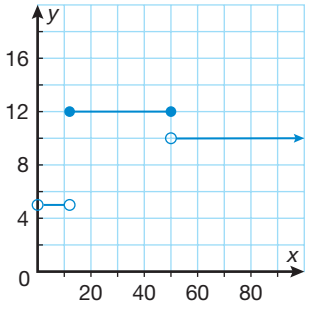
SOLUTION The domain of the function is divided into three different intervals:

Under age 12: $x < 12$

Ages 12 to 50: $12 \leq x \leq 50$

Over age 50: $x > 50$

For the interval $x < 12$, the range is 5. For the interval $12 \leq x \leq 50$, the range is 12. For the interval $x > 50$, the range is 10. The graph of the function is a step function.



A **step function** is a piecewise function that is constant for each interval in its domain. The graph of the museum admission prices is a step function. Notice that the step function passes the vertical line test. Piecewise functions have all the attributes of a function.

To evaluate a function for a given value of x, find the domain interval to which x belongs. Then find the corresponding range value.

Math Language

The **step function** gets its name from its graph which sometimes resembles steps on a staircase.

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Example 2 Evaluating a Piecewise Function

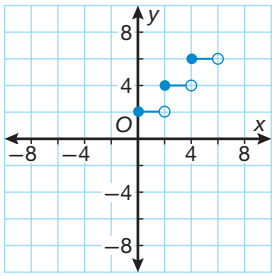
Evaluate the piecewise function for $x = 0$ and $x = 5$.

$$f(x) = \begin{cases} 12, & \text{if } x < 0 \\ -5, & \text{if } 0 \le x < 5 \\ 2, & \text{if } x \ge 5 \end{cases}$$

SOLUTION When $x = 0$, $f(x) = -5$, so $f(0) = -5$.

When $x = 5$, $f(x) = 2$, so $f(5) = 2$.

Example 3 Writing a Piecewise Function



Write a piecewise function rule for the graph.

SOLUTION The function is a step function comprised of three linear pieces that are represented by line segments on the graph. For $0 \le x < 2$, the graph is the line segment $y = 2$. For $2 \le x < 4$, the graph is the line segment $y = 4$. For $4 \le x < 6$, the graph is the line segment $y = 6$. The function rule is:

$$f(x) = \begin{cases} 2, & \text{if } 0 \le x < 2 \\ 4, & \text{if } 2 \le x < 4 \\ 6, & \text{if } 4 \le x < 6 \end{cases}$$

Reading Math

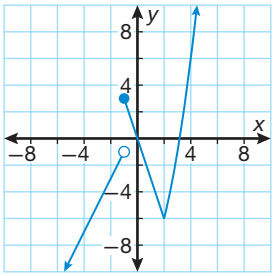
Open circles are used when the symbols < or > are in the domain. Closed circles are used for ≤ and ≥.

Example 4 Graphing a Piecewise Function

Graph the function $f(x) = \begin{cases} 2x + 1, & \text{if } x < -1 \\ -3x, & \text{if } -1 \le x < 2. \\ x^2 - 10, & \text{if } x \ge 2 \end{cases}$

SOLUTION The function is comprised of two linear pieces and a quadratic piece. Use a table of values to graph each piece of the function.

x	$f(x) = 2x + 1$	$f(x) = -3x$	$f(x) = x^2 - 10$
–2	–3		
–1	–1	3	
0		0	
2		–6	–6
3			–1
4			6
5			15



Caution

Do not choose x-values which are not included in the domain of the function.

Plot the points $(-2, -3)$ and $(-1, -1)$ and draw a ray whose endpoint is $(-1, -1)$. Plot the points $(-1, 3)$, $(0, 0)$, and $(2, -6)$ and draw a line segment with endpoints $(-1, 3)$ and $(2, -6)$. Plot the points $(2, -6)$, $(3, -1)$, $(4, 6)$, and $(5, 15)$ and draw a curve connecting the points. Draw an open circle at $(-1, -1)$ and a closed circle at $(-1, 3)$. No circle is needed at $(2, -6)$ because the function is continuous at that point.

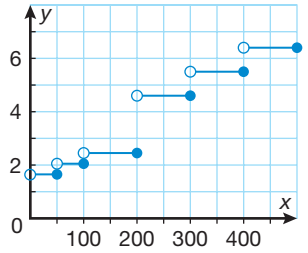
Example 5 Application: Postal Service Rates

The United States Postal Service insurance rates for up to $500 of coverage are shown in the table. Write and graph a piecewise function for the cost of x dollars of insurance coverage.

USPS Insurance Rates for Domestic Mail	
Fee	**Insurance Coverage**
$1.65	up to $50
$2.05	$50.01 to $100
$2.45	$100.01 to $200
$4.60	$200.01 to $300
$5.50	$300.01 to $400
$6.40	$400.01 to $500

SOLUTION The information in the table represents a step function. The function rule is:

$$f(x) = \begin{cases} 1.65, & \text{if } x \le 50 \\ 2.05, & \text{if } 50 < x \le 100 \\ 2.45, & \text{if } 100 < x \le 200 \\ 4.60, & \text{if } 200 < x \le 300 \\ 5.50, & \text{if } 300 < x \le 400 \\ 6.40, & \text{if } 400 < x \le 500 \end{cases}$$



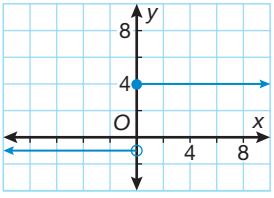
Lesson Practice

a. *(Ex 1)* The information in the table shows the admission prices to an amusement park. Graph the piecewise function showing the admission cost for someone x years of age.

Amusement Park Admission Prices	
Age	**Price**
6 and under	$10
7–18	$20
over 18	$25

b. *(Ex 2)* Evaluate $f(x) = \begin{cases} 5x - 2, & \text{if } x < -3 \\ 2x + 2, & \text{if } -3 \le x < 10 \\ x^2 + 4, & \text{if } x \ge 10 \end{cases}$ for $x = -3$ and $x = 10$.

c. *(Ex 3)* Write a piecewise function rule for the graph.



d. *(Ex 4)* Graph the piecewise function $f(x) = \begin{cases} x + 1, & \text{if } x \le 1 \\ 1, & \text{if } 1 < x < 2 \\ 4, & \text{if } x \ge 2 \end{cases}$

e. *(Ex 5)* A plumber charges $50 per hour for the first 3 hours and $25 per hour after that. Write and graph a piecewise function for the cost of x hours of service.

Practice Distributed and Integrated

1. (55) **Probability** A bag contains 3 white marbles and 7 purple marbles. What is the probability of picking a white marble, keeping it out of the bag, and then picking a purple marble?

Solve.

2. (70) $\sqrt[3]{x} - 10 = -3$

3. (70) $\sqrt[3]{x-3} + 2 = 4$

***4.** (77) **Generalize** What information must be known to be able to use the Law of Cosines to find a missing measure in a triangle?

Solve.

***5.** (78) $x^4 - 81 = 0$

***6.** (78) $12x = 9x^2 + 4$

***7.** (78) $8x^3 - 392x = 0$

8. (46) **Multi-Step** **a.** A box in the shape of a rectangular prism has a 4 ft × 4 ft square base, and a volume of $64\sqrt{6}$ cubic feet. Find the height of the box.

b. Find the length of the longest stick that can fit inside the box.

c. When this stick is placed inside the box, what is the measure of the angle it forms with the vertical face of the box?

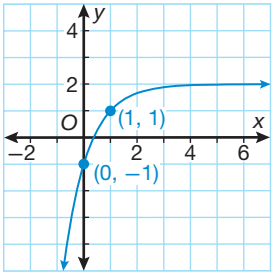
9. (Inv 5) **Verify** Suppose you are conducting an experiment consisting of rolling a number cube 40 times to find the probability of rolling a 2 seven times. Is this a binomial experiment? State why or why not.

10. (53) **Multi-Step** Let $f(x) = 2x$ and $g(x) = x^2$.

a. Find the composition functions $(f \circ g)(x)$ and $(g \circ f)(x)$.

b. Evaluate the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$ for $x = 2$.

11. (57) **Multiple Choice** Which of the following equations represents the graph shown?



A $y = -3^x + 2$

B $y = -3^{x+1} + 2$

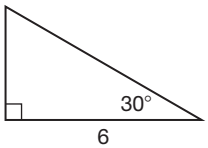
C $y = \left(\frac{1}{3}\right)^{x+1} - 2$

D $y = -\left(\frac{1}{3}\right)^{x-1} + 2$

12. (59) Simliify the expression $2^{\frac{1}{3}} \cdot 4^{\frac{1}{6}} \cdot 8^{\frac{1}{9}}$.

13. (60) **Soccer** Of the 65 students going on the soccer trip, 43 are players and 12 are left-handed. Only 5 of the left-handed players are soccer players. What is the probability that one of the students on the trip is a soccer player or is left-handed?

14. (52) Find the unknown lengths of the triangle shown.



15. (20) Given $f(x) = 3x - 5$ and $g(x) = x + 4$, find $(fg)(x)$.

***16.** (65) **Graphing Calculator** Solve $2x^2 + x - 4 = 0$ by using a graphing calculator.

17. (63) **Error Analysis** Describe and correct the student's error in finding the area of a sector with a radius of 6 centimeters and a central angle of 40°.

$$A = \frac{1}{2}(6)^2(40) = 720 \text{ cm}^2$$

18. (73) **Analyze** How is using a table of random digits to form a sample of 20 individuals different from using it to form a sample of 150 individuals?

19. (69) **Probability** For i^a, if a is randomly chosen from the numbers 20 through 30 inclusive, what is the probability that the value of the expression is 1 or –1?

***20.** (79) **Justify** Explain why $f(x) = \begin{cases} x + 1, & \text{if } x < -2 \\ 3, & \text{if } -2 \le x < 8 \\ 15, & \text{if } x \ge 8 \end{cases}$ is not a step function.

21. (66) **Marketing** The logo for a new math product includes the graph of the polynomial $3x^3 + 5x^2 - 6x - 8 = 0$. What coordinates will the project manager give the art production team to be sure the curve crosses the x-axis at the correct locations?

***22.** (79) **Delivery Charges** A garden center delivers up to 10 items for a $25 delivery charge. There is an additional charge of $5 per item for each additional item delivered. Write a piecewise function for the cost having x items delivered.

23. (67) **Analyze** Describe how the range of the inverse sine function and the range of the inverse tangent function are different. Explain why they are different.

Use the discriminant to find all values for which the equation has (a) two real solutions, (b) one real solution, and (c) two imaginary solutions.

***24.** (74) $x^2 - 4x - c = 0$

***25.** (74) $-4x^2 - 10x + c = 0$

26. (68) **Multiple Choice** A coin purse contains 6 dimes, 2 nickels, and 10 pennies. When selecting three coins, one at a time, without replacement, what is the approximate probability of randomly selecting 3 pennies?

A 12.3% **B** 14.7% **C** 20.4% **D** 29.4%

27. (71) **Formulate** Show how to derive the Law of Sines from the three expressions for the area of a triangle: $\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$.

28. (32) Find A^{-1} for $A = \begin{bmatrix} 1 & -1 \\ 4 & -6 \end{bmatrix}$.

29. (75) **Aviation** As an airplane is taking off from a runway, the force of lift on the plane can be calculated using the equation shown below. The plane begins to fly when the force of lift is equal to the weight of the plane. Find the radical function that can be used to find the minimum speed for flight.

$$\text{Lift} = kv^2, \text{ for constant } k \text{ and velocity } v$$

***30.** (79) Evaluate the piecewise function for $x = -3$ and $x = 4$.

$$f(x) = \begin{cases} 3x - 1 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \le x < 4 \\ 1 - x & \text{if } 4 \le x \end{cases}$$

LAB
12

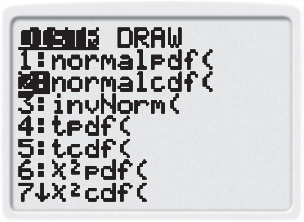
Calculating Normal Distribution Areas and Z-Scores

Graphing Calculator Lab (*Use with Lesson 80*)

A graphing calculator can be used to find the area under the normal curve given the bounds, mean, and standard deviation. Use a normal distribution with a mean of 26 and a standard deviation of 3 to find the probability of a value between 26 and 30.

Calculating the Area under the Normal Curve

1. Calculate an area under the normal curve with upper and lower bounds.

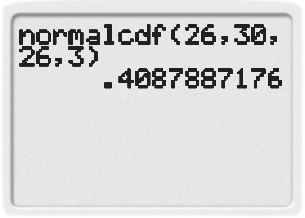


   a. Press 2nd VARS to open the **DISTR** menu.

   b. Press ▼ to select [**2: normalcdf (**] and press ENTER.

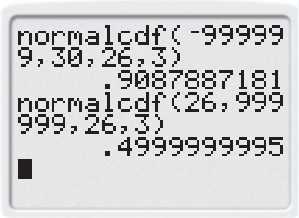
   c. To calculate the area, the data needs to be entered in this order: (lower, upper, mean, standard deviation). To do this press: 2 6 , 3 0 , 2 6 , 3).

   d. Now press ENTER to calculate the area.



2. To calculate an area under the curve to the left of an upper bound use [**2: normalcdf (**]. Enter −999999 as a lower bound, 30 as the upper bound, 26 as the mean, and 3 as the standard deviation. Refer to step **1c.**

3. To calculate an area under the curve to the right of an lower bound use [**2: normalcdf (**]. Enter 26 as a lower bound, 999999 as the upper bound, 26 as the mean, and 3 as the standard deviation. Refer to step **1c**.



4. Graphing the area under the normal curve

   a. Adjust the viewing window to Xmin=15, Xmax=40, Xscl=1, Ymin=−0.1, Ymax=0.2, Yscl=0.05, and Xres=1.

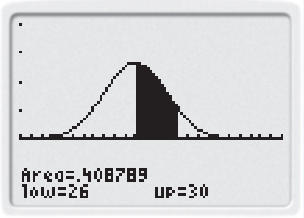
   b. From the home screen, press 2nd VARS to open the **DISTR** menu.

   c. Press ▶ to select the **DRAW** menu.

   d. Press ENTER to select [**1: ShadeNorm** ( ].

   e. In this order, enter (lower, upper, mean, standard deviation). Refer to step **1c**.

   f. Now press ENTER to generate the graph.



Graphing Calculator Tip

The area under the curve cannot be calculated directly, so use −999999 and 999999 as lower and upper bounds to obtain an approximate answer.

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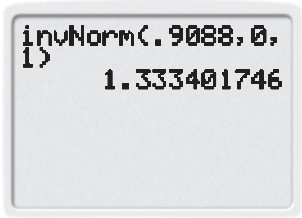
Calculating a Z-Score Given a Corresponding Area

5. Since the area entered into [**3: invNorm (**] is interpreted as area left of a given upper bound, use the area calculated in step **2.**

a. From the home screen, press 2nd VARS to open the **DISTR** menu.

b. Press ▼ to select [**3: invNorm (**] and press ENTER.



c. To calculate a z-score, the following data needs to be entered within the parenthesis: total area left of the given upper bound value, mean, and the standard deviation. To do this enter:

. 9 0 8 8 , 0 , 1).

d. Now press ENTER to calculate the z-score.

Graphing Calculator

When calculating the z-score using the **invNorm** function, the default value for mean is 0 and the default value for standard deviation is 1.

Lab Practice

a. Find the area under the normal curve for values between 115 and 130, given the mean is 130 and the standard deviation is 50.

b. Use the graphing calculator to graph area under the normal curve for values up to 180, given the mean is 130 and the standard deviation is 50. Use −9999 as the lower bound. Then find the corresponding z-value for this area.

LESSON
80

Finding the Normal Distribution

Warm Up

(25) **1.** **Vocabulary** For a set of data, the _____ [mean/median/mode] is the average of the data.

(55) **2.** True or False: The probability of an event must be a number between 0 and 1.

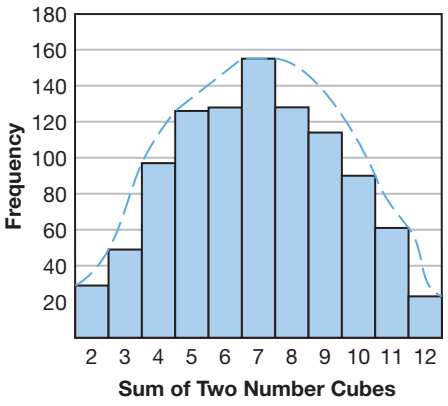
(25) **3.** True or False: The standard deviation of {1, 3, 5, 7, 10} is less than the standard deviation of {3, 4, 5, 6, 7}.

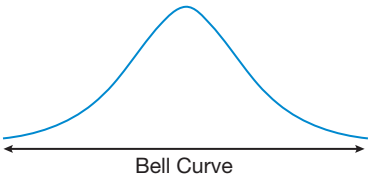
(25) **4.** Find the mean and median of 1, 2, 87.

New Concepts

The table and histogram below show the frequencies of the sum of two number cubes, each numbered 1–6, after 1000 rolls.

Sum	Frequency
2	29
3	49
4	97
5	126
6	128
7	155
8	128
9	114
10	90
11	61
12	23



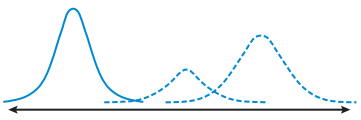


Math Language

A **bell curve** can be called a normal curve. It always has one peak. The width and height depend on the standard deviation of the data. The **standard deviation** σ is a measure of the spread of the data from the mean.

The curve in this histogram, which is formed by drawing a line through the tops of the histogram bars, is referred to as a **bell curve.** Data that is bell shaped has a **normal distribution,** which means that the data is symmetric about the mean. The curve is symmetric about the mean because the mean, median, and mode are equal. Data that is approximately normally distributed can be modeled by the bell curve.

The mean μ and standard deviation σ determine the location, spread, and the height of the bell curve. As the curve widens, more data values are farther from the mean than closer to the mean.



3 normal distributions, with different means and standard deviations

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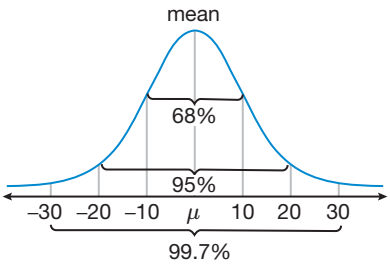
Since the area under the curve contains 100% of the data, the bell curve can be used to interpret values and draw conclusion. The 68-95-99.7 rule states that for any normal distribution the following properties apply:

- About 68% of the data lie within 1 standard deviation of the mean.
- About 95% of the data lie within 2 standard deviations of the mean.
- About 99.7% of the data lie with 3 standard deviations of the mean.

Math Language

The Greek letter μ is read "mu" and the Greek letter σ is read "sigma".



The sign of σ tells whether the number of standard deviations is above or below the mean.

Example 1 Using the Mean and Standard Deviation of a Normal Distribution

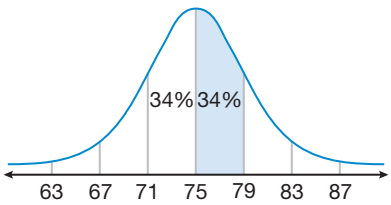
A set of test scores are normally distributed with a mean of 75 and a standard deviation of 4.

a. What percent of the scores are between 67 and 83?

SOLUTION Find how many deviations 67 and 83 are away from the mean. $75 - 67 = 8$, $83 - 75 = 8$, and $8 \div 4 = 2$. The scores are two standard deviations away from the mean, so about 95% of the scores are between 67 and 83.

b. What percent of the scores are between 75 and 79?

SOLUTION 68% of the scores are between 71 and 79. Because the data is distributed symmetrically about the mean, about half of 68%, or 34%, of the scores are between 75 and 79.

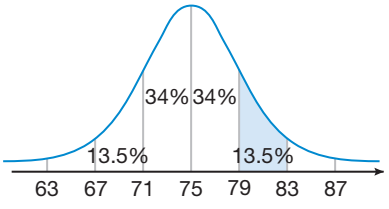


Math Reasoning

Analyze What percent of the scores are less than 83?

c. What percent of the scores are between 79 and 83?

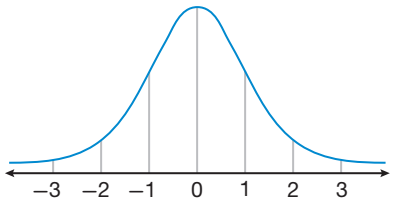
SOLUTION 95% of the scores are between 67 and 83. That means half, or 47.5%, is above the mean. Since 34% of the scores are between 75 and 79, then only 47.5% − 34%, or 13.5%, of the scores are between 79 and 83.



A normal distribution can be standardized, which is helpful in identifying how far a data value is from the mean.

Standard Normal Distribution

A standard normal distribution is a normal distribution that has a mean of 0 and a standard deviation of 1.



The value of a data point on the standard normal distribution is a ***z*-score.** It is a standardized value and its absolute value is equal to the number of standard deviations it is away from the mean. The sign tells if it is above or below the mean. A z-score of -2.5 is 2.5 standard deviations *below* the mean. A z-score of 1.8 is 1.8 standard deviations *above* the mean.

Standardizing different normal distributions onto a common scale allows for them to be compared using z-scores. To convert any given data value taken from a normal distribution to a z-score, subtract the mean from the data value and divide by the standard deviation.

***z*-score**

Given that x is a data value of a normally distributed set of data, the

$$z\text{-score} = \frac{x - \text{mean}}{\text{standard deviation}}.$$

Example 2 Finding *z*-Scores

The weights of cans produced in a factory have a mean of 6.5 ounces and a standard deviation 0.25 ounces.

a. A randomly selected can weighs 6.9 ounces. How many standard deviations is this above the mean?

SOLUTION Find the z-score, the standardized weight.

$$z\text{-score} = \frac{6.9 - 6.5}{0.25} = \frac{0.4}{0.25} = 1.6$$

The standardized weight is 1.6; the can is 1.6 standard deviations above the mean.

b. A randomly selected can weighs 6.4 ounces. How many standard deviations is this below the mean?

SOLUTION Find the z-score, the standardized weight.

$$z\text{-score} = \frac{6.4 - 6.5}{0.25} = \frac{-0.1}{0.25} = -0.4$$

The standardized weight is -0.4; the can is 0.4 standard deviations below the mean.

Math Reasoning

Estimate What is the actual weight of a can if its standardized weight is -2.82? Round the answer to the nearest tenth.

If the absolute value of a z-score for a data value is extremely large, then the data value may be an outlier.

Example 3 Application: Making Comparisons

Student A received a score of 63 on a test where the scores are normally distributed with a mean of 50 and a standard deviation of 8. Student B received a score of 32 on a test where the scores are normally distributed with a mean of 28 and a standard deviation of 1.8. Which student scored better compared to the rest of their class?

SOLUTION

1. **Understand** Two normal distributions are being compared.

2. **Plan** Standardize each score. The student with the greater standardized score did better.

3. **Solve**

Student A

$$z\text{-score} = \frac{63 - 50}{8} = \frac{13}{8} = 1.625$$

Student B

$$z\text{-score} = \frac{32 - 28}{1.8} = \frac{4}{1.8} \approx 2.22$$

Student B has the greater standardized score and did better compared to the rest of their class than did Student A.

4. **Check** On Student A's test, a score of 66 is two standard deviations above the mean and on Student B's test, 31.6 is two standard deviations above the mean. Student A's score of 63 is *within* two standard deviations, and Student B's score of 32 is *above* two standard deviations.

Hint

Student A's score was 1.625 standard deviations above the mean. Student B's score was 2.22 standard deviations above the mean.

Lesson Practice

A set of test scores are normally distributed with a mean of 300 and a standard deviation of 18.

(Ex 1) **a.** Between what two scores do 99.7% of the data fall?

(Ex 1) **b.** What percent of the scores are between 264 and 300?

(Ex 1) **c.** What percent of the scores are between 282 and 336?

(Ex 1) **d.** What percent of the scores are greater than 318?

The weights of hammers produced in a factory have a mean of 12.2 ounces and a standard deviation 0.08 ounces.

(Ex 2) **e.** A randomly selected hammer weighs 12 ounces. How many standard deviations is this below the mean?

(Ex 2) **f.** A randomly selected hammer weighs 12.3 ounces. How many standard deviations is this above the mean?

(Ex 3) **g.** Student A received a score of 445 on a test where the scores are normally distributed with a mean of 450 and a standard deviation of 12. Student B received a score of 170 on a test where the scores are normally distributed with a mean of 175 and a standard deviation of 15. Which student scored better compared to the rest of their class?

Practice Distributed and Integrated

1. (50) Find an equation for the inverse of $y = x^2 - 7$. Determine whether each relation is a function.

***2.** (80) A set of results are normally distributed with a mean of 88 and a standard deviation of 4. Find the percent of results between 80 and 96.

3. (60) Given a deck of cards, what is the probability of randomly drawing two ace cards without replacing the first card?

4. (57) The number of lemmings grew exponentially. At first there were 400,000. Three weeks later there were 1,200,000. How many would there be after 8 weeks?

***5.** (80) A set of scores have a mean of 43 and a standard deviation of 2. Find the z-score for an outcome of 37.

6. (52) **Astronomy** The binary star system 12 Bootis consists of two stars with nearly equal mass, and therefore a nearly equal gravitational pull. The distance between the two stars changes slightly over time, but is on average about 18.7 million kilometers. Suppose a spaceship crew wants to position their spaceship at a point equidistant from both stars, and so that the gravitational pulls were perpendicular to each other. How far from each star would the spaceship have to be?

7. (56) P is a point on the terminal side of θ in standard position. Find the exact value of the six trigonometric functions for θ where $P(2, 9)$.

8. (62) **Analyze** What are the values of a and b in the imaginary unit i?

9. (59) Simplify $\dfrac{13^{\frac{1}{3}} \cdot 13^{\frac{1}{4}}}{13^{\frac{1}{12}} \cdot 13^{\frac{1}{2}}}$.

10. (63) **Multiple Choice** What is the radian measure of a 90° angle?

A $-\frac{\pi}{2}$ radian **B** -1 radian **C** $\frac{\pi}{2}$ radian **D** 1 radian

11. (70) **Biology** The formula $SA = \sqrt{\frac{mk}{36}}$ approximates the surface area SA in square meters of a human being with a height of m meters and a mass of k kilograms. Approximate the mass of a person who has a surface area of 1.64 square meters and a height of 1.7 meters.

12. (64) **Multi-Step** **a.** Write the equation $y = \log_3 \frac{1}{9}$ in exponential form. What value of y makes both equations true?

b. Complete the table of values for the function $y = \log_3 x$.

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$y = \log_3 x$					

c. Plot the points and sketch the graph of the function.

13. *(65)* **Geometry** The length of the hypotenuse of a right triangle is 75 inches. Find the lengths of the legs given that the perimeter is 168 inches.

***14.** *(78)* **Write** Explain why the equation $16x^4 - 16 = 0$ can be solved by factoring the difference of two squares.

15. *(66)* **Error Analysis** Find and correct the error a student made in determining the possible rational roots of $2x^3 + 7x^2 - 5x - 4 = 0$.

$$\frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 4} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2$$

16. *(67)* **Navigation** "Lines" of longitude are actually circles that get closer together as you travel north or south from the equator. The formula for the approximate length l of a 1° arc of longitude in miles is $l \approx 69.033 \cos \theta$, where θ is the latitude in degrees at some point on the 1° arc of longitude.

a. At what latitude, to the nearest degree, is the length of a 1° arc of longitude approximately 53.6 miles?

b. The latitude at the equator is 0° and the latitude at the Arctic Circle is about 66°N. To the nearest mile, how much longer is the length of a 1° arc of longitude at the equator than at the Arctic Circle?

17. *(73)* **Error Analysis** A student said that a cluster sample is a type of simple random sample because every cluster has an equal chance of being chosen. Explain the error in the student's thinking.

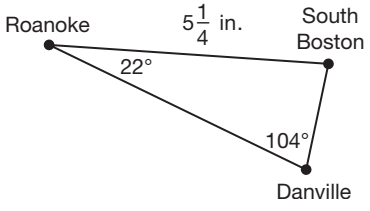
18. *(68)* A box of colored pencils only holds primary colors. It has 5 red, 8 yellow, and 3 blue. Find the probability that a student will randomly grab 2 yellow pencils without replacement.

19. *(74)* **Probability** Given that the discriminant is nonnegative and chosen randomly from the following list, find the probability that the equation it was derived from has exactly one real solution.

$$7, 0, -1, 4, -3, 5$$

20. *(69)* **Electrical Engineering** The flow of a current, I, the resistance to the flow, Z, and the voltage, E, are related by $E = I(Z)$. Find the voltage of a circuit if the current is $40 - 36i$ amps and the resistance is $8 + 3i$ ohms.

21. *(71)* **Geography** On a map, line segments connecting three southwest Virginia cities form a triangle as shown below. To the nearest tenth of a mile, what is the distance from Danville to South Boston if 1 inch = 13.5 miles?



22. (75) **Write** What happens to the graph of $y = \sqrt{x^2 + 200}$ for very large values of x? What can you conclude about functions of the form $y = \sqrt[n]{x^n + c}$, for constants n and c?

***23.** (76) **Graphing Calculator** Use a graphing calculator to test this hypothesis: If a cubic function has two roots, one of the roots must have a multiplicity of 2.

***24.** (80) **Multiple Choice** Which z-score is closest to the mean of a standardized normal distribution?

A -2.36 **B** -0.45 **C** 0.78 **D** 2.9

25. (72) **Analyze** What pair of inverse functions contain the ordered pairs $(1.5, 3^{1.5})$ and $(3^{1.5}, 1.5)$?

***26.** (80) **Generalize** Write $<$, $=$, or $>$.

When standardizing a data value selected from a normally distributed set of data,

$z \bigcirc 0$ when the mean is greater than the data value.

$z \bigcirc 0$ when the mean is less than the data value.

***27.** (77) **Estimate** In $\triangle PQR$, $p = 7.8935$, $q = 3.456$, and $r = \sqrt{37}$. Estimate the area of the triangle.

***28.** (79) **Comparison Shopping** At a party shop, balloons cost \$2 each if you buy fewer than 12, \$1.75 each if you buy between 12 and 24, inclusive, and \$1.50 each if you buy more than 24. Write a piecewise function for the cost of buying x balloons.

Evaluate each piecewise function for $x = \frac{1}{2}$ and $x = -10$.

***29.** (79) $f(x) = \begin{cases} 7, & \text{if } x < 0 \\ x^2, & \text{if } 0 \le x < 10 \\ -x^2, & \text{if } x \ge 10 \end{cases}$

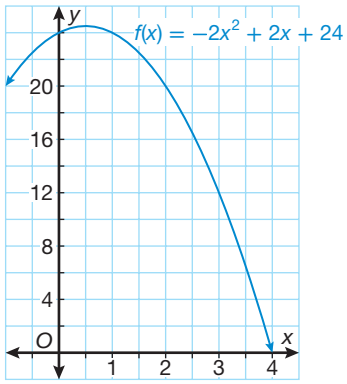
***30.** (79) $f(x) = \begin{cases} \frac{1}{2}, & \text{if } x < 1 \\ x - 4, & \text{if } 1 \le x < 10 \\ -x - 9, & \text{if } x \ge 10 \end{cases}$

INVESTIGATION

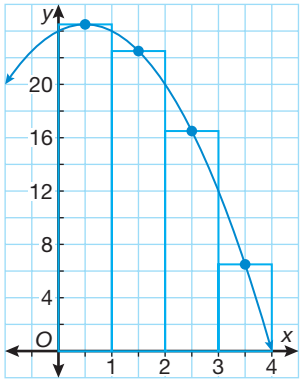
8

Finding the Area Under a Curve

Finding the area under a curve requires using calculus. But you can estimate the area under a curve with a good deal of accuracy without using calculus. Suppose you want to find the area under the curve shown below from $0 \le x \le 4$.



1. To begin the process of estimating the area, we will partition the region from $0 \le x \le 4$ into four rectangular regions, as shown. What is the width of each rectangular region?

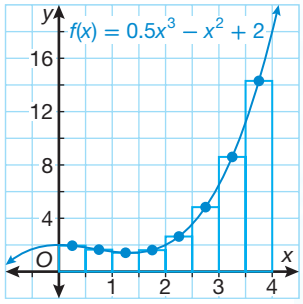


2. The height of each rectangle is the value of $f(x)$ at the points indicated on the graph. Use a calculator to find the height of each rectangle for the points shown. (Height is calculated at the value of x, the midpoint of a given interval. For example, if the first interval is from 0 to 1, then x starts at 0.5 continuing with increments of 1. Likewise, if the first interval is from 0 to 0.5, then x starts at 0.25 and continues with increments of 0.5.)

x	$f(x) = -2x^2 + 2x + 24$
0.5	
1.5	
2.5	
3.5	

3. Find and add the areas of the rectangular regions to estimate the total area under the curve from $0 \le x \le 4$.
4. Now calculate the table with x values at increments of 0.5 with a total of 8 rectangular regions. Find and add the areas of the rectangular regions to estimate the total area under the curve from $0 \le x \le 4$.
5. The exact area of the curve from $0 \le x \le 4$ is $69\frac{1}{3}$. Compare your estimate to the actual value.
6. Is there a way of improving the accuracy of the estimated area using the rectangle technique? Explain.

You can use a summation formula to find the estimated area under a curve. The graph of the function shown is partitioned into eight rectangles.



The summation formula for finding the estimated area is

$$\text{Total Area} = 0.5\left(\sum_{i=1}^{8} f(x_i)\right)$$

7. Each term in the series is multiplied by 0.5. What is the significance of this term?
8. Extend and complete the table to $i = 8$ to find each term in the series.

i	x_i	$f(x_i) = 0.5x^3 - x^2 + 2$
1	0.25	≈ 1.95
2	0.75	
3		
4		

9. Calculate the estimated area.

Investigation Practice

a. Estimate the area under the curve $y = -x^2 + 5x + 7$ from $0 \le x \le 5$. Use five partitions.

Estimate the area under the curve $y = -x^3 + 11x^2 + 7x + 1$ from $0 \le x \le 10$.

b. Write the summation formula to find the estimated area with ten partitions.

c. Use a table to calculate each term in the series to find the estimated area.

LESSON 81

Using Natural Logarithms

Warm Up

1. (64) **Vocabulary** In the expression $\log 20$, the _____ is 10.
2. (72) Write $\log xy$ as a sum or difference of terms.
3. (72) Write $\log \frac{x}{y}$ as a sum or difference of terms.
4. (64) **Multiple Choice** Which is $9^2 = 81$ in logarithmic form?

A $\log_2 9 = 81$ **B** $\log_2 81 = 9$

C $\log_9 81 = 2$ **D** $\log_{81} 9 = 2$

New Concepts

Since a logarithm is the exponent applied to a base to obtain a given value, $b^x = a$ is equivalent to $\log_b a = x$.

When the base is e, the logarithm is called the **natural logarithm,** denoted ln, and the equations above are written as follows.

$$e^x = a \Leftrightarrow \ln a = x$$

The Inverse Properties of Logarithms was previously used to simplify logarithmic expressions. In this lesson the Inverse Property of Logarithms will be used to simplify logarithmic expressions where the base is the constant e.

Reading Math

$\log_{10} a$ is denoted $\log a$. $\log_e a$ is denoted $\ln a$, and is read "natural log of a."

Inverse Properties of Logarithms
$e^{\ln x} = x$ and $\ln e^x = x$, where $x > 0$

Example 1 Simplifying Exponential and Logarithmic Expressions

a. Simplify $e^{\ln p}$.

SOLUTION

$e^{\ln p} = p$ Inverse Properties of Logarithms

b. Simplify $\ln e^{3c^2+1}$.

SOLUTION

$\ln e^{3c^2+1} = 3c^2 + 1$ Inverse Properties of Logarithms

Hint

The Inverse Properties of Exponents and Logarithms involving base e are special cases.

Properties of logarithms are the same for base e.

Properties of Natural Logarithms		
Product Property	$\ln ab = \ln a + \ln b$	for any positive numbers a and b.
Quotient Property	$\ln \frac{a}{b} = \ln a - \ln b$	for any positive numbers a and b.
Power Property	$\ln a^p = p \ln a$	for any positive number a and real number p.

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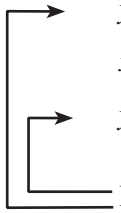
The exponential function with base e and the natural logarithmic function are inverse functions, as shown below.



$y = e^x$ Exponential function with base e

$x = e^y$ Interchange the variables.

$y = \ln x$ Solve for y using the definition of logarithm.

Inverse functions

Example 2 Applying Properties of Natural Logarithms

Rewrite each expression as a sum or difference of terms if possible. Then simplify.

a. $\ln 6e$

SOLUTION

$\ln 6e = \ln 6 + \ln e$ Product Property

$= \ln 6 + 1$ $\ln e = 1$ because $e^1 = e$.

b. $\ln \frac{3e}{x}$

SOLUTION

$\ln \frac{3e}{x} = \ln 3e - \ln x$ Quotient Property

$= \ln 3 + \ln e - \ln x$ Product Property

$= \ln 3 + 1 - \ln x$ $\ln e = 1$ because $e^1 = e$.

$= \ln 3 - \ln x + 1$

c. $\ln e^{5x^2}$

SOLUTION

$\ln e^{5x^2} = 5x^2 \ln e$ Power Property

$= 5x^2 \cdot 1$ $\ln e = 1$ because $e^1 = e$.

$= 5x^2$

d. $\ln\left(\frac{2x^3}{e^4}\right)^2$

SOLUTION

$\ln\left(\frac{2x^3}{e^4}\right)^2 = 2\ln\left(\frac{2x^3}{e^4}\right)$ Power Property

$= 2(\ln 2x^3 - \ln e^4)$ Quotient Property

$= 2(\ln 2 + \ln x^3 - \ln e^4)$ Product Property

$= 2(\ln 2 + 3\ln x - 4)$ Power Property, $\ln e^a = a$

$= 2\ln 2 + 6\ln x - 8$ Distribute the 2.

Recall the exponential growth function $A = Pe^{rt}$ that is used to model the amount in an account where interest is compounded continuously. This function can be written in a general way to represent other growth and decay situations.

Natural Base Functions
$N(t) = N_0e^{kt}$ is a natural base exponential growth function and $N(t) = N_0e^{-kt}$ is a natural base exponential decay function, where $N_0 = N(0)$ is the original amount ($t = 0$) and $k > 0$ is a constant.

In order to solve a related exponential equation, where there is a variable in the exponent, the logarithm of both sides of the equation must be taken.

Example 3 Application: Medicine

Thallium is a radioactive substance that can be injected into the blood as part of a health test for the heart. It is then tracked through the bloodstream by a special camera. Thallium has a half-life of 73.1 hours. How long will it take for 1 gram of thallium to decay to 0.1 gram?

SOLUTION

Step 1: Find the constant k for thallium.

$N(t) = N_0e^{-kt}$ — Write the natural decay function.

$0.5 = 1e^{-k(73.1)}$ — Substitute 0.5 for $N(t)$, 1 for N_0, and 73.1 for t.

$\ln 0.5 = \ln e^{-k(73.1)}$ — Take the natural logarithm of both sides.

$\ln 0.5 = -73.1k$ — Inverse Properties of Logarithms.

$$k = \frac{\ln 0.5}{-73.1} \approx 0.00948$$

Hint

Substitute 0.5 for $N(t)$ and 73.1 for t because 50% of the initial amount remains after 73.1 hours. Substitute 1 for N_0 because 100% is the initial amount.

Step 2: Write the decay function and solve for t.

$N(t) = N_0e^{-0.00948t}$ — Substitute 0.00948 for k.

$0.1 = 1e^{-0.00948t}$ — Substitute 1 gram for the initial amount and 0.1 gram for the amount at time t.

$\ln 0.1 = \ln e^{-0.00948t}$ — Take the natural logarithm of both sides.

$\ln 0.1 = -0.00948t$ — Inverse Properties of Logarithms

$$t = \frac{\ln 0.1}{-0.00948} \approx 242.9$$

Hint

By taking the natural logarithm of both sides of the equation, the inverse property of exponents and logarithms can be used to eliminate e.

It will take about 242.9 hours for 1 gram of thallium to decay to 0.1 gram.

Lesson Practice

a. *(Ex 1)* Simplify $e^{\ln(3d-1)}$.

b. *(Ex 1)* Simplify $\ln e^{u^3+u}$.

c. *(Ex 2)* Rewrite $\ln 4e$ as a sum or difference of terms if possible. Then simplify.

d. *(Ex 2)* Rewrite $\ln \frac{2x}{e}$ as a sum or difference of terms if possible. Then simplify.

e. (Ex 2) Rewrite $\ln 8e^{-x^3}$ as a sum or difference of terms if possible. Then simplify.

f. (Ex 2) Rewrite $\ln\left(\frac{5e^2}{y^3}\right)^3$ as a sum or difference of terms if possible. Then simplify.

g. (Ex 3) Determine how long it will take for 200 mg of chromium-49, which has a half-life of about 42.3 minutes, to decay to 5 mg.

Practice Distributed and Integrated

1. (50) Find an equation for the inverse of $y = x^2 + 3$. Determine whether each relation is a function.

***2.** (80) Weights have a mean of 35 lbs and a standard deviation of 0.5 lbs. Find the z-score for a weight of 36.8 lbs.

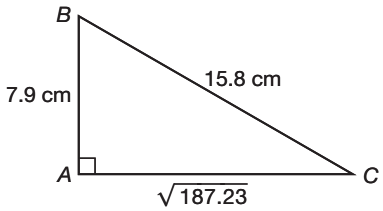
Determine the domain and range.

3. (75) $y = -4\sqrt{x+2}$

4. (75) $y = \frac{1}{2}\sqrt{x}$

***5.** (81) Rewrite $\ln 14e$ as a sum or difference of terms if possible. Then simplify.

6. (52) **Geometry** The lengths of the three sides of a right triangle ABC are 7.9 cm, 15.8 cm and $\sqrt{187.23}$ cm respectively. Find the measure of the angle C.



7. (72) **Write** Explain why $e^{\ln x} = x$.

8. (53) Let $p(u) = 2u + 5$ with domain $\{1, 3, 5, 7\}$ and $r(u) = u - 3$ with domain $\{6, 8\}$. What are the domain and range of $p(r(u))$?

9. (61) **Verify** Show by synthetic division that $x + 9$ is a factor of $6x^3 + 51x^2 - 27x$.

10. (69) **Fractals** Some fractals are produced using the formula $Z_{n+1} = (Z_n)^2 + c$. Find Z_2 if $Z_1 = 0.3 + 0.8i$ and $Z_2 = (Z_1)^2 + 0.5$.

11. (60) **Error Analysis** A box contains 16 red balls numbered 1–16, 10 green balls numbered 27–36, and 14 yellow balls numbered 47–60. A ball is randomly selected from the box. The work below is used to find the probability that the ball selected is yellow or a one-digit number. Explain why the work is incorrect.

Y = event of yellow ball selected D = event of one-digit number selected

$P(Y) = \frac{14}{40} = \frac{7}{20}$ $P(D) = \frac{9}{60} = \frac{3}{20}$

$P(Y \text{ or } D) = P(Y) \cdot P(D) = \frac{7}{20} \cdot \frac{3}{20} = 0.0525$

12. (65) **Multi-Step** A 5 inch by 7 inch photo is framed and the total area of the photo and the frame is 50 square inches.

a. Write a quadratic expression in standard form to represent the situation.

b. Find the width of the frame to the nearest hundredth of an inch.

Solve by completing the square.

13. (78) $2d^2 = 8 + 10d$

14. (78) $-3x^2 + 18x = -30$

15. (68) **Housing** The table shows estimated housing occupancy for three states for the year 2006. Find the probability that a randomly selected house from one of these states is vacant given that the house is in Idaho. Write the probability as a percent rounded to the nearest hundredth.

	Occupied Houses	Vacant Houses
Idaho	548,555	67,148
Florida	7,106,042	1,425,818
Illinois	4,724,252	475,491

16. (23) **Running** The expression representing a runner's speed is $120x^2 - 60x - 240$ feet per minute. Find the expression for the runner's speed in feet per second.

***17.** (78) **Multiple Choice** What are the roots of the equation $2x^2 = 50$?

A 0 and 50 **B** 0 and 25 **C** 5 and -5 **D** 25 and -25

18. (73) **Multi-Step** Explain why the question below leads the reader to vote *no*. Then rewrite the question in a way that leads the reader to vote *yes*.

"A new shopping mall will bring more traffic to our already congested streets, increasing the noise and pollution levels and the time it takes to drive anywhere. Do you want a new shopping mall in our area?"

19. (70) **Multiple Choice** Solve $\sqrt{3x} + 7 = 9$.

A $\frac{3}{4}$ **B** $\frac{4}{3}$ **C** $\frac{16}{3}$ **D** $\frac{256}{3}$

***20.** (81) **Analyze** What pair of inverse functions contain the ordered pairs $\left(e^{\frac{1}{2}}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, e^{\frac{1}{2}}\right)$?

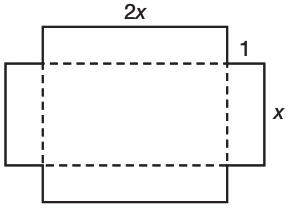
21. (74) **Ranching** A rancher has 380 feet of fencing material to enclose a rectangular area for her llamas. She wants the fenced-in area to be 9025 square feet. The equation $w(190 - w) = 9025$ gives the width of a rectangle that meets these requirements. Use the discriminant to explain if the rancher has enough material to build a fence that meets these requirements.

***22.** (81) **Paleontology** Carbon 14 is an isotope that is used to determine ages of fossils. It has a half-life of about 5730 years, which means that 50% of an initial amount will decay in 5730 years. To the nearest year, how long will it take for 51% of an initial amount to decay? (Use the natural decay function $N(t) = N_0e^{-kt}$.)

23. (67) **Write** Explain why the inverses of the trigonometric functions are not functions.

***24.** (81) Given $N(t) = 0.3$, $N_0 = 1$, $k = 0.0974$, write the decay function and solve for t.

25. (76) **Construction** Concrete is going to be poured into a container in the shape of rectangular prism. The container is constructed from the form shown. If the volume of concrete is 300, what are the actual dimensions of the container to the nearest hundredth?



***26.** (81) **Probability** An experiment consists of choosing a random integer from the integers 1 through 10 inclusive. Five trials are conducted. What is the probability that an integer greater than 2 is chosen on every trial?

*27. (77) **Graphing Calculator** Find c in $\triangle BCD$ given $b = 34.729$, $d = 21.65$, and $m\angle C = 125.37°$. Round to the nearest thousandth.

*28. (80) **Model** The heights of a group of children are normally distributed with a mean of 42 inches and a standard deviation of 2.2 inches. Draw a bell curve and label 3 standard deviations above and below the mean.

29. (67) Evaluate the inverse trigonometric function $\text{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)$. Give you answer in both radians and degrees.

*30. (80) A set of results are normally distributed with a mean of 26 and a standard deviation of 2. Find the percent of results > 22.

LESSON
82

Graphing the Sine and Cosine Functions

Warm Up

1. (4) **Vocabulary** For a given function $f(x)$, the function $f(x) + 1$ is a _____ of the original function.

2. (4) True or False. For a given function $f(x)$, the function $f(x - 1)$ is a vertical shift of the original function.

3. (4) True or False. For a given function $f(x)$, the function $f(x) + 1$ is a vertical shift of the original function.

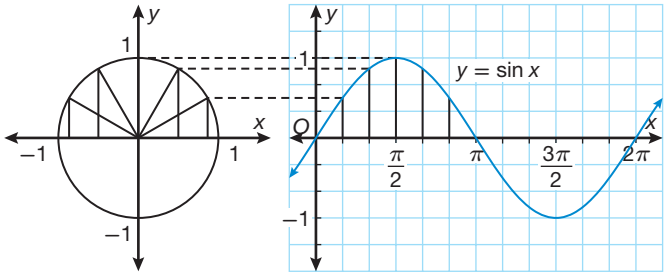
4. (46) **Multiple Choice** Which is the same as $\sin \frac{\pi}{6}$?

A 0 **B** 0.5 **C** 1 **D** $\frac{\sqrt{2}}{2}$

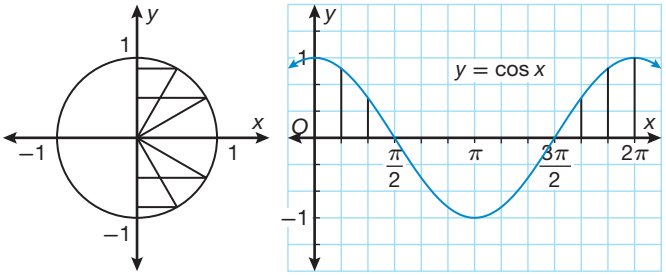
New Concepts

A **periodic function** is a function that repeats exactly in regular intervals, called periods. A **cycle of a periodic function** is the shortest repeating part of a periodic graph or function. The **period of a periodic function** is the length of a cycle measured in units of the independent variable.

A sine function is an example of a periodic function. The sine function $y = \sin x$ can be graphed by matching the measure x of an angle in standard position with the y-coordinate of the intersection point of the terminal side of the angle and the unit circle.



The cosine function is graphed in a similar manner with the measure x of an angle in standard position matching with the x-coordinate of the intersection point of the terminal side of the angle and the unit circle.



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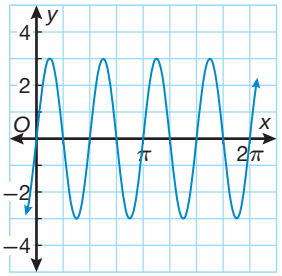
Looking at the graphs you can see that one cycle of each function is completed in 2π radians; so the period of the sine and cosine function is 2π.

The **amplitude** of a period function is half the distance between the maximum and minimum values of the range. The amplitude is always positive. For the sine and cosine functions, the maximum value of the range is 1 and the minimum value of the range is -1, with a distance of 2 between. So, the amplitude is $\frac{2}{2} = 1$.

Example 1 Finding the Amplitude and Period Graphically

a. Use the graph of $y = 3 \sin 4x$ to find the amplitude and period of the function.

SOLUTION



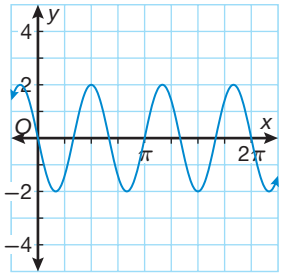
The range has a maximum value of 3 and a minimum value of 3 with a distance of 6 between them.

The amplitude is $\frac{6}{2} = \mathbf{3}$.

There are 4 cycles graphed in 2π. The period is the length of 1 cycle.

The period is $\frac{2\pi}{4} = \mathbf{\frac{\pi}{2}}$.

b. Use the graph of $y = -2 \sin 3x$ to find the amplitude and period of the function.



SOLUTION

The range has a maximum value of 2 and a minimum value of 2 with a distance of 4 between them.

The amplitude is $\frac{4}{2} = \mathbf{2}$.

There are 3 cycles graphed in 2π. The period is the length of 1 cycle.

The period is $\mathbf{\frac{2\pi}{3}}$.

Math Reasoning

Generalize How is the number of cycles related to the coefficient of x in the equation?

Math Reasoning

Generalize How is the amplitude related to the coefficient of the trig function?

How does a negative value of the coefficient of the trig function affect the graph?

Properties of Sine and Cosine Graphs

For the graphs of $y = a \sin bx$ or $y = a \cos bx$ where x is in radians and $a \neq 0$

- the amplitude is $|a|$.
- the number of cycles in the interval from 0 to 2π is b.
- the period is $\frac{2\pi}{b}$.

To graph a sine or cosine function, locate five points equally spaced through one cycle. These points will be the zeros, the maximums, and the minimums. For the sine function if $a > 0$, the pattern of these points is zero-max-zero-min-zero and if $a < 0$, the pattern of these points is zero-min-zero-max-zero. For the cosine function if $a > 0$, the pattern of these points is max-zero-min-zero-max and if $a < 0$, the pattern of these points is min-zero-max-zero-min.

Math Reasoning

Formulate What is the period for $y = \sin(ax)$?

Example 2 Graphing $y = a\sin(bx)$

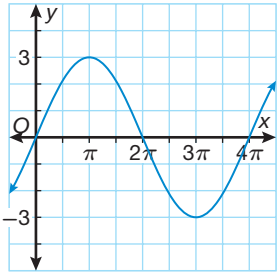
Graph $y = 3\sin\frac{1}{2}x$.

SOLUTION

Step 1: Find the amplitude $a = 3$. $|a| = 3$, so the amplitude is 3.

Step 2: Find the period of the function.

$\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$, so the period is 4π.



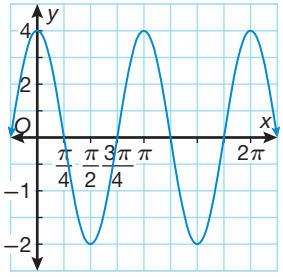
Step 3: Divide the period into fourths.

$\frac{1}{4}(4\pi) = \pi, \frac{2}{4}(4\pi) = 2\pi, \frac{3}{4}(4\pi) = 3\pi$

Step 4: Since a > 0 , plot the pattern zero-max-zero-min-zero using the values of the amplitude and the period.

Example 3 Graphing $y = a\cos(bx)$

Graph $y = 4\cos 2x$.



SOLUTION

Step 1: Find the amplitude $a = 4$. $|a| = 4$, so the amplitude is 4.

Step 2: Find the period of the function.

$b = 2$, so there are 2 cycles in 2π. $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$, so the period is π.

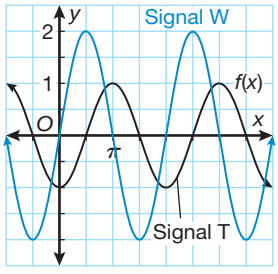
Step 3: Divide the period into fourths.

$\frac{1}{4}(\pi) = \frac{\pi}{4}, \frac{2}{4}(\pi) = \frac{\pi}{2}, \frac{3}{4}(\pi) = \frac{3\pi}{4}$.

Step 4: Since $a > 0$, plot the pattern max-zero-min-zero-max using the values of the amplitude and the period.

Example 4 Application: Radio Signals

Two radio signals of the same period are shown on the graph. Write an equation for Signal W in terms of the sine function.



SOLUTION

To write an equation, first find a and b. Find the amplitude, a, by dividing the distance between the maximum range value and the minimum range value by 2.

$|a| = \frac{4}{2} = 2$ Since the sine pattern is zero-max-zero-min-zero, the value of a is positive.

Find b by determining the length of the period from the graph and using the relationship between the period and b.

$$\text{period} = \frac{2\pi}{b}$$

$$2\pi = \frac{2\pi}{b}$$

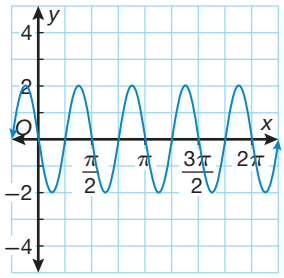
$$b = 1$$

The equation is $y = 2 \sin 1x$ or $y = 2 \sin x$.

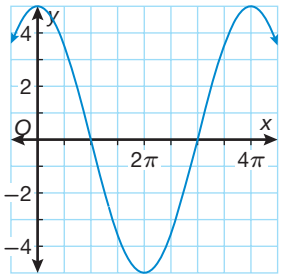
Lesson Practice

Use the graph of each function to determine the amplitude and the period.

a. (Ex 1) $y = -2 \sin 4x$



b. (Ex 1) $y = 5 \cos \frac{1}{2}x$



Graph each function.

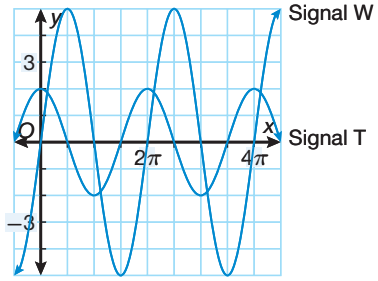
c. (Ex 2) $y = 4 \sin 2x$

d. (Ex 2) $y = -3 \sin \frac{1}{4}x$

e. (Ex 3) $y = 3 \cos 2x$

f. (Ex 3) $y = -3 \cos \frac{1}{4}x$

g. (Ex 4) Write an equation for Signal T in terms of the cosine function.



Practice Distributed and Integrated

1. (76) Which of the following is not a factor of $x^3 - 15x^2 + 74x - 120$?

A $(x - 6)$ **B** $(x - 5)$ **C** $(x - 4)$ **D** $(x - 3)$

2. (57) **Bacteria** A colony of bacteria begins with a population of 28, and after 45 minutes the population is 572. How long will it take the colony to grow to 1,200 bacteria?

***3.** (82) Given $h(x) = -2\sin(x)$, identify the amplitude and period. Graph the function.

*4. (65) **Graphing Calculator** Solve $x^2 + 2x + 1 = 0$ by using a graphing calculator.

5. (76) Find the roots and give the multiplicity of each root: $x^3 + 2x^2 - 3x = 0$.

6. (52) Find the exact values of x and y in an isosceles right triangle with a hypotenuse 21 units long and legs x and y.

*7. (81) Write $\ln\left(\frac{4x^2}{e^3}\right)^3$ as a sum or difference of terms if possible. Then simplify.

Find the area of $\triangle ABC$ with the given side lengths and included angle.

8. (71) $B = 124°$, $a = 9$, $c = 11$

9. (71) $A = 68°$ and $b = 13$, $c = 7$

10. (66) **Geometry** The volume of a pyramid is given by the formula $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height of the pyramid. A pyramid with a square base has a volume of 108 cubic units.

a. Write a polynomial equation in standard form that can be used to find the length of one side of the base if the height is 3 units longer than the base.

b. Solve the polynomial equation to find the length of one side. (Hint: The length is less than 10 units.)

*11. (80) In a study of 748 subjects, the mean height was 168.8 centimeters and the standard deviation was 8.9 centimeters. Assuming the distribution is normal, what is the range of heights that the middle 68% of the subjects fell into?

12. (72) **Aquarium** The recommended range for the pH of aquarium water for most fish is from 6 to 8. Use the formula $\text{pH} = -\log[\text{H}^+]$, and write the equivalent range for $[\text{H}^+]$, the concentration of hydrogen ions in moles per liter. Write your answer without exponents.

13. (67) **Error Analysis** A student incorrectly solved the equation $\cos\theta = 0.8$, for $0 \le \theta \le \pi$, as shown. $\cos\theta = 0.8 \rightarrow \theta = \text{Cos}^{-1}(0.8) \rightarrow \theta = \frac{1}{\cos(0.8)} \rightarrow \theta \approx 1.44$. What is the error? Find the correct solution.

Find the roots of the polynomial function.

14. (76) $f(x) = 12x^3 + 4x^2 - 2x$

15. (76) $f(x) = (x - 2)(x^2 + 6) - (x - 2)(7x + 4)$

16. (68) **Probability** A used car lot contains 16 red cars, 14 gold cars, 9 black cars, and 6 blue cars. A customer needs 5 cars for their new company. Find the probability that they will randomly choose 5 gold cars.

*17. (82) Graph $y = 2\cos 3x$.

18. (60) **Multiple Choice** Choose the answer that best represents the probability of flipping two coins five times and always showing tails.

A 0.5 **B** 0.0005 **C** 0.0001 **D** 0.01

19. (69) **Verify** Show that the product of two imaginary numbers (bi and ci) can be either positive or negative.

20. (73) **Generalize** Suppose a researcher captures and tags n animals, goes back, captures $n - 1$ animals, and finds that all of the $n - 1$ animals are tagged. What is the estimated animal population?

Multiply. Write the result in the form $a + bi$.

21. (69) $2(3i)$

22. (69) $-4(5i)$

***23.** (81) **Write** Explain why $e^{\ln(x-3)} = x - 3$.

24. (71) **Surveying** A surveyor stands on the south end of a roof and finds that the angle formed by the roof and the line of sight with a flagpole is 33°. The surveyor then walks to the north end of the roof and finds the angle formed by the roof and the flagpole to be 61°. Estimate the distance from the roof to the flagpole, given that the roof has a length of 50 feet.

25. (78) **Vertical Motion** The height of a free-falling object (neglecting air resistance) is given by the function $h(t) = -16t^2 + v_0t + h_0$, where h_0 is the initial height in feet, v_0 is the initial velocity in feet per second, and $h(t)$ is the height in feet at time t seconds. How long does it take for an object to hit the ground after it is dropped from a height of 49 feet?

26. (74) **Multi-Step** **a.** Write $5x - 3x^2 = 10$ in standard form.

b. Find the discriminant of the equation in part a.

c. Use the discriminant to describe the solutions of the equation in part a.

27. (70) **Pendulum** The period of a pendulum, T, is the time it takes for the pendulum to make one back-and-forth swing, and is given by $T = 2\pi\sqrt{\frac{L}{9.8}}$, where L is the length of the pendulum in meters. Find the length of a pendulum whose period is 4 seconds.

***28.** (82) How many cycles of the function $y = 3\cos(2x)$ are in the interval 0 to 2π.

29. (77) **Write** Without writing a formula, explain how to find the area of a triangle given its three side lengths.

***30.** (81) **Radioisotopes** Thallium, a radioisotope used to track blood flow, has a half-life of about 73.1 hours. How long will it take for 1 mg of thallium to decay to 0.7 mg? Use the natural decay function $N(t) = N_0e^{-kt}$.

LESSON
83

Writing Quadratic Equations from Roots

Warm Up

1. (19) **Vocabulary** To multiply two binomials use the _____ method.
2. (23) Factor the quadratic expression $x^2 - 7x + 12$.
3. (19) Multiply the binomials $(2x + 4)(x - 5)$.

New Concepts

Recall that the Zero Product Property states that if $ab = 0$, then $a = 0$ or $b = 0$. The converse of the Zero Product Property is also true.

Math Language

The **converse** of an if-then statement exchanges the words that follow *if* with the words that follow *then*.

Converse of the Zero Product Property
If $a = 0$ or $b = 0$, then $ab = 0$.

Use the converse to write a quadratic function when given the roots of the function.

Example 1 Deriving Quadratic Equations with Two Roots

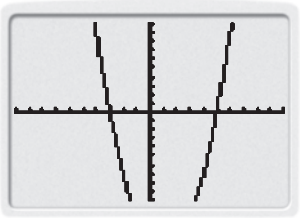
a. Write a quadratic equation whose roots are -3 and 5.

SOLUTION Work backwards from solving a quadratic equation.

$x = -3$ or $x = 5$	Write the roots as solutions of two equations.
$x + 3 = 0$ or $x - 5 = 0$	Make the right side of each equation 0.
$(x + 3)(x - 5) = 0$	Converse of the Zero Product Property
$x^2 - 2x - 15 = 0$	Multiply the binomials.

The equation $x^2 - 2x - 15 = 0$ has roots of -3 and 5.

Check: The zeros of $y = x^2 - 2x - 15$ occur at -3 and 5.

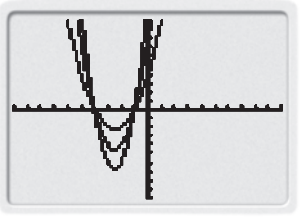


b. Write three quadratic equations whose roots are -4 and -1.

SOLUTION Write the roots as solutions of two equations: $x = -4$ or $x = -1$, and make one side equal to 0: $x + 4 = 0$ or $x + 1 = 0$. Use the Converse of the Zero Product Property to find the basic equation. Add a constant factor to find other quadratic equations with the same roots.

$(x + 4)(x + 1) = 0$	$2(x + 4)(x + 1) = 0$	$3(x + 4)(x + 1) = 0$
$x^2 + 5x + 4 = 0$	$2x^2 + 10x + 8 = 0$	$3x^2 + 15x + 12 = 0$

Graph $y = x^2 + 5x + 4$, $y = 2x^2 + 10x + 8$, and $y = 3x^2 + 15x + 12$. They have the same zeros.



Online Connection
www.SaxonMathResources.com

Every quadratic equation has two roots. However, when the roots are identical, it is more common to say that the equation has one root.

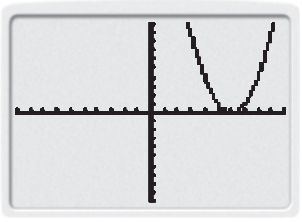
Example 2 Deriving Quadratic Equations with One Root

a. Write a quadratic equation whose root is 6.

SOLUTION Treat the single root of 6 as two identical roots of 6.

$x = 6$ or $x = 6$	Write the roots as solutions of two equations.
$x - 6 = 0$ or $x - 6 = 0$	Make the right side of each equation 0.
$(x - 6)(x - 6) = 0$	Converse of the Zero Product Property
$x^2 - 12x + 36 = 0$	Multiply the binomials.

The equation $x^2 - 12x + 36 = 0$ has one real root of 6.



Check The zero of $y = x^2 - 12x + 36$ occurs at 6.

b. Write three quadratic equations so that the root of each is -2.

SOLUTION Treat the single root as two roots and write them as solutions of two equations: $x = -2$ or $x = -2$. Make one side equal to 0: $x + 2 = 0$ or $x + 2 = 0$. Use the Converse of the Zero Product Property to find the basic equation. Add a constant factor to find other quadratic equations with the same roots.

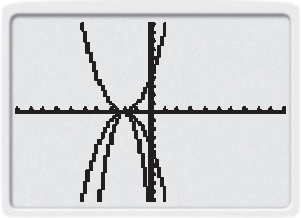
Hint

The constant factor can be positive or negative. It can be a whole number or a fraction.

$(x + 2)(x + 2) = 0$ $\quad -(x + 2)(x + 2) = -1(0)$ $\quad -3(x + 2)(x + 2) = -3(0)$

$x^2 + 4x + 4 = 0$ $\quad -x^2 - 4x - 4 = 0$ $\quad -3x^2 - 12x - 12 = 0$

Graph $y = x^2 + 4x + 4$, $y = -x^2 - 4x - 4$, and $y = -3x^2 - 12x - 12$. They have the same zero.



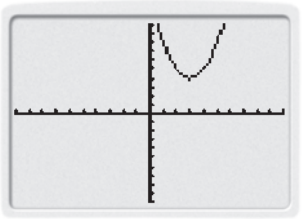
Example 3 Deriving Quadratic Equations with Complex Roots

a. Write a quadratic equation whose roots are $3 + 2i$ and $3 - 2i$.

SOLUTION Write the roots as a solution to one equation.

$x = 3 \pm 2i$	Use $\pm$ symbol.
$x - 3 = \pm 2i$	Isolate the imaginary term.
$x - 3 = \pm\sqrt{-4}$	Write the imaginary term as a square root.
$(x - 3)^2 = -4$	Square both sides.
$x^2 - 6x + 9 = -4$	Expand the binomial.
$x^2 - 6x + 13 = 0$	Add 4 to each side.

The function $y = x^2 - 6x + 13$ has no real zeros.



b. Write three quadratic equations whose roots are $3i$ and $-3i$.

SOLUTION Write the roots as a solution to one equation.

$x = \pm 3i$ Use $\pm$ symbol.

$x = \pm\sqrt{-9}$ Write the imaginary term as a square root.

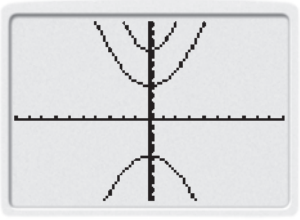
$x^2 = -9$ Square both sides.

$x^2 + 9 = 0$ Add 9 to each side.

Multiply each side by a non-zero constant.

$$x^2 + 9 = 0$$
$$2(x^2 + 9) = 2(0) \rightarrow 2x^2 + 18 = 0$$
$$-1(x^2 + 9) = -1(0) \rightarrow -x^2 - 9 = 0$$

The graphs of $y = x^2 + 9$, $y = 2x^2 + 18$, and $y = -x^2 - 9$ are shown. None cross the x-axis.



Math Reasoning

Analyze What characteristic is common to the graphs of the equations whose roots are $3i$ and $-3i$?

Example 4 Application: Arches

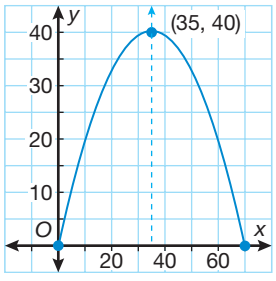
The bases of an arch are 70 feet apart. The arch has a height of 40 feet. Write a quadratic function to approximate the arch.

SOLUTION

1. **Understand** An arch is parabolic. A parabola's function can be determined when given the vertex and the coordinates of another point.

2. **Plan** Set the x-intercepts at (0, 0) and (70, 0) to represent the bases. The x-coordinate of the vertex is halfway between them at 35. The y-coordinate is the height 40. The vertex is located at (35, 40).

3. **Solve** Use the vertex form of a quadratic function: $y = a(x - h)^2 + k$. Substitute 35 for h and 40 for k. Substitute the x- and y-coordinates of either x-intercept for x and y. Then solve for a.



$$0 = a(0 - 35)^2 + 40$$
$$-40 = a(1225)$$
$$\frac{-40}{1225} = a$$
$$\frac{-8}{245} = a$$

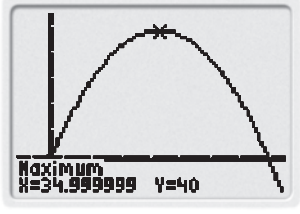
Write the function:

$$y = \frac{-8}{245}(x - 35)^2 + 40$$

Expand the binomial and distribute to write the quadratic in standard form.

$$y = \frac{-8}{245}x^2 + \frac{16}{7}x$$

4. **Check** Graph the function on a graphing calculator.



Caution

The y-intercept, -0.425, is not 0 because of rounding.

Lesson Practice

(Ex 1) **a.** Write a quadratic equation whose roots are -7 and 3.

(Ex 1) **b.** Write three quadratic equations whose roots are 2 and 3.

(Ex 2) **c.** Write a quadratic equation whose root is -5.

(Ex 2) **d.** Write three quadratic equations so that the root of each is 4.

(Ex 3) **e.** Write a quadratic equation whose roots are $1 + 4i$ and $1 - 4i$.

(Ex 3) **f.** Write three quadratic equations whose roots are $7i$ and $-7i$.

(Ex 4) **g.** The bases of an arch are 50 feet apart. The arch has a height of 30 feet. Write a quadratic function to approximate the arch.

Practice Distributed and Integrated

Simplify each expression.

(59) **1.** $(8 \cdot 216)^{\frac{1}{3}}$

(59) **2.** $\dfrac{6^{\frac{14}{9}}}{6^{\frac{6}{9}}}$

(68) **3.** A box of crayons has 3 purples, 5 reds, 8 greens, and 2 oranges. Find the probability that a student will randomly choose 1 purple, 2 red, and 1 green crayon in that order without replacement.

Factor each expression completely.

(61) **4.** $32z^5 - 2z$

(61) **5.** $a^5b^2 - a^2b^4 + 2a^4b - 2ab^3 + a^3 - b^2$

(60) ***6.** **Multi-Step** A doctor organized data about her patients' ages and genders in the chart below.

Gender of Client

Age	Male	Female
18–35	39	41
36–55	45	52
56–75	54	40
76+	47	56

a. Are the events male client under 36 or male client over 75 mutually exclusive or inclusive? Explain why.

b. Find the probability that a patient is male under 36 or male over 75. Round to the hundredths place.

c. Use the probability found in part b to find the probability that three clients, selected randomly with replacement, are male under 36 or male over 75. Round to the hundredths place.

(53) **7.** Let $f(x) = 2x + 5$ and $g(x) = 3x$. Evaluate the composition $(f \circ g)(4)$ using two different methods.

(65) **8.** **Generalize** The Quadratic Formula can be written as $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$. How does this form show the symmetry of a parabola?

(72) **9.** **Multiple Choice** Which is equal to $\log_{20} 20^{a-b}$?

A $a - b$ **B** $20(a - b)$ **C** $\dfrac{a}{b}$ **D** $\dfrac{20^a}{20^b}$

(66) ***10.** **Graphing Calculator** Solve $2x^3 + 4x^2 - 39x - 45 = 0$ by using a graphing calculator.

11. (75) **Analyze** Suppose that $f(x) = \sqrt[m]{g(x)} + \sqrt[n]{h(x)}$. If mn is an odd number, what can you conclude about the domain and range of $f(x)$? Why?

***12.** (83) **National Parks** The Landscape Arch, in Arches National Park, is the longest natural stone arch in the world. Its length is about 290 feet and its height is about 78 feet. Write a quadratic function to approximate the arch.

13. (67) **Estimate** Use what you know about the unit circle and special angles to estimate $\text{Sin}^{-1}(0.02)$ in degrees. Do not use a calculator. Explain your method.

14. (69) **Electrical Engineering** The flow of a current, I, the resistance to the flow, Z, and the voltage, E, are related by $E = I(Z)$. Find the voltage if the current is $24 - 28i$ amps and the resistance is $5 + i$ ohms.

15. (78) **Write** Explain why the equation $x^4 - 625 = 0$ can be solved by factoring the difference of two squares.

16. (70) **Geometry** The formula $d = \sqrt{l^2 + w^2 + h^2}$ gives the length of a diagonal d of a $l \times w \times h$ rectangular prism. Prism A has a width of 3 inches, a length of 7 inches, and a diagonal whose length is about 9.1 inches. Prism B has a width of 2 inches, a length of 8 inches, and a diagonal whose length is about 10.2 inches. Which prism has the greater height and by how much?

17. (71) **Analyze** When can you find the area of a triangle without knowing the height of the triangle?

18. (73) **Government** Suppose a politician must visit 4 states on the East Coast. His human resource personnel form a simple random sample by listing each state on the East Coast and numbering it as shown below. If they use the line of random digits shown below, which 4 states will the politician visit?

10: ME	11: NH	12: MA	13: RI	14: CT	15: NY	16: NJ
17: DE	18: MD	19: VA	20: NC	21: SC	22: GA	23: FL

63157	48257	29223	66713	89022	43069	91349	33111	63682	48864	53097

19. (74) **Multi-Step** **a.** Write $-18x^2 - 84x = 98$ in standard form.

b. Factor the left side of the equation in part a.

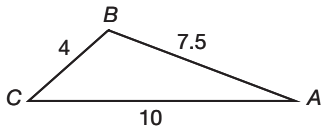
c. Find and use the discriminant to describe the solutions of the equation in part A.

20. (76) **Physics** A satellite in orbit accelerates according to the equation below, for orbital speed v and distance from the Earth's center r. Two satellites are in orbit, one at a speed of v and the other at a speed of $5v$. How far from Earth's center is the faster satellite compared to the slower one if they both have the same acceleration?

$$\text{acceleration} = \frac{v^2}{r}$$

21. (77) **Error Analysis** Explain and correct the error a student made in finding $m\angle C$.

$$7.5^2 = 4^2 + 10^2 - 2(4)(10)\cos C$$
$$56.25 = 116 - 80\cos C$$
$$56.25 = 36\cos C$$
$$1.5625 = \cos C$$
Error



***22.** (83) **Multiple Choice** Which equation is a quadratic equation with a single root of 7?

A $x(x+7)=0$ **B** $x^2-7=0$ **C** $(x-7)(x-7)=0$ **D** $(x+7)(x+7)=0$

***23.** (82) **Error Analysis** Two students are finding the period of $y = 4\sin(\frac{1}{\pi}x)$ but get different results. Which student made the mistake?

Student A	Student B
The period is where $y = 0$	The period is where $y = 0$
$\frac{1}{\pi}x = 2\pi$	$\frac{1}{\pi}x = \pi$
$x = 2\pi^2 \leftarrow$ Period	$x = \pi^2 \leftarrow$ Period

24. (71) Find the area of $\triangle ABC$ with the given side lengths and included angle. $a = 19$, $c = 8$, $B = 75°$

***25.** (81) **Economics** The formula for continuously compounded interest is $A = Pe^{rt}$, where A is the total amount, P is the principal, r is the annual interest rate, and t is the time in years. To the nearest hundredth of a percent, what annual interest rate would be necessary for the value of an investment to double in 10 years if the interest is compounded continuously?

Evaluate each piecewise function for $x = 0$ and $x = 2$.

26. (79) $f(x) = \begin{cases} 5, \text{ if } x < 0 \\ x + 2, \text{ if } 0 \le x \le 8 \\ 2x^2 + 1, \text{ if } x > 8 \end{cases}$

27. (79) $f(x) = \begin{cases} 9, \text{ if } x < -1 \\ 12, \text{ if } -1 \le x \le 0 \\ 20, \text{ if } x > 0 \end{cases}$

***28.** (82) **Multiple Choice** Which of the following has a period of π?

A $y = \sin(2\pi x)$ **B** $y = \sin(\pi x)$ **C** $y = \sin(x)$ **D** $y = \sin(2x)$

***29.** (83) **Probability** The roots of a quadratic equation are $4 \pm 5i$.

a. What is the probability that the related function crosses the x-axis?

b. What is the probability that the related function has a minimum or maximum at $x = 4$?

***30.** (83) Write a quadratic equation whose roots are 6 and -2.

LESSON
84

Solving Rational Equations

Warm Up

1. (28) **Vocabulary** The quotient of two polynomials is a _____ (***rational*/*radical***) expression.

2. (37) Find the LCM of $4x^2$ and $6x^3y$.

3. (37) Add $\frac{5}{3x+6} + \frac{x}{x+2}$.

4. (37) **Multiple Choice** Which is the LCD of $\frac{3x^2}{x^2-8x+16}$ and $\frac{2}{5x^2-15x-20}$?

A $6x^2$ **B** $(x-4)(x+1)$

C $5(x-4)(x+1)$ **D** $5(x-4)^2(x+1)$

5. (23) Simplify $\frac{x^2+4x-12}{6x-12}$.

New Concepts

A **rational function** is a function whose rule is a rational expression.

An equation that contains at least one rational expression is a **rational equation.** It is important to notice that rational equations sometimes have extraneous solutions. Since dividing by 0 is undefined in mathematics, any solution that yields a fraction with a denominator of 0 when it is substituted into the original equation is extraneous.

There are several methods that can be used to solve rational equations. Cross-multiplying can be used when the rational equation is expressed as a proportion.

Example 1 Solving Rational Equations with Cross Multiplication

Solve each equation.

a. $\frac{30}{x^2-25} = \frac{3}{x-5}$

SOLUTION Cross multiply to solve for x.

$$\frac{30}{x^2-25} = \frac{3}{(x-5)}$$

$30(x-5) = 3(x^2-25)$	Cross multiply.
$30x - 150 = 3x^2 - 75$	Distribute.
$0 = 3x^2 - 30x + 75$	Addition Property of Equality.
$0 = 3(x^2 - 10x + 25)$	Factor.
$0 = 3(x-5)(x-5)$	Factor.
$x = 5$	Solve for x.

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Now substitute $x = 5$ back into the equation to check for an extraneous solution.

$$\frac{30}{x^2 - 25} = \frac{3}{x - 5}$$
$$\frac{30}{5^2 - 25} = \frac{3}{5 - 5}$$
$$\frac{30}{0} = \frac{3}{0}$$

Each side of this equation is undefined, so the solution $x = 5$ is extraneous and the equation has no solution.

b. $\frac{9}{5 - x} = \frac{x}{x - 5}$

SOLUTION

$$\frac{9}{5 - x} = \frac{x}{x - 5}$$

$9x - 45 = 5x - x^2$	Cross multiply, and distribute.
$0 = -x^2 - 4x + 45$	Addition Property of Equality
$0 = (-x + 5)(x + 9)$	Factor.
$x = 5$ or -9	Solve.

Check each solution.

$$\frac{9}{(5 - 5)} = \frac{5}{(5 - 5)} \qquad \frac{9}{5 - (-9)} = \frac{-9}{(-9) - 5}$$
$$\frac{9}{5} = \frac{5}{0} \qquad \frac{9}{14} = \frac{9}{14}$$

The solution $x = 5$ is extraneous because it makes the denominators of the original equation equal to zero. Since -9 makes the equation true, the only solution to this rational equation is $x = -9$.

When a rational equation contains one fraction, the equation can first be simplified by multiplying both sides of the equation by the fraction's denominator.

Example 2 Solving Rational Equations by Multiplication

Solve each equation.

a. $\frac{4x}{x - 9} = 15$

SOLUTION Multiply both sides of the equation by the fraction's denominator.

$$\frac{10x}{x - 9} = 15$$

$10x = 15x - 135$	Multiply both sides by $(x - 9)$.
$-5x = -135$	Addition Property of Equality
$x = 27$	Solve.

Substitute the solution into the original equation to check for an extraneous answer. Since 27 makes the original equation true, the solution is $x = 27$.

b. $\frac{8}{x} + 2 = 2$

SOLUTION

Multiply both sides of the equation by the fraction's denominator.

$$\frac{8}{x} + 2 = x$$

$$x\left(\frac{8}{x} + 2\right) = x(x)$$

$$8 + 2x = x^2$$

$$0 = x^2 - 2x - 8$$

$$0 = (x - 4)(x + 2)$$

$$x = 4 \text{ or } x = -2$$

Substitute both solutions into the original equation to check for extraneous answers. Since 4 and –2 both make the original equation true the solutions are $x = 4$ and $x = -2$.

Hint

Instead of finding a least common denominator, you can also multiply by each factor of each denominator in the problem one-by-one.

When the rational equation has the sum or difference of two rational expressions, the equation can first be simplified by multiplying both sides of the equation by the least common denominator.

Example 3 Solving Rational Equations with a Common Denominator

Solve for x.

$$\frac{x + 5}{x + 3} + \frac{28}{x^2 + 8x + 15} = 4$$

SOLUTION

Factor the polynomial and find a common denominator. In this problem, the common denominator is $(x + 3)(x + 5)$.

$$\frac{x + 5}{x + 3} + \frac{28}{x^2 + 8x + 15} = 4$$

$$\frac{x + 5}{x + 3} + \frac{28}{(x + 3)(x + 5)} = 4$$

Now multiply both sides of the equation by this common denominator.

$$(x + 3)(x + 5)\left(\frac{x + 5}{x + 3} + \frac{28}{(x + 3)(x + 5)}\right) = 4(x + 3)(x + 5)$$

$$(x + 5)^2 + 28 = 4(x^2 + 8x + 15)$$

$$x^2 + 10x + 25 + 28 = 4x^2 + 32x + 60)$$

$$0 = 3x^2 + 22x + 7$$

Use the quadratic formula to solve this equation.

$$x = \frac{-22 \pm \sqrt{22^2 - 4(3)(7)}}{6} = -\frac{1}{3} \text{ and } -7.$$

Check for extraneous solutions. Since $-\frac{1}{3}$ and -7 both make the original equation true, the solution is $x = -\frac{1}{3}$ or $x = -7$.

Example 4 Application: Distance, Rate, and Time

An athlete runs 12 kilometers on Saturday and bikes 30 kilometers on Sunday. The total time spent working out was 4 hours. Find the time spent working out each day if the rate of travel on Sunday was twice what it was on Saturday.

SOLUTION Because $d = rt$, $r = \frac{d}{t}$. Write an expression for the rate of travel on each day. If t = time on Saturday, then $4 - t$ = time on Sunday.

The rate on Sunday was twice what it was on Saturday.

$$\frac{30}{4 - t} = 2\left(\frac{12}{t}\right)$$

$$t(4 - t)\left(\frac{30}{4 - t}\right) = t(4 - t)\left(\frac{24}{t}\right)$$

$$30t = 96 - 24t$$

$$54t = 96$$

$$t = 1.\overline{7}$$

The time was about 1.8 hours on Saturday and $4 - 1.8 = 2.2$ hours on Sunday.

Math Reasoning

Analyze Show the same answer is found when t is the time on Sunday and $4 - t$ is the time on Saturday.

Lesson Practice

Solve each equation.

a. *(Ex 1)* $\frac{2x - 2}{3x} = \frac{5}{2}$

b. *(Ex 1)* $\frac{30}{x} = \frac{x + 4}{2}$

c. *(Ex 2)* $\frac{3x}{x + 3} = -1$

d. *(Ex 2)* $5 - \frac{4}{x} = x$

e. *(Ex 3)* $\frac{8x}{x - 1} - \frac{x + 33}{x^2 + 2x - 3} = 9$

f. *(Ex 3)* $\frac{3x}{x^2 - 1} = \frac{4}{x}$

g. *(Ex 4)* A biker biked 18 miles on pavement and 6 miles on dirt. The total biking time was 5 hours. Find the time spent biking on each surface if the rate of biking on the dirt was three-fourths of what it was biking on the pavement.

Practice Distributed and Integrated

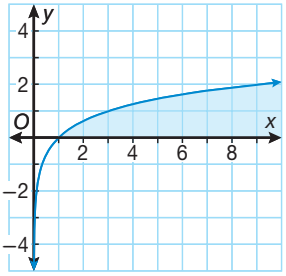
1. (56) Given $\theta = 504°$, find the measure of the reference angle.

Find all the possible values.

2. (67) $\cos^{-1}\left(\frac{1}{2}\right)$

3. (67) $\tan^{-1}(1)$

4. (64) **Coordinate Geometry** The graph of the logarithmic function $y = \log_3 x$ is shown. Estimate the shaded area.



5. (74) Use the discriminant to find all values of c for which $x^2 - x + c = 0$ has (a) two real solutions, (b) one real solution, and (c) two complex solutions.

6. (58) Complete the square and factor the resulting perfect square trinomial: $x^2 - 24x$.

7. (80) **Analyze** The z-score of an unknown data value is -1.4. Find the data value given that the mean of the data is 60 and the standard deviation is 12.

8. (77) **Geography** A map of southwest Illinois shows that Waterloo is 34 miles from Chester and 44 miles from Pinckneyville. Pinckneyville is 27 miles from Chester. Find the number of square miles enclosed in the triangle formed by the lines connecting these three cities. Round to the nearest square mile.

***9.** (65) **Graphing Calculator** Solve $2x^2 + 4x - 3 = 0$ by using a graphing calculator.

***10.** (81) **Multiple Choice** Which is equal to $e^{\ln 4x^2}$?

A $8x$ **B** $\ln 8x$ **C** $2 \ln 4x$ **D** $4x^2$

11. (68) **Housing** The table shows estimated housing occupancy for three states for the year 2006. Find the probability that a randomly selected house from one of these states is in Texas given that it is occupied. Write the probability as a percent rounded to the nearest hundredth.

	Occupied Houses	Vacant Houses
Virginia	2,905,071	325,750
Texas	8,109,388	1,115,532
Utah	814,028	87,294

12. (78) **Analyze** Describe how to solve the equation $\frac{1}{2}x^2 - 2 = 0$ by factoring the difference of two squares.

13. (69) **Error Analysis** Which student made an error in finding $|2 - i|$? Describe the error.

Student A: $\sqrt{2^2 + 0^2} \rightarrow \sqrt{4} \rightarrow 2$ Student B: $\sqrt{2^2 + (-1)^2} \rightarrow \sqrt{4 + 1} \rightarrow \sqrt{5}$

***14.** (83) Write a quadratic equation whose root is 7.

15. (70) **Diameter** The formula $d = \frac{\sqrt{15w}}{\pi}$ gives the diameter in inches of the rope needed to lift w tons. How much more weight can be lifted with a rope of diameter 3 inches than with a rope of diameter 2.5 inches?

16. (72) **Sound** Two measures of sound are loudness and intensity. They are related by the function $L(I) = 10\log \frac{I}{I_0}$, where $L(I)$ is loudness in decibels, I is intensity in watts per square meter, and $I_0 \approx 10^{-12}$ watt per square meter, the intensity of a barely audible sound. The sound intensity near a speaker is 0.01 watt per square meter. What is the loudness in decibels?

***17.** (83) **Multi-Step** Write a quadratic equation with roots of $5 + 3i$ and $5 - 3i$ so that the parabola of the related function opens up. Then write one so that the parabola opens down and is wider than the one that opens up.

18. (73) **Multiple Choice** Which type of sampling is *not* a type of probability sampling?

A simple random sampling **B** voluntary response sampling

C cluster sampling **D** systematic sampling

Solve.

***19.** (84) $2 + \frac{1}{x} = 4$

***20.** (84) $\frac{2}{x+2} + \frac{8}{x-2} = \frac{14}{x^2-4}$

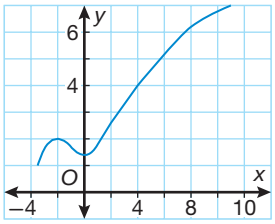
***21.** (84) $\frac{-6}{x-3} = 1$

22. (74) **Generalize** What will be the value of the discriminant of an equation if the vertex of the related function is located in Quadrant III and the parabola opens upward? Why?

***23.** (83) Write a quadratic equation whose roots are 6 and -1.

24. (60) **Students** Of the 70 students going on the soccer trip, 58 are players and 8 are left-handed. Six of the left-handed students are soccer players. What is the probability that one of the students on the trip is a soccer player or is left-handed?

25. (75) **Analyze** The graph of a radical function $f(x) = \sqrt[n]{g(x)}$ is shown. What can you conclude about n?



***26.** (82) **Formulate** Suppose that $f(x) = \sin(ax)$ and $g(x) = \sin(bx)$. If $f(x)$ completes four periods when $g(x)$ completes one, what is the relationship between a and b?

27. (76) **Geometry** The volume of a rectangular prism is $3x^3 + 5x^2 - 12x$. What are the dimensions of the rectangular prism?

28. (79) **Multi-Step** At a florist shop, roses cost \$3.50 each if you buy fewer than 12, \$3.00 each if you buy between 12 and 24, and \$2.50 each if you buy more than 24. Write a piecewise function for the cost per rose when buying x roses. Then, graph the function.

29. (77) In $\triangle RST$, $ST = 8.9$, $RT = 12.2$, and the $m\angle T = 78°$. Find $m\angle S$.

***30.** (84) **Error Analysis** A student said that the equation $\frac{3}{x^2+7x+10} + \frac{1}{x^2+10x+25} = 24$ can be solved by multiplying both sides of the equation by $(x + 2)(x + 5)$. Why is this not true?

LESSON
85

Finding Polynomial Roots II

Warm Up

1. (78) **Vocabulary** The expression $x^2 - 4$ is a _____ and has factors $(x + 2)$ and $(x - 2)$.

2. (78) Factor the expression $16x^4 - 81$.

3. (65) Solve using the quadratic formula. $3x^2 - 29x - 84 = 0$

New Concepts

Cubic polynomials that are either the sum or difference of cubes can be factored as follows,

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$
$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

Look for such patterns when you need to factor a cubic binomial.

Example 1 Using the Sums and Differences of Cubes

a. Find the real roots of the polynomial function.

$y = x^3 - 27$

SOLUTION

Since $27 = 3^3$, this is a difference of cubes.

$y = x^3 - 27 = (x - 3)(x^2 + 3x + 9)$

To see if the quadratic term has real roots, find the discriminant.

$3^2 - 4(1)(9) = -27 \rightarrow$ complex roots

The polynomial function $y = x^3 - 27$ has one real root: $x = 3$.

b. Find the real roots of the polynomial function.

$y = x^3 + 64$

SOLUTION

Since $64 = 4^3$, this is a sum of cubes.

$y = x^3 + 64 = (x + 4)(x^2 - 4x + 16)$

To see if the quadratic term has real roots, find the discriminant.

$(-4)^2 - 4(1)(16) = -48 \rightarrow$ complex roots

The polynomial function $y = x^3 + 64$ has one real root: $x = -4$.

Math Reasoning

Verify Show that if a polynomial is a sum or difference of cubes, there is only one real root.

Some non-cubic functions can still follow the patterns of the sums and differences of cubes. Look for polynomials $f(x)$ and $g(x)$ that are of the form,

$$f(x) = g^3(x) + a^3 = (g(x) + a)(g^2(x) - ag(x) + a^2)$$
$$f(x) = g^3(x) - a^3 = (g(x) - a)(g^2(x) + ag(x) + a^2)$$

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Example 2 Using Cubic Sums and Difference Patterns

a. Find the real roots of the polynomial equation.

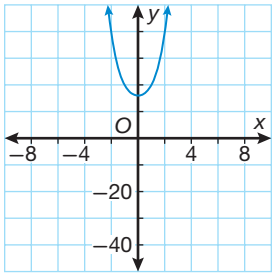
$$x^6 - 64 = 0$$

SOLUTION

Note that $x^6 = (x^2)^3$ and $64 = 4^3$. Therefore the polynomial can be rewritten as a difference of cubes.

$$f(x) = (x^2)^3 - 4^3 = (x^2 - 4)((x^2)^2 + 4(x^2) + 16)$$
$$= (x^2 - 4)(x^4 + 4x^2 + 16)$$

The factor $(x^2 - 4)$ is a difference of squares with real roots $x = \pm 2$. Graph the quartic factor to see if it intersects the x-axis.



The polynomial function $f(x) = x^6 - 64$ has two real roots: $x = 2$ and $x = -2$.

b. Find the real roots of the polynomial equation.

$$x^9 + 27 = 0$$

SOLUTION

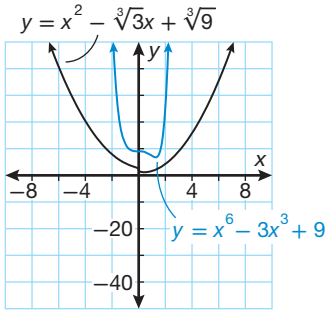
Note that $x^9 = (x^3)^3$ and $27 = 3^3$. Therefore the polynomial can be rewritten as a sum of cubes.

$$f(x) = (x^3)^9 + 3^3 = (x^3 + 3)((x^3)^2 - 3(x^3) + 9)$$
$$= (x^3 + 3)(x^6 - 3x^3 + 9)$$

The factor $(x^3 + 3)$ is itself a sum of cubes and can be further factored.

$$f(x) = (x + \sqrt[3]{3})(x^2 - \sqrt[3]{3}x + \sqrt[3]{9})(x^6 - 3x^3 + 9)$$

Graphs of the two polynomial factors show there are no other roots.



The real root is $x = -\sqrt[3]{3}$.

Math Reasoning

Verify Show that if $x^6 - 64$ is factored as a difference of squares, the results are the same as factoring it as a difference of cubes.

Graphing Calculator Tip

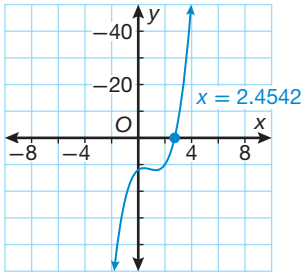
For help in finding the zeros of graphs, see Graphing Calculator Lab 3 on page 84.

Example 3 Solving Cubic Equations

Find the real roots of $3x^3 - 7x^2 + 4x - 12 = 0$.

SOLUTION To find the real roots of the equation, graph the polynomial function

$$y = 3x^3 - 7x^2 + 4x - 12$$



The real root is approximately 2.4542.

Rational Root Theorem

If a polynomial equation has rational roots, then use the **Rational Root Theorem** to find these roots. To do so, follow these steps.

1. Write the polynomial in this form, where the term with the highest power, $a_n x^n$, is the leading term and the constant term, a_0, is the last term.

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$$

2. Find all the factors of a_n and a_0.

3. Let q be one of the factors of a_n and p be one of the factors of a_0. List all possible rational numbers $\frac{p}{q}$.

4. Test if $\left(x - \frac{p}{q}\right)$ is a factor of the polynomial using synthetic division.

If a polynomial has a rational root, then it will be one of the $\frac{p}{q}$ terms.

Example 4 Using the Rational Root Theorem

a. Find the rational roots for the following polynomial equation.

$$x^3 + 2x^2 - x - 2 = 0$$

SOLUTION

Use the Rational Root Theorem to find any roots that are rational roots. The terms that need to be factored are 1 and −2. The possible values of p are $\pm 1, \pm 2$ and for q are ± 1. So, $\frac{p}{q} = \{2, -2, 1, -1\}$. Use synthetic division to test roots.

2	1	2	−1	−2
		2	8	14
	1	4	7	12

✗

−2	1	2	−1	−2
		−2	0	2
	1	0	−1	0

✓

$$x^3 + 2x^2 - x - 2 = (x + 2)(x^2 - 1) = (x + 2)(x - 1)(x + 1)$$

All the real roots are rational: $x = -2$, 1, and −1.

Caution

Remember that rational roots can be positive or negative.

b. Find the real roots for the following polynomial equation.

$$x^4 - x^3 - 8x + 8 = 0$$

SOLUTION

Step 1: Use the Rational Root Theorem.

Factors of 1: ± 1 ← Values of q Factors of 8: $\pm 1, \pm 2, \pm 4, \pm 8$ ← Values of p

$$\frac{p}{q} = \{\pm 1, \pm 2, \pm 4, \pm 8\}$$

Find the first root that yields a factor:

$$\begin{array}{r|rrrrr} 1 & 1 & -1 & 0 & -8 & 8 \\ & & 1 & 0 & 0 & -8 \\ \hline & 1 & 0 & 0 & -8 & 0 \end{array} \checkmark$$

$$x^4 - x^3 - 8x + 8 = (x - 1)(x^3 - 8)$$

Step 2: Use the Rational Root Theorem to find additional roots, or use a polynomial pattern. $(x^3 - 8)$ is a difference of cubes.

$$(x - 1)(x - 2)(x^2 + 2x + 4)$$

The discriminant of the quadratic term is negative yielding complex roots. The only real roots are $x = 1$ and $x = 2$.

c. Find the real roots for the following polynomial equation.

$$2x^4 - 9x^3 + 2x^2 + 9x - 4 = 0$$

SOLUTION

Step 1: Use the Rational Root Theorem.

Factors of 2: $\pm 1, \pm 2$ ← Values of q Factors of 4: $\pm 1, \pm 2, \pm 4$ ← Values of p

$$\frac{p}{q} = \left\{\pm\frac{1}{2}, \pm 1, \pm 2, \pm 4\right\}$$

Find the first rational number that yields a factor:

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & -9 & 2 & 9 & -4 \\ & & 1 & -4 & -1 & 4 \\ \hline & 2 & -8 & -2 & 8 & 0 \end{array} \checkmark$$

$$2x^4 - 9x^3 + 2x^2 + 9x - 4 = \left(x - \frac{1}{2}\right)(2x^3 - 8x^2 - 2x + 8)$$

$$= (2x - 1)(x^3 - 4x^2 - x + 4)$$

Step 2: Use the Rational Root Theorem on the cubic term.

$$\frac{p}{q} = \{\pm 1, \pm 2, \pm 4\}$$

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -1 & 4 \\ & & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & 0 \end{array} \checkmark$$

$$(2x - 1)(x - 1)(x^2 - 3x - 4)$$

Step 3: Use the Quadratic Formula.

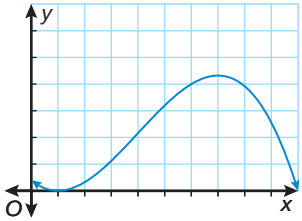
$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2} = \frac{3 \pm 5}{2} = 4 \text{ and } -1$$

The real roots are $x = \frac{1}{2}$, 1, -1, and 4.

Example 5 Application: Roller Coaster Design

A portion of a roller coaster is modeled by the polynomial function $y = -0.4x^3 + 4.8x^2 - 8.4x + 4$. Graph the function for positive values only and find the positive roots.

SOLUTION



Step 1: You can use the Rational Root Theorem even when the terms are decimals by multiplying by powers of 10.

$$-0.4x^3 + 4.8x^2 - 8.4x + 4 = 0$$
$$-4x^3 + 48x^2 - 84x + 40 = 0$$
$$-x^3 + 12x^2 - 21x + 10 = 0$$

Factors of -1: ± 1 ← Values of q

Factors of 10: $\pm 1, \pm 2, \pm 5, \pm 10$ ← Values of p

$$\frac{p}{q} = \{\pm 1, \pm 2, \pm 5, \pm 10\}$$

Find the first rational number that yields a factor:

1	-1	12	-21	10	
		-1	11	-10	✓
	-1	11	-10	0	

$$-x^3 + 12x^2 - 21x + 10 = (x - 1)(-x^2 + 11x - 10)$$

Step 2: Use the Quadratic Formula.

$$x = \frac{-11 \pm \sqrt{11^2 - 4(-1)(-10)}}{2(-1)} = \frac{-11 \pm \sqrt{81}}{-2}$$

$$= \frac{11 \pm 9}{2} = 10 \text{ and } 1$$

The roots are $x = 10$ and 1.

Lesson Practice

a. *(Ex 1)* Find the roots of $x^3 - 8 = 0$.

b. *(Ex 1)* Find the roots of $x^3 + 125 = 0$.

c. *(Ex 2)* Find the roots of $x^9 - 512 = 0$.

d. *(Ex 2)* Find the roots of $x^6 - 729 = 0$.

e. *(Ex 3)* Find the roots of the cubic equation $5x^3 + 2x - 1 = 0$.

f. (Ex 4) Find the rational roots for $4x^3 - 4x^2 + 9x - 9 = 0$.

g. (Ex 4) Find the rational roots for $x^4 - 2x^3 - 27x + 54 = 0$.

h. (Ex 4) Find the rational roots for $4x^3 + 4x^2 - 29x + 21 = 0$.

i. (Ex 5) Find the rational roots for $0.05x^3 - 1.3x^2 + 9.6x - 14.4 = 0$.

Practice Distributed and Integrated

Find the roots of the given equations.

1. (76) $y = 6x^2 + 7x - 1$

2. (76) $y = x^4 - 27x$

***3.** (85) Which of the following is not a root of $f(x) = 3x^3 - 19x^2 + 22x + 24$?

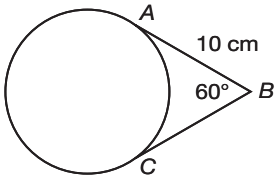
A $-\frac{2}{3}$ **B** $\frac{2}{3}$ **C** 3 **D** 4

Determine whether each of the following are step functions.

4. (79) $f(x) = \begin{cases} 12, \text{ if } x < 1 \\ x + 12, \text{ if } x \geq 1 \end{cases}$

5. (79) $f(x) = \begin{cases} 0, \text{ if } x < 0 \\ 5, \text{ if } x \geq 0 \end{cases}$

6. (63) **Geometry** $\overline{AB}$ is tangent to the circle at point A and $\overline{BC}$ is tangent to the circle at point C. What is the length of $\overset{\frown}{AC}$? Give the answer in exact form and to the nearest tenth of a centimeter.



7. (78) **Gravity on the Moon** The height of a free-falling object on the moon (neglecting air resistance) is given by the function $h(t) = -0.8t^2 + v_0t + h_0$, where h_0 is the initial height in meters, v_0 is the initial velocity in meters per second, and $h(t)$ is the height in meters at time t seconds. How long does it take for an object to hit the surface of the moon after it is dropped from a height of 125 meters? (Hint: Multiply both sides of a quadratic equation by a number so that the coefficient of the quadratic term is a perfect square.)

8. (68) **Error Analysis** Find and correct the error a student made in finding $P(\text{AM}|\text{Female})$.

$$P(\text{AM}|\text{Female}) = \frac{4}{12} = \frac{1}{3} = 33\tfrac{1}{3}\%$$

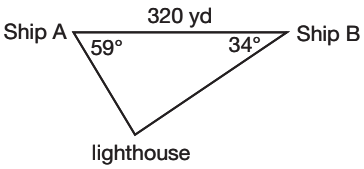
	Male	Female
AM	16	4
PM	8	12

9. (70) **Pendulum** The period of a pendulum, T, is the time it takes for the pendulum to make one back-and-forth swing, and is given by $T = 2\pi\sqrt{\frac{L}{9.8}}$, where L is the length of the pendulum in meters. Find the length of a pendulum whose period is 1.5 seconds.

10. (55) Ty tosses a quarter three times. What is the probability that the quarter will land on tails all three times?

***11.** (83) **Write** How is deriving a quadratic equation from two complex roots different from deriving a quadratic equation from two real roots?

12. (71) **Navigation** Two ships 320 yards apart spot the same lighthouse, as shown in the diagram. How much farther is ship B from the lighthouse than ship A?



13. (78) **Error Analysis** A student solved the equation $2x^3 - 72x = 0$ as shown below. What is the error?

$2x^3 - 72x = 0 \rightarrow 2x(x^2 - 36) = 0 \rightarrow 2x(x + 6)(x - 6) = 0 \rightarrow x = -6$ or $x = 6$

14. (72) **Geology** The Richter magnitude M of an earthquake is expressed by $M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$, where E is the energy released in ergs. The earthquake of greatest magnitude, 9.5, occurred in 1960 in the Pacific, near the coast of Chile. Find the energy released by that earthquake.

***15.** (84) **Graphing Calculator** Use a graphing calculator to estimate the solution(s) of $\frac{x^4}{2} + \frac{1}{x^2 + 1} = x + 3$.

16. (73) **Write** Explain what is meant by voluntary response sampling and why it is not a form of probability sampling.

17. (80) **Multi-Step** The scores on a test are normally distributed with a mean of 86 and a standard deviation of 3.

a. What percent of the scores are between 83 and 89?

b. Tell how you can use the answer in part a to find the percent of scores that are either less than 83 or greater than 89. Then find the percent.

c. Tell how you can use the answer in part b to find the percent of scores that are less than 83. Then find the percent.

Find the restricted values for the rational equations.

***18.** (84) $\frac{1}{5} - \frac{1}{x + 5} = \frac{4}{3x^2}$

***19.** (84) $\frac{x}{3} + \frac{x + 3}{x - 1} = \frac{4}{x - 1}$

20. (74) **Estimate** Estimate the discriminant of $43.876x^2 + \frac{4}{5}x - 4.128 = 0$.

***21.** (85) Let $f(x) = x^3 - 27$ and $g(x) = x + 2$. Find the roots of $f(g(x))$.

22. (75) **Physics** The equation $h = -4.9t^2 + v_o t + h_o$ can be used to find the distance from the ground at time t for a ball thrown from a height h_o at an initial speed of v_o. Use the technique of completing the square to derive the function that can be used to calculate t.

23. (52) Let $f(x) = 3x$, $g(x) = x - 5$, and $h(x) = x^2 - 2$. Find the composite function $f(g(h(x)))$.

24. (77) **Analyze** In $\triangle ABC$, $m\angle A = 63°$, $b = 15$, and $c = 8$. To find $m\angle A$, which should be used, the Law of Sines or the Law of Cosines. Why?

*25. (81) **Probability** A machine on an assembly line makes an acceptable product 90% of the time.

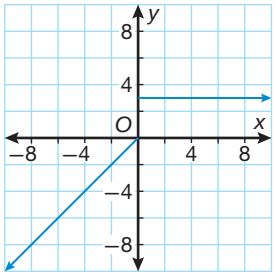
a. Ten samples are chosen at random. What is the probability that all 10 samples are acceptable?

b. At what number of samples is the probability 10% or less? Let n represent the number of samples. Write an equation and use natural logarithms to solve it.

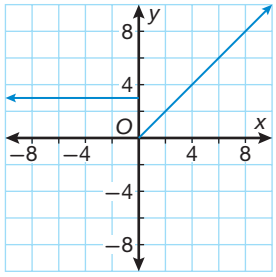
*26. (Inv 7) **Write** A radio station asks for listeners to call in and vote for their favorite high school football team. Identify the type of sample and determine whether it is biased. Explain your answer.

27. (79) **Multiple Choice** Which is the graph of the function $f(x) = \begin{matrix} -x, \text{ if } x < 0 \\ 3, \text{ if } x \geq 0 \end{matrix}$?

A



B



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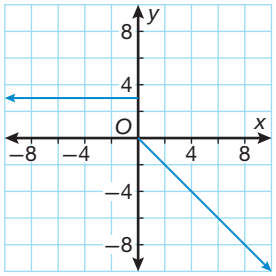

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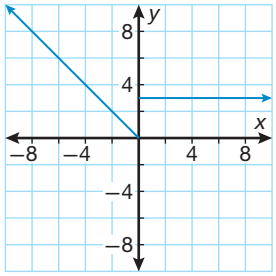


Identify all the real roots.

*28. (85) $x^3 - 729 = 0$

*29. (85) $x^4 - 13x^3 + 55x^2 - 81x + 18 = 0$

*30. (82) Graph $y = -2 \sin\left(\frac{1}{4}x\right)$

LESSON 86

Translating Sine and Cosine Functions

Warm Up

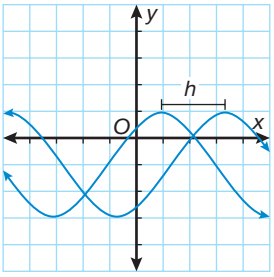
1. (82) **Vocabulary** The function $y = \sin(x)$ is an example of a _____ function.

2. (82) What is the amplitude of $y = 5\sin(x)$?

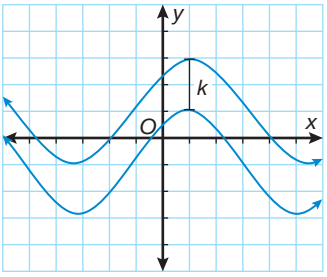
3. (82) What is the period of $y = \cos(2x)$?

New Concepts

Periodic functions can be translated horizontally and vertically with the same methods used for other functions like the quadratic function in Lesson 30.



$g(x)$: horizontal translation of $f(x)$
$g(x) = f(x - h)$



$h(x)$: vertical translation of $f(x)$
$h(x) = f(x) + k$

A **phase shift** is a horizontal translation of a periodic function. A periodic function $f(x - h)$ is shifted h units to the right if $h > 0$ and h units to the left if $h < 0$.

Hint

The parent functions $y = \cos(x)$ and $y = \sin(x)$ have periods of 2π.

Math Reasoning

Verify Create a table of values comparing $\cos(x)$ and $\sin\left(x + \frac{\pi}{2}\right)$ to verify that sine and cosine are phase shifted by $\frac{\pi}{2}$.

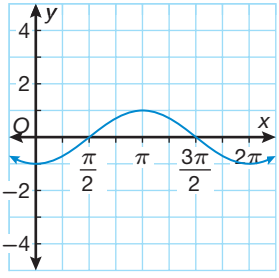
Example 1 Graphing Phase Shifts

Describe the phase shift of each function and then graph it.

a. $y = \sin\left(x - \frac{\pi}{2}\right)$

SOLUTION

Write the function in standard form. $h = \frac{\pi}{2}$ and $h > 0$; the graph is shifted $\frac{\pi}{a}$ radians to the right. To graph begin with the graph of the parent function $y = \sin x$ and shift the graph $\frac{\pi}{2}$ radians to the right.



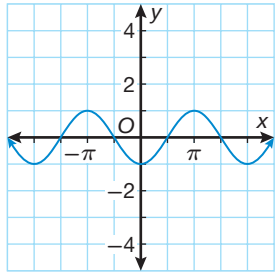
b. $y = \cos(x + \pi)$

SOLUTION

Write the function in standard form first.

$y = \cos(x - (-\pi))$

$h = -\pi$ and $h < 0$; the graph is shifted π units to the left. To graph begin with the graph of the parent function $y = \cos x$ and shift the graph π radians to the left.



Online Connection
www.SaxonMathResources.com

In the function $f(x) + k$, the value of k determines the vertical shift of the graph. If $k > 0$, the graph is shifted up k units from $f(x)$ and if $k < 0$, the graph is shifted down k units from $f(x)$.

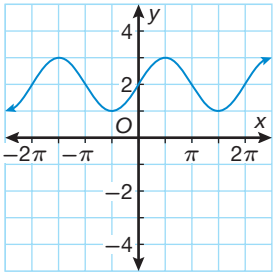
Example 2 Graphing Vertical Shifts

Describe the vertical shift of each function and then graph it.

a. $y = \sin x + 2$

SOLUTION

$k = 2$ and $k > 0$; the graph is shifted 2 units up. To graph, begin with the graph of the parent function $y = \sin x$ and shift the graph up 2 units.



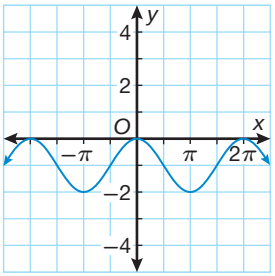
b. $y = \cos x - 1$

SOLUTION

Write the function in standard form first.

$y = \cos x + (-1)$.

$k = -1$ and $k < 0$; the graph is shifted 1 unit down. To graph, begin with the graph of the parent function $y = \cos x$ and shift the graph 1 unit down.



A translation can also be formed by combining a vertical shift and a horizontal shift.

Example 3 Graphing a Combination of Shifts

Describe the shifts of the function and then graph it.

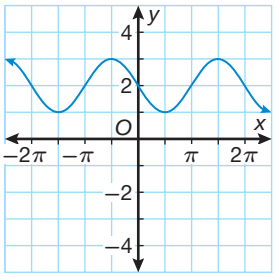
$y = \sin(x + \pi) + 2$

SOLUTION

Write the function in standard form first.

$y = \sin\big(x - (-\pi)\big) + 2$

$h = -\pi$ and $h < 0$; the graph is shifted π radians to the left. $k = 2$ and $k > 0$; the graph is shifted 2 units up. To graph, begin with the graph of the parent function $y = \sin x$ and shift the graph π radians to the left and 2 units up.



To graph a more complicated translated function, a simpler member of the sine or cosine functions can be used.

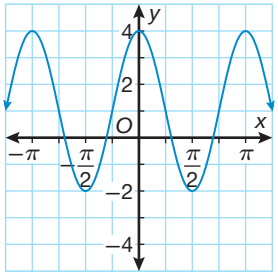
Sine and Cosine Functions in Standard Form	
$y = a\sin b(x - h) + k$	$y = a\cos b(x - h) + k$
The measure of angle x is in radians. $\lvert a\rvert$ = amplitude, the vertical stretch or shrink of the graph of the function, $\frac{2\pi}{b}$ = period, where $b > 0$, h = horizontal or phase shift, and k = vertical shift.	

The sine and cosine functions in standard form can be graphed using transformations of the functions $y = a\sin bx$ and $y = a\cos bx$.

Example 4 Graphing $y = a\cos(b(x - h)) + k$

Graph $y = 3\cos(2(x - \pi)) + 1$

SOLUTION



Begin by graphing the simpler function $y = 3\cos 2x$ involving only amplitude and period changes.

The amplitude is the $\lvert a\rvert = \lvert 3\rvert = 3$ and the period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

Since the amplitude is positive the pattern to graph is max-zero-min-zero-max.

$h = \pi$ and $h > 0$; so the graph is shifted π radians to the right.

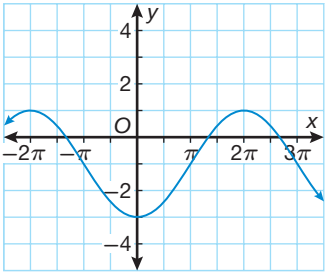
$k = 1$ and $k > 0$; so the graph is shifted 1 units up.

Shift the graph of $y = 3\sin 2x$ π radians to the right and 1 units up.

Example 5 Graphing $y = a\sin(b(x + h)) - k$

Graph $y = -2\sin\left(\frac{1}{2}(x + \pi)\right) - 1$

SOLUTION



Begin by graphing the simpler function $y = -2\sin\frac{1}{2}x$ involving only amplitude and period changes.

The amplitude is the $\lvert a\rvert = \lvert -2\rvert = 2$ and the period is $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$.

Since the amplitude is negative, the pattern to graph is zero-min-zero-max-zero.

Write in standard form to determine h and k.

$$y = -2\sin\left(\frac{1}{2}(x - (-\pi))\right) + (-1)$$

$h = -\pi$ and $h < 0$; so the graph is shifted π radians to the left.

$k = -1$ and $k < 0$; so the graph is shifted 1 unit down.

Shift the graph of $y = -2\sin\frac{1}{2}x$ π radians to the left and 1 unit down.

Lesson Practice

Describe the phase shift of each function and then graph it.
(Ex 1)

a. $y = \sin(x - \pi)$

b. $y = \cos(x + 2\pi)$

Describe the vertical shift of each function and then graph it.
(Ex 2)

c. $y = \sin x - 3$

d. $y = \cos x + 2$

e. (Ex 3) Describe the shifts of the function $y = \sin(x + \pi) - 2$ and then graph it.

Graph each function.

f. (Ex 4) $y = 2\cos\big(4(x - \pi)\big) + 3$

g. (Ex 5) $y = -1\sin\big(2(x + \pi)\big) - 3$

Practice Distributed and Integrated

1. (60) Given a deck of cards, what is the probability of randomly drawing a face card, and then a number card without replacement of the first card? (The ace is not a face or number card)

Solve.

2. (70) $\sqrt[3]{8x^3 - 1} = 2x - 1$

3. (70) $\sqrt[3]{12x - 5} - \sqrt[3]{8x + 15} = 0$

4. (66) Find the roots and give the multiplicity of each root: $x^3 + 12x^2 + 36x = 0$

***5.** (8) Determine the period of the function $y = 3\cos\left(\frac{\pi}{2}x\right)$.

6. (67) **Jewelry** A jeweler is about to finish his sales for the year. He has 8 diamonds, 6 tanzanite stones, 9 opal stones, and 3 sapphires. A customer comes in needing three stones for a ring setting. Find the probability that the customer will choose 2 tanzanite stones and 1 sapphire.

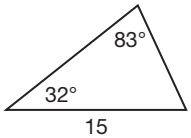
7. (25) Find the range and standard deviation for the following set of data;

{35, 67, 21, 16, 24, 51, 18, 32}.

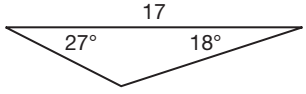
***8.** (86) Consider the function $f(x) = 3\cos(2x)$. Determine the period and amplitude of the function.

***9.** (71) **Multiple Choice** Which triangle cannot be solved by using the Law of Sines?

A



B



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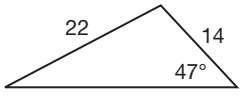

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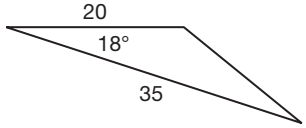


10. (80) **Generalize** Given a data value that is to be standardized and the mean of the data set, how can you tell if the z-score will be positive or negative without performing any calculations.

11. (77) An equilateral triangle has a side length of 12.7 centimeters. Find the area of the triangle by using Heron's formula.

***12.** (86) Which of the following has a vertical shift of 2 up?

A $y = \cos(2x) + 2$ **B** $y = \cos(2\pi x) - 2$

C $y = 2\cos(x)$ **D** $y = \cos\left(\frac{x}{2}\right)$

13. (74) **Geometry** The perimeter of a rectangle is 52 units.

a. Write an expression for the length if the width is represented by w.

b. Write a quadratic equation in standard form to show that the product of the length and width is 169 square units.

c. Find the discriminant of the equation. Explain what it means in terms of the rectangle.

***14.** (85) **Graphing Calculator** Use a graphing calculator to test this hypothesis: If a cubic function has only one rational root, then the other roots are complex.

15. (75) Determine the domain and range of the function $f(x) = \sqrt{x - 4} + 3$ and its inverse.

16. (58) Complete the square and factor the resulting perfect square trinomial: $x^2 + 30x$.

***17.** (76) **Physics** Acceleration due to gravity is different on different planets. The equation below can be used to find the distance from the ground at time t for a ball thrown from a height h_o at an initial speed of v_o. Use the table to determine how long it will take a ball thrown with an initial speed of 20 at a height of 35 to hit the ground on Mars.

$$h = -\frac{1}{2}at^2 + v_o t + h_o$$

Planet	Acceleration (a)
Mercury	3.59
Venus	8.87
Mars	3.77
Jupiter	25.95
Saturn	11.08
Uranus	10.67
Neptune	14.07

18. (84) **Data Analysis** The harmonic mean x of y and z is given by the formula $\frac{2}{x} = \frac{1}{y} + \frac{1}{z}$. Find z when $x = 7.5$ and $y = 6$.

19. (81) A radioactive material has a half life of 46 hours. Find the decay constant.

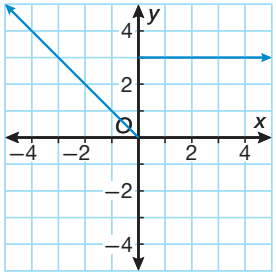
20. (78) **Multi-Step** Complete parts a-b to help graph the quadratic function $y = \frac{1}{4}x^2 - \frac{9}{4}$.

a. Find the points in which the graph intersects the x-axis.

b. Find the vertex.

c. Graph the function.

21. *(79)* **Error Analysis** Jake graphed the piecewise function $f(x) = \begin{cases} x, \text{ if } x < 0 \\ 3, \text{ if } x \geq 0 \end{cases}$ incorrectly. Explain Jake's error.



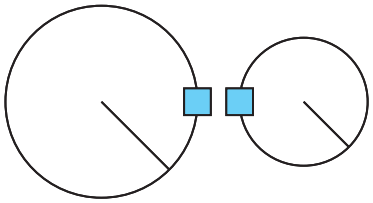
22. *(57)* **Multi-Step** Suppose that the population of a certain species decreases at the rate of 5% per year and that the current population of that species in a certain habitat is 150.

a. Write an exponential function that gives the population as a function of x, the number of elapsed years. (Hint: $0 < b < 1$).

b. Write a logarithmic function that gives the number of elapsed years as a function of x, the population.

c. To the nearest tenth, how many years will it take for the population to drop below 100?

***23.** *(86)* **Physics** Two wheels spin in opposite directions. The larger wheel completes a cycle in 4 seconds and the smaller wheel completes a cycle in 9 seconds. The two squares attached to the wheels are facing each other at time $t = 0$. How many seconds elapse before they are back in sync?



24. *(56)* P is a point on the terminal side of θ in standard position. Find the exact value of the six trigonometric functions for θ where $P(-2, -1)$.

25. *(83)* **Write** What is the converse of the Zero-Product Property? What can it be used for?

26. *(73)* **Government** Suppose a politician must visit 6 counties in Massachusetts. Her human resource personnel form a simple random sample by listing each county and numbering it as shown below. If they use the line of random digits shown below, which 6 counties will the politician visit?

01: Barnstable	02: Berkshire	03: Bristol	04: Dukes	05: Essex
06: Franklin	07: Hampden	08: Hampshire	09: Middlesex	10: Nantucket
11: Norfolk	12: Plymouth	13: Suffolk	14: Worcester	

62010	60482	89824	08705	57913	46815	76221	25650	89337	53603	73597

27. *(82)* **Analyze** Why is the amplitude of $\sin(\sin(x))$ 0.84 and not 1?

Identify all the real roots:

***28.** *(85)* $x^3 - 216$

***29.** *(85)* $x^9 + 125 = 0$

30. *(41)* Find the distance between the points $(2, -1)$ and $(5, 4)$.

LESSON
87

Evaluating Logarithmic Expressions

Warm Up

1. (81) **Vocabulary** The type of logarithm that has e as the base is the _____ logarithm.
2. (81) Simplify $\ln(exy)$.
3. (81) Simplify $\ln\left(\frac{x}{ey}\right)$.
4. (72) Express as a common logarithm $\log_7(100)$.

New Concepts

Properties of logarithms can be used to evaluate expressions. Recall the log rules:

$$\log_b(mn) = \log_b(m) + \log_b(n)$$
$$\log_b(m^n) = n\log_b(m)$$

Example 1 Evaluating Expressions of the Form $\log_a(bc)^d$

a. Use the properties of logarithms to evaluate $\log_4(16x)^3$ when $x = 256$.

SOLUTION

Step 1: Use the second rule to write the expression as $3 \cdot \log_4(16x)$.

Step 2: Apply the first rule:

$3 \cdot (\log_4 16 + \log_4 x)$. Since $\log_4 16 = 2$, then

$3 \cdot (\log_4 16 + \log_4 x) = 3 \cdot (2 + \log_4 x)$ or $6 + 3\log_4 x$.

Step 3: Evaluate when $x = 256$,

$6 + 3\log_4 256 = 6 + 3 \cdot 4 = 6 + 12 = 18$.

b. Use the properties of logarithms to evaluate $\ln(7e)^2$.

SOLUTION

Step 1: Use the second rule to write the expression as:

$\ln(7e)^2 = 2 \cdot \ln(7e)$.

Step 2: Use the first rule to expand.

$2 \cdot \ln(7e) = 2(\ln 7 + \ln e)$

Step 3: Evaluate.

$2(\ln 7 + \ln e) \approx 2(1.95 + 1) = 2(2.95) = 5.9$

Caution

When the exponent becomes a multiplier, it must be distributed to all terms in the log expression.

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Sometimes when evaluating and solving logarithmic equations, the change of base formula is utilized. Recall $\log_b x = \frac{\log_a x}{\log_a b}$, where a is the new base.

Example 2 Using the Change of Base Formula

a. Use the change of base formula to convert $\log_{100}(10x)^2$ to base 10. Then evaluate when $x = 1000$.

SOLUTION

$$\log_{100}(10x)^2 = 2 \cdot \log_{100}(10x)$$

$$2 \cdot \log_{100}(10x) = 2\left(\frac{\log 10x}{\log 100}\right) \qquad \text{Use the change of base formula.}$$

$$2\left(\frac{\log 10x}{\log 100}\right) = 2\left(\frac{\log 10 + \log x}{\log 100}\right)$$

$$= 2\left(\frac{\log 10 + \log 1000}{\log 100}\right) \qquad \text{Evaluate for } x = 1000.$$

$$= 2\left(\frac{1+3}{2}\right) = 4$$

b. Use the change of base formula to convert $\log_4(2x)^3$ to base e. Then evaluate when $x = 6$.

SOLUTION

$$\log_4(2x)^3 = 3 \cdot \log_4(2x)$$

$$3 \cdot \log_4(2x) = 3\left(\frac{\ln 2x}{\ln 4}\right) \qquad \text{Use the change of base formula.}$$

$$3\left(\frac{\ln 2x}{\ln 4}\right) = 3\left(\frac{\ln 2 + \ln x}{\ln 4}\right) \qquad \text{Use the log rules.}$$

$$= 3\left(\frac{\ln 2 + \ln 6}{\ln 4}\right) \qquad \text{Evaluate for } x = 6.$$

$$\approx 3\left(\frac{0.6931 + 1.7918}{1.3863}\right) \approx 5.3774$$

Hint

The change of base formula can also be used with the natural log function.

Example 3 Solving Logarithmic Equations Using the Change of Base Formula

Solve $1000^{9x} = 100$ for x.

SOLUTION

$$1000^{9x} = 100$$

$$\log_{1000} 100 = 9x \qquad \text{Simplify using log rules.}$$

$$\left(\frac{\log 100}{\log 1000}\right) = 9x \qquad \text{Use the change of base formula.}$$

$$\frac{2}{3} = 9x \qquad \text{Simplify and solve for } x.$$

$$x = \frac{2}{27}$$

Example 4 Application: Newton's Law of Cooling

Newton's Law of Cooling states that the rate at which an object cools is related to the difference in temperature between the object and its environment. The formula $T = e^{kt+C} + R$ can be used to determine the desired temperature of the object, T, the cooling time in minutes, t, or the temperature of the environment of the object, R. Solve the formula for t and then determine the amount of time needed to cool a cup of water from 80°C to 55°C in a room temperature of 25°C. Use $k = -0.200064$ and $C = 4.0073$.

SOLUTION

$$T = e^{kt+C} + R$$
$$e^{kt+C} = T - R$$
$$\ln(T - R) = kt + C$$
$$kt = \ln(T - R) - C$$
$$t = \frac{\ln(T - R) - C}{k}$$
$$t = \frac{\ln(55 - 25) - 4.0073}{-0.200064}$$
$$t \approx 3.03$$

The water would take approximately 3.03 minutes to cool.

Lesson Practice

a. *(Ex 1)* Use the properties of logarithms to evaluate $\log_6(36x)^2$ when $x = 216$.

b. *(Ex 1)* Use the properties of logarithms to evaluate $\ln(10e)^3$.

c. *(Ex 2)* Use the change of base formula to convert $\log_7(5x)^3$ to base 10. Then evaluate when $x = 5$.

d. *(Ex 2)* Use the change of base formula to convert $\log_8(11x)^2$ to base e. Then evaluate when $x = 10$.

e. *(Ex 3)* Solve $5^{3x} = 20$ for x.

f. *(Ex 4)* The magnitude of an earthquake can be found on the Richter scale using the equation $R = \log \frac{I}{I_o}$ where I represents intensity and I_o represents the intensity of a zero-level earthquake. The 1989 California earthquake's intensity was $10^{7.1}$ times I_o. Evaluate to find R, the magnitude on the Richter scale.

Practice Distributed and Integrated

1. *(60)* A cabinet contains 12 blue cups, 12 red cups, and 20 green cups. What is the probability that a green cup and a blue cup are chosen at random in that order with replacement?

Simplify.

2. *(69)* $\dfrac{8+2i}{1+3i}$

3. *(69)* $i^{22}+i^{15}$

4. *(53)* Let $f(x)=x^2+1$ and $g(x)=7-x$. What are the domain and range of $(f\circ g)(x)$?

5. *(75)* Which of the following is equivalent to $y=\left(\sqrt{x^2+3x+2}\right)^{\frac{1}{3}}$?

A $y=\sqrt[3]{x^2+3x+2}$ **B** $\sqrt{\sqrt[3]{x^2+3x+2}}$

C $y=\sqrt[3]{\sqrt{x^2+3x+2}}$ **D** $y=\sqrt[3]{(x+2)(x+1)}$

6. **Multi-Step** *(59)* Use $\dfrac{13^{\frac{2}{3}}\cdot 13^{\frac{1}{4}}}{13^{\frac{1}{12}}\cdot 13^{\frac{1}{2}}}$.

a. Which property of rational exponents should be applied first to simplify the expression, explain?

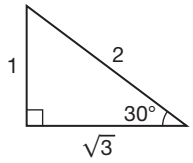
b. Which property of rational exponents should be applied next?

c. Simplify the expression.

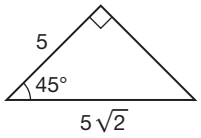
***7.** **Model** *(86)* Graph $y=2\cos(2x)-3$

Find the sine, cosine, and tangent of the given angle.

8. *(52)*



9. *(52)*



10. **Multiple Choice** *(83)* Which equation does *not* have the exact same roots as the others?

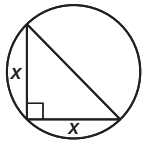
A $(x+1)(x-7)=0$ **B** $2(x+1)(x-7)=0$

C $4(x+1)3(x-7)=0$ **D** $x(x+1)(x-7)=0$

11. **Radio** *(68)* A radio station needs 4 different songs to complete the song set for the end of their show. They can pick from 16 eighties hits, 20 nineties hits, and 18 current top hits. Find the probability that the DJ will select 2 current hits and 2 eighties hits in that order for his show.

***12.** **Earthquakes** *(87)* Solve $R=\log\frac{I}{I_o}$ for I and then determine the intensity of an earthquake with a magnitude, R, of 7. Your answer should be in terms of I_o.

13. **Probability** *(78)* The right triangle is inscribed in the circle. Suppose a point is chosen at random inside the circle. What is the probability that the chosen point is inside the triangle?



***14.** **Justify** *(87)* The solution of $100^{2x}=1000$ is $x=\frac{3}{4}$.

15. (79) **Postal Rates** The rates for first-class mail (retail letters and cards) are shown in the table. Write a piecewise function that represents the cost of mailing a first-class letter.

Fee	Weight
\$0.41	up to 1 ounce
\$0.58	greater than 1 ounce and up to 2 ounces
\$0.75	greater than 2 ounces and up to 3 ounces
\$0.92	greater than 3 ounces and up to 3.5 ounces

16. (81) **Error Analysis** To rewrite $\ln\frac{5e}{x^2}$, a student wrote $\ln\frac{5e}{x^2} = \frac{\ln 5e}{\ln x^2} = \frac{\ln 5 + \ln e}{2\ln x} = \frac{\ln 5 + 1}{2\ln x}$. What is the error? Rewrite $\ln\frac{5e}{x^2}$ as a sum or difference of terms correctly.

***17.** (85) **Generalize** Suppose there are two cubic functions: $f(x) = x^3 - a^3$ and $g(x) = x^3 - b^3$. If $f(x) \cdot g(x)$ crosses the x-axis at only one point, what can you conclude about a and b?

***18.** (80) **Graphing Calculator** The weights of bridge tokens are normally distributed with a weight of 4.85 g and a standard deviation of 0.068 g. Find the range of weights for the middle 99.7% of bridge tokens.

19. (36) Find the equation of the line perpendicular to the graph of $y = 2x - 4$ that crosses $(2, -8)$.

20. (74) **Farming** A farmer has 820 feet of fencing material to enclose a rectangular grazing area for his livestock. He wants the area to be at least 50,000 square feet. The equation $w(410 - w) = 50{,}000$ gives the width of a rectangle that meets these requirements. Use the discriminant to explain if the farmer has enough material to build a fence that meets these requirements.

***21.** (87) **Error Analysis** A student rewrote the expression $\log_7(2x)^4$ as $4\log_7 2 + \log_7 x$.

a. Describe the student's error.

b. Rewrite the original expression.

Simplify.

22. (81) $\ln e^{7x}$

23. (81) $\ln e^{x+4}$

24. (70) **Geometry** The formula $r = \sqrt{\frac{S}{4\pi}}$ gives the radius of a sphere with a surface area S. How much greater is the surface area of a sphere with a radius of 20 inches than a radius of 15 inches? Give your answer in terms of π.

***25.** (86) **Physics** Every two seconds a radio beacon emits a signal. If the magnitude of the signal is 10, write the periodic function that models the signal using the form $y = a\cos(bx)$.

***26.** (84) **Analyze** What are the steps to solving a rational equation algebraically?

Solve for x.

***27.** (87) $1000^{6x} = 10$

***28.** $100^{3x} = 10$

29. (81) **Write** Explain why $\ln e^{x^2+3} = x^2 + 3$.

30. (82) Given $g(x) = 4\sin(3x)$, identify the period.

LESSON
88

Solving Abstract Equations

Warm Up

1. (37) **Vocabulary** For the fractions $\frac{1}{r}$ and $\frac{3}{2s}$, the _____ is $2rs$.

2. (7) Solve for x. $2x + 3 = x - 17$

3. (23) Solve for x by factoring. $4x^2 - 35x - 50 = 0$

4. (51) **Multiple Choice** Which of these is a factor of $3x^4 - 6x^3 + 6x^2 + 3x - 30$?

A $x - 6$ **B** $x + 6$

C $3x - 6$ **D** $3x + 6$

5. (40) Simplify the expression $\sqrt[4]{x^8} \cdot \sqrt[3]{x^4}$.

New Concepts

An **abstract equation**, also called a literal equation, has two or more variables. Formulas are literal equations. It is often useful to solve a literal equation for one of its variables.

Math Language

The word **literal** is derived from the Latin word *littera*, which means *letter*.

Example 1 Solving an Abstract Linear Equation

A formula that relates Fahrenheit temperature and Celsius temperature is $F = \frac{9}{5}C + 32$. Solve the formula for C.

SOLUTION

$$F = \frac{9}{5}C + 32$$

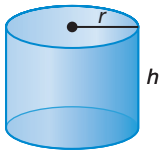
$$F - 32 = \frac{9}{5}C$$ Subtract 32 from both sides.

$$\frac{5}{9}(F - 32) = \frac{5}{9} \cdot \frac{9}{5}C$$ Multiply both sides by $\frac{5}{9}$, the reciprocal of $\frac{9}{5}$.

$$\frac{5}{9}(F - 32) = C$$

Example 2 Solving Abstract Polynomial Equations

a. The formula for the surface area S of a right circular cylinder is $S = 2\pi r^2 + 2\pi rh$. Solve the formula for h.



SOLUTION

$$S = 2\pi r^2 + 2\pi rh$$

$$S - 2\pi r^2 = 2\pi rh$$ Subtract $2\pi r^2$ from both sides.

$$\frac{S - 2\pi r^2}{2\pi r} = h$$ Divide both sides by $2\pi r$.

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Hint

In the equation, g is the acceleration due to gravity, v_0 is the initial velocity, and h_0 is the initial height.

b. The height h of a free-falling object (neglecting air resistance) is given by the equation $h = \frac{1}{2}gt^2 + v_0t + h_0$. Solve the equation for t.

SOLUTION

Use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve for t.

$$h = \frac{1}{2}gt^2 + v_0t + h_0$$

$$0 = \frac{1}{2}gt^2 + v_0t + h_0 - h$$ Write the equation in the form $0 = ax^2 + bx + c$.

$$0 = \frac{1}{2}gt^2 + v_0t + h_0 - h$$ Identify coefficients in the quadratic equation.

$$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4\left(\frac{1}{2}g\right)(h_0 - h)}}{2\left(\frac{1}{2}g\right)}$$ Substitute $a = \frac{1}{2}g$, $b = v_0$, and $c = h - h_0$.

$$t = \frac{-v_0 \pm \sqrt{v_0^2 - 2g(h_0 - h)}}{g}$$ Simplify.

$$t = \frac{-v_0 \pm \sqrt{v_0^2 - 2gh_0 + 2gh}}{g}$$ Distribute $-2g$.

Example 3 Solving Abstract Rational Equations

The total resistance R is given by the formula $R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$. Solve the formula for R_1.

SOLUTION To simplify a complex fraction, multiply the numerator and denominator of the complex fraction by the LCD of the fractions of the numerator and denominator.

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \cdot \frac{R_1R_2}{R_1R_2}$$ Since R_1R_2 is the least common denominator of $\frac{1}{R_1}$ and $\frac{1}{R_2}$, multiply by $\frac{R_1R_2}{R_1R_2}$ to simplify the complex fraction.

$$R = \frac{R_1R_2}{\frac{R_1R_2}{R_1} + \frac{R_1R_2}{R_2}}$$ Distribute.

$$R = \frac{R_1R_2}{R_2 + R_1}$$ Simplify.

$$R(R_2 + R_1) = R_1R_2$$ Multiply both sides by the new LCD, $R_2 + R_1$.

$$RR_2 + RR_1 = R_1R_2$$ Distribute R.

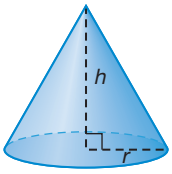
$$RR_2 = R_1R_2 - RR_1$$ Collect all R_1 terms on one side.

$$RR_2 = R_1(R_2 - R)$$ Factor out R_1.

$$\frac{RR_2}{R_2 - R} = R_1$$ Divide both sides by $R_2 - R$.

Example 4 Solving an Abstract Radical Equation

A formula for the lateral surface area L of a right circular cone is $L = \pi r\sqrt{r^2 + h^2}$. Solve the formula for h.



SOLUTION

$$L = \pi r\sqrt{r^2 + h^2}$$

$$\frac{L}{\pi r} = \sqrt{r^2 + h^2}$$ Divide both sides by πr.

$$\frac{L^2}{\pi^2 r^2} = r^2 + h^2$$ Square both sides.

$$\frac{L^2}{\pi^2 r^2} - r^2 = h^2$$ Subtract r^2 from both sides.

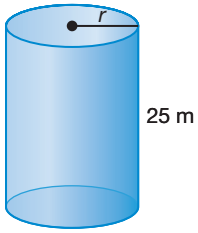
$$\sqrt{\frac{L^2}{\pi^2 r^2} - r^2} = h$$ Take the square root of both sides.

Math Reasoning

Write Squaring both sides of an equation can introduce an extraneous solution if negative numbers are involved. Explain why that does not apply in this case.

Example 5 Application: An Aquarium

The AquaDom in Berlin, Germany is a cylindrical aquarium with an elevator tube inside. It is the largest acrylic glass cylinder in the world, with a height of 25 meters and a volume of approximately 2400 cubic meters (including the elevator tube). What is the approximate radius of the AquaDom?



SOLUTION

Solve the formula for the volume of a cylinder for r.

$$V = \pi r^2 h$$ Write the formula for the volume of a cylinder.

$$\frac{V}{\pi h} = r^2$$ Divide both sides by πh.

$$\sqrt{\frac{V}{\pi h}} = r$$ Take the square root of both sides.

$$\sqrt{\frac{2400}{\pi \cdot 25}} = r$$ Substitute the known values for V and h.

$$r \approx 5.5$$ Use a calculator to find the approximate value of r.

The approximate radius of the AquaDom is 5.5 meters.

Check Substitute 5.5 for r and 25 for h in the volume formula.

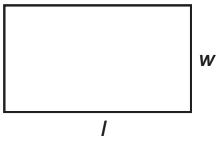
$V = \pi r^2 h \approx 3.14(5.5)^2(25) \approx 2375$, which is close to 2400. So the answer is reasonable.

Hint

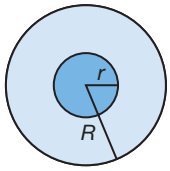
An approximation for the irrational number π is 3.14, but the π key on a calculator gives a better approximation.

Lesson Practice

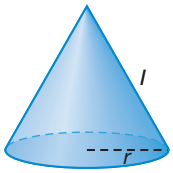
(Ex 1) **a.** The formula for the perimeter of a rectangle is $P = 2l + 2w$. Solve the formula for w.



b. (Ex 2) The formula for the area of a circular ring is $A = \pi R^2 - \pi r^2$. Solve the formula for R.

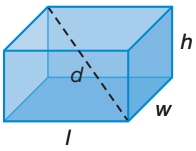


c. (Ex 2) A formula for the total surface area S of a right circular cone is $S = \pi r^2 + \pi rl$. Solve the formula for r. (Hint: All quantities are positive.)

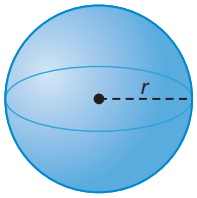


d. (Ex 3) The focal length f for a camera lens is given by the formula $f = \frac{d_1 d_2}{d_1 - d_2}$, where d_1 is the distance from the lens to the object being photographed and d_2 is the distance from the lens to the film. Solve the formula for d_1.

e. (Ex 4) The length d of a diagonal of a rectangular prism is $d = \sqrt{l^2 + w^2 + h^2}$. Solve the equation for h.



f. (Ex 5) Earth is nearly spherical with an approximate surface area of 197,000,000 square miles. The surface area S of a sphere is $S = 4\pi r^2$. Solve the formula for r and then use your result to find the approximate radius of Earth.



Practice Distributed and Integrated

Write the exponential equation in logarithmic form.

1. (63) $5^3 = 125$

2. (63) $20^3 = 8000$

Identify all the real roots.

3. (85) $x^3 + 8x^2 - 23x - 30 = 0$

4. (85) $6x^3 + 12x^2 - 18x = 0$

Write the expression as a complex number in standard form.

5. (62) $(1 - 9i)(1 - 4i)(4 - 3i)$

6. (62) $(3 + 2i) + (5 - i) + 6i$

7. (81) **Analyze** What pair of inverse functions contain the ordered pairs $(3, e^3)$ and $(e^3, 3)$?

8. (72) **Chemistry** Hydrochloric acid is classified as a strong acid, with a pH of 1. The pH of a solution is given by the formula $\text{pH} = -\log [\text{H}^+]$, where $[\text{H}^+]$ represents the concentration of hydrogen ions in moles per liter. What is the hydrogen ion concentration in hydrochloric acid?

***9.** (87) **Multiple Choice** The expression $\log_2(8x)^2$ when $x = 16$ is equivalent to:

A 7 **B** 12 **C** 14 **D** 16

10. (73) **Probability** A researcher visits a pond and captures and tags 75 fish. On a return visit, the researcher captures 130 fish, and 68 of them are found to be tagged. What is the probability that a randomly chosen fish in the pond was marked on the first day?

***11.** (88) **Write** Explain why $A = \frac{1}{2}h(b_1 + b_2)$ and $h = \frac{2A}{b_1 + b_2}$ describe the same relationship among the variables A, h, b_1, and b_2.

12. (74) **Write** How can finding the discriminant of a quadratic equation help you decide which method you will use to solve the equation?

Solve each formula for the indicated variable.

***13.** (88) Given $S = 2\pi r^2 + 2\pi rh$, solve for r.

***14.** (88) Given $\frac{5}{9}(F - 32) = C$, solve for F.

15. (75) **Physics** According to Einstein's theory of relativity, as an object moves faster, it appears shorter in length to a stationary observer. The equation below shows the relationship between the original length of the object and the new length, where v is the speed of the object and c is the speed of light. Suppose the ratio of the original length to the new length is 2:1. How fast is the object going as a factor of c?

$$\text{New length} = (\text{Original length})\sqrt{1 - \frac{v^2}{c^2}}$$

16. (84) **Multiple Choice** Solve $\frac{2}{3x + 15} + \frac{1}{3} + \frac{x}{x + 5} = \frac{4}{9}$.

A -0.125 **B** 1 **C** -6.625 **D** no solution

17. (76) **Geometry** An isosceles triangle with base angles of 45° and base measuring $x + 1$ has the same area as a square with sides x. What integer value of x is a solution to this problem?

***18.** (87) **Multi-Step** **a.** Use the change of base formula to convert $\log_2(6x)^2$ to base e.

b. Evaluate when $x = 1$.

19. (77) **Error Analysis** Explain and correct the error a student made in using Heron's formula to find the area of a triangle with side lengths of 3, 10, and 11.

$$A = \sqrt{24(21)(14)(13)} \approx 302.9 \text{ sq. units}$$

Identify the period and amplitude.

***20.** (82) $y = \frac{1}{2}\cos(2x)$

***21.** (82) $y = \frac{2}{3}\cos 4x$

22. (78) **Estimate** Without using a calculator, estimate the root(s) of the equation $x^2 - 62 = 0$ to the nearest integer(s). Explain your answer by discussing x-intercepts.

***23.** (87) Use the change of base formula to solve $\log_6 \frac{24}{5}$.

24. (79) **Delivery Charges** An appliance store delivers up to 4 items for a $50 delivery charge. There is an additional charge of $15 per item for each additional item delivered. Write a piecewise function for the cost having x items delivered.

*25. (86) Describe the phase shift of the function $y = 2 \sin\left(x - \frac{\pi}{2}\right) + 7$

26. (80) **Multi-Step** IQ test scores are often adjusted to have a mean of 100 and a standard deviation of 15.

a. What percent of people have an IQ between 70 and 130?

b. Suppose a person scores 150 on an IQ test. How many standard deviations is their score above the mean?

27. (82) **Radio** A radio signal generates the following data. Use the data to model the periodic function.

x	0	$\frac{\pi}{4}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$
y	0	−0.707	0	1

*28. (83) **Graphing Calculator** Write a quadratic equation with roots of −2.58 and 1.002. Check your work by graphing the related function and finding the zeros.

29. (70) **Error Analysis** Describe and correct the error in solving the equation.

$$(x + 7)^{\frac{1}{2}} = 5$$
$$[(x + 7)^{\frac{1}{2}}]^2 = 5$$
$$x + 7 = 5$$
$$x = -2$$

*30. (88) **Statistics** In a normal distribution a z-score is the number of standard deviations a particular data value lies above or below the mean. In a normal distribution with mean $\bar{x}$ and standard deviation σ, the z-score of a data value x is given by $z = \frac{x - \bar{x}}{\sigma}$.

a. Solve the formula for x.

b. A data value in a certain normal distribution is 2 standard deviations below the mean of 76. The standard deviation is 5.5. What is the data value?

LESSON 89

Solving Quadratic Inequalities

Warm Up

1. (39) **Vocabulary** The _____ of the graph of an inequality is dashed for $>$ and $<$ and solid for $\geq$ and $\leq$.

2. (39) Is $(0, 5)$ a solution of $y < 2x + 5$?

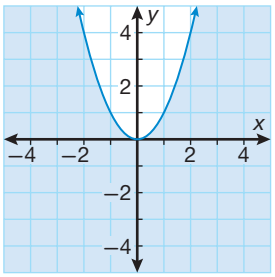
3. (65) Solve $24x^2 - 14x = 3$.

New Concepts

A **quadratic inequality in two variables** is an inequality that can be written in the standard form of $y < ax^2 + bx + c$, where $a \neq 0$, and $<$ can be replaced with $>$, $\leq$, or $\geq$.

Graphing Quadratic Inequalities

The graph of a quadratic inequality divides the plane into a region that contains the solutions and a region that does not contain the solutions. In a quadratic inequality, the boundary is a parabola. The curve is dashed if it does not contain solutions ($<$ or $>$) and solid if it does contain solutions ($\leq$ or $\geq$).



Example 1 Graphing Quadratic Inequalities

Graph $y > -2x^2 + 12x - 16$.

SOLUTION Graph $y = -2x^2 + 12x - 16$ with a dashed line and then shade above the parabola.

Step 1: Find the x-coordinate of the vertex point: $\frac{-b}{2a} = \frac{-12}{2(-2)} = 3$.
Find the y-coordinate of the vertex point: $y = -2(3)^2 + 12(3) - 16 = 2$.
Plot the vertex $(3, 2)$.

Step 2: Plot the y-intercept $(0, c)$: $(0, -16)$.

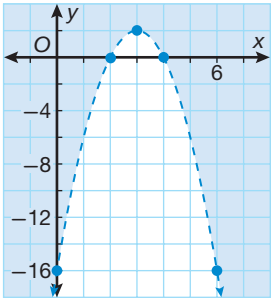
Step 3: Find the x-intercepts:

$0 = -2x^2 + 12x - 16$

$0 = -2(x^2 - 6x + 8)$

$0 = -2(x - 2)(x - 4)$

The x-intercepts are 2 and 4.



Step 4: Shade above the parabola because in standard form, the symbol is $>$.

Check the test point $(0, 0)$: $y \stackrel{?}{>} -2x^2 + 12x - 16$

$0 > -16$ **True**

Math Reasoning

Justify How do you know $(6, -16)$ is a point on the parabola?

Online Connection
www.SaxonMathResources.com

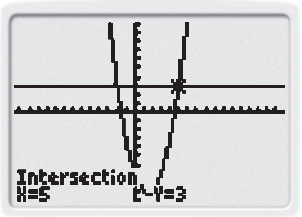
A **quadratic inequality in one variable,** $ax^2 + bx + c < d$, where $a \neq 0$, is an inequality that can be formed by replacing y with d and $<$ with $\leq$, $>$, or $\geq$. It can be solved either graphically or algebraically.

Example 2 Solving Quadratic Inequalities with a Graphing Calculator

Solve each inequality with a graphing calculator.

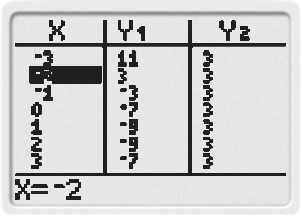
a. $x^2 - 3x - 7 \leq 3$

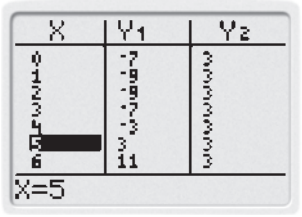
SOLUTION Graph Y1 $= x^2 - 3x - 7$ and Y2 $= 3$. The solutions are those x-values for which the parabola is at or below the line $y = 3$. To find these x-values, find the points of intersection.



The graph of Y1 $= x^2 - 3x - 7$ equals 3 when $x = -2$ and $x = 5$. The graph is below $x = 3$ for all the x-values between -2 and 5. The solution set can be written as $-2 \leq x \leq 5$.

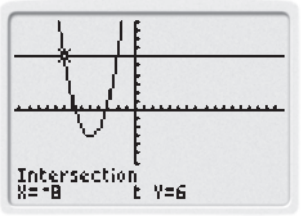
Use the Table function as a check. Look for where Y1 $\leq$ Y2.





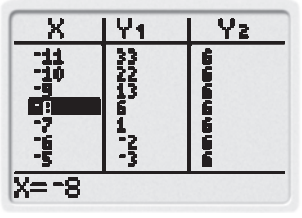
b. $x^2 + 10x + 22 > 6$

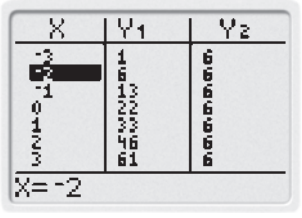
SOLUTION Graph Y1 $= x^2 + 10x + 22$ and Y2 $= 6$. The solutions are those x-values for which the parabola is above the line. Find the points of intersection.



The graph of Y1 $= x^2 + 10x + 22$ equals 6 when $x = -8$ and $x = -2$. The graph is above $x = 6$ for all the x-values less than -8 or greater than -2. The solution set can be written as $x < -8$ or $x > -2$.

Use the Table function as a check. Look for where Y1 $>$ Y2.

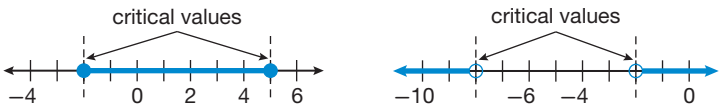




Math Reasoning

Analyze Give an example of a quadratic inequality with one critical point.

The solutions in Example 2 can be graphed on a number line. Notice that in both cases, there are two values called *critical points* or *critical values*, which separate the number line into three regions which either contain or do not contain solutions.



Example 3 Solving Quadratic Inequalities Algebraically

a. Solve $x^2 + x > 20$ algebraically.

SOLUTION Write the related equation in standard form and solve for x to find the critical values.

$$x^2 + x = 20$$
$$x^2 + x - 20 = 0$$
$$(x - 4)(x + 5) = 0$$

The critical values are 4 and -5. They separate the number line into the three regions, $x < -5$, $-5 < x < 4$, and $x > 4$. Test a point in each region to see if that region does or does not contain solutions.

Interval	Test Value	Is the test value a solution?
$x < -5$	-6	$(-6)^2 + -6 \stackrel{?}{>} 20$ $30 > 20$ Yes
$-5 < x < 4$	0	$(0)^2 + 0 \stackrel{?}{>} 20$ $0 > 20$ No
$x > 4$	5	$(5)^2 + 5 \stackrel{?}{>} 20$ $30 > 20$ Yes

The two "outer regions" are solutions.

The solution is $x < -5$ or $x > 4$.

Hint

Remember that *and* statements are intersections and *or* statements are unions.

b. Solve $x^2 - 3x - 18 \leq 0$ algebraically.

SOLUTION Write the related equation and solve for x to find the critical values.

$$x^2 - 3x - 18 = 0$$
$$(x + 3)(x - 6) = 0$$

The critical values are -3 and 6. They separate the number line into the three regions, $x \leq -3$, $-3 \leq x \leq 6$, and $x \geq 6$. Test a point in each region to see if that region does or does not contain solutions.

Interval	Test Value	Is the test value a solution?
$x \leq -3$	-4	$(-4)^2 - 3(-4) - 18 \stackrel{?}{\leq} 0$ $10 \leq 0$ No
$-3 \leq x \leq 6$	0	$(0)^2 - 3(0) - 18 \stackrel{?}{\leq} 0$ $-18 \leq 0$ Yes
$x \geq 6$	7	$(7)^2 - 3(7) - 18 \stackrel{?}{\leq} 0$ $10 \leq 0$ No

The "inner region" contains the solutions.

The solution is $-3 \leq x \leq 6$.

As with a system of linear inequalities, the solution of a system that includes quadratic inequalities is contained in the overlapping region.

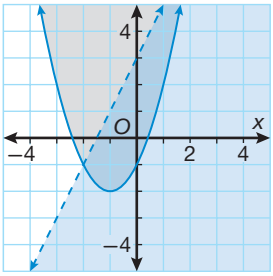
Math Reasoning

Verify Show algebraically that (1, 2) is a solution.

Example 4 Solving Systems that Include Quadratic Inequalities

Solve $\begin{matrix} y \geq x^2 + 2x - 1 \\ y < 2x + 3 \end{matrix}$ by graphing. Name one point that is a solution and one that is not.

SOLUTION Graph each inequality on the same coordinate plane.



The solutions are contained in the overlapping region.

(0, 0) is a solution.

(−3, 1) is not a solution.

Example 5 Application: Business

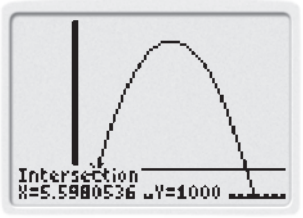
The profit per trip, in dollars, for an adventure company offering tours is modeled by $y = -14x^2 + 650x - 2200$, where x is the number of people on the tour. Trips are cancelled if a minimum of \$1000 cannot be made in profit. How many people must be on a trip for the company to earn \$1000 or more?

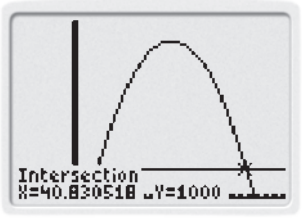
SOLUTION

1. Understand The answer is the solution of $-14x^2 + 650x - 2200 \geq 1000$.

2. Plan Graph Y1 $= -14x^2 + 650x - 2200$ and Y2 $= 1000$. Find the x-values for which the first graph has greater y-values than the second.

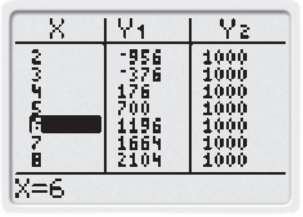
3. Solve Round the x-values to whole numbers because they represent numbers of people. Think: If about 5.6 is the solution, there must be greater than 5 people. If 40.8 is a solution, there must be less than 41 people.

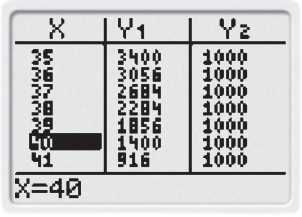




There must be between 6 and 40 people inclusivey.

4. Check Use the Table function to see the actual amounts of profit earned for each number of people.





Lesson Practice

a. (Ex 1) Graph $y \geq x^2 + 8x + 7$.

Solve each inequality with a graphing calculator.

b. (Ex 2) $-x^2 + 2x + 11 < -4$

c. (Ex 2) $14 \geq x^2 - 8x + 26$

d. (Ex 3) Solve $x^2 + 8x < 9$ algebraically.

e. (Ex 3) Solve $2x^2 - 9x - 1 \geq 4$ algebraically.

f. (Ex 4) Solve $\begin{matrix} y < x^2 - 3 \\ y \geq 2x^2 - 6 \end{matrix}$ by graphing. Name one point that is a solution and one that is not.

g. (Ex 4) The profit per trip, in dollars, for an adventure company offering tours is modeled by $y = -16x^2 + 720x - 1950$, where x is the number of people on the tour. How many people must be on a trip for the company to earn at least \$3000 in profit?

Practice Distributed and Integrated

1. (67) Evaluate the inverse trigonometric function $\text{Tan}^{-1}\left(\sqrt{3}\right)$. Give your answer in both radians and degrees.

Solve algebraically.

***2.** (89) $3x^2 - 13x > 10$

***3.** (89) $5x^2 - 6x - 2 < 0$

4. (63) Find the arc length and area of a sector with the given radius r and central angle θ. $r = 12$ ft, $\theta = 150°$

5. (68) A prize box contains 5 stickers, 13 erasers, 20 pencils and 2 highlighters. Find the probability that a student randomly chooses a sticker and a pencil when choosing 2 prizes.

6. (83) **Error Analysis** Explain and correct the error a student made in writing a quadratic equation with a root of 9. $\begin{matrix} x = 9 \\ x - 9 = 0 \end{matrix}$

7. (79) **Write** Explain how to write a piecewise function rule from a graph.

8. (74) **Multiple Choice** A quadratic equation has 1 real solution. Which could be the discriminant of the equation?

A -6 **B** -4 **C** 0 **D** 4

***9.** (86) **Formulate** Consider the function $f(x) = \cos(x - \theta)$. For what value of θ is the graph of $f(x)$ the same as the graph of $y = \sin(x)$?

10. (75) Graph the cube root function and its inverse. $y = -2\sqrt[3]{x + 5} + 5$

***11.** (89) **Justify** Explain why the solution of $x^2 - 4x + 7 > -5$ is the set of all real numbers.

12. (76) **Physics** Acceleration due to gravity is different on different planets. The equation below can be used to find the distance from the ground at time t for a ball thrown from a height h_o at an initial speed of v_o. Use the table to determine how long it will take a ball thrown with an initial speed of 35 at a height of 100 to hit the ground on Mercury.

$$h = -\frac{1}{2}at^2 + v_o t + h_o$$

Planet	Acceleration (a)
Mercury	3.59
Venus	8.87
Mars	3.77
Jupiter	25.95
Saturn	11.08
Uranus	10.67
Neptune	14.07

***13.** (88) **Analyze** Solve the equation $A = \pi r^2$ for r. Determine whether the solutions represent possible values for the radius of a circle. Explain.

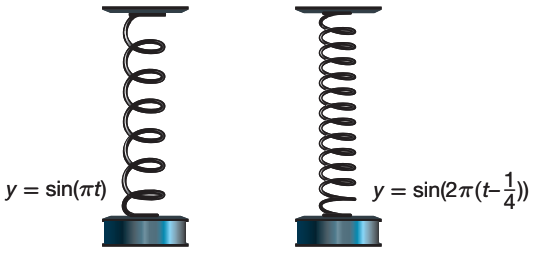
14. (77) **Geometry** Three sides of a triangle measure 46 inches, 22 inches, and 50 inches. Classify the triangle by its side and angle measures.

15. (80) **Multiple Choice** The weights of a group of children are normally distributed with a mean of 68 pounds and a standard deviation of 3 pounds. What is the range of the middle 95% of the weights?

A 59 lbs to 77 lbs **B** 62 lbs to 74 lbs **C** 65 lbs to 71 lbs **D** 68 lbs to 77 lbs

16. (71) Solve $\triangle MAP$ given that $m\angle A = 12°$, $m\angle P = 93°$, and $m = 16$.

17. (82) **Physics** Two springs have two different weights attached at the ends. The movement of each weight is modeled by the periodic functions shown, where t is the time in seconds. How often will the weights be at the top of their movement at the same time?



18. (84) **Multi-Step** Worker A can paint a room in 3 hours and Worker B can paint the same size room in 4 hours.

a. If it takes x hours to complete a job, the work rate is $\frac{1}{x}$ because $\frac{1}{x}$ of the job can be completed in one hour. Write a work rate for each worker.

b. Workers A and B worked together and painted a room (same size as above) in h hours. Write expressions to show the amount of work done by each worker.

c. Add the expressions in part b together and set them equal to 1 to represent one room. Solve the equation for h. How long did it take the workers to paint the room working together?

19. (81) **Economics** The formula for continuously compounded interest is $A = Pe^{rt}$, where A is the total amount, P is the principal, r is the annual interest rate, and t is the time in years. To the nearest tenth, how many years would it take for the value of an investment to double at an annual interest rate of 6%, compounded continuously?

20. (57) A savings account earns interest at an annual rate of 4.5%, compounded continuously. If the account begins with a principal amount of $3000, what will its value be after 5 years?

21. (85) **Error Analysis** Two students are solving for the roots of $y = x^{12} - 1$ but get different results. Which student made the mistake?

Student A	Student B
$x^{12} - 1 = (x^4)^3 - 1$ $= (x^4 - 1)((x^4)^2 + x^4 + 1)$ $= (x^2 + 1)(x^2 - 1)(x^8 + x^4 + 1)$ $= (x + 1)(x - 1)(x^2 + 1)(x^8 + x^4 + 1)$ The roots of $y = x^{12} - 1$ are ± 1.	$x^{12} - 1 = (x^4)^3 - 1$ $= (x - 1)((x^4)^2 + x^4 + 1)$ $= (x - 1)(x^8 + x^4 + 1)$ The root of $y = x^{12} - 1$ is 1.

*22. (87) **Data Analysis** The Ehrenberg relation $\log(w) = 0.8h + 0.4 \pm 0.04$ shows the relationship between height h and weight w in young children ages 5 through 13. Complete the table for the range of weight values.

h (m)	0.91	0.93	0.96	0.99	1.01
w (kg)					

23. (73) **Write** Tell if the sampling method performed was a probability sample. If not, explain why not. If so, state the type of probability sample.

A person running for local office asks drivers in a local gas/convenience station if they agree with the new gas tax being proposed. He asks every other driver who pulls into the station.

Solve.

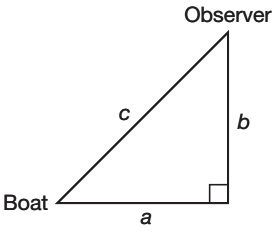
*24. (88) $5 - \sqrt{x + 7} = 3$

*25. (88) $\sqrt{x + 2} = 2 - \sqrt{x}$

*26. (89) **Graphing Calculator** Use a graphing calculator to estimate the solutions of $-23.4 < x^2 + 7.56x - 40$. Round to the nearest hundredth.

27. (78) **Vertical Motion and Gravity** The height of a free-falling object (neglecting air resistance) is given by the function $h(t) = -4.9t^2 + v_0t + h_0$, where h_0 is the initial height in meters, v_0 is the initial velocity in meters per second, and $h(t)$ is the height in meters at time t seconds. How long does it take for an object to hit the ground after it is dropped from a height of 44.1 meters?

*28. (88) **Cliff Height** The Cliffs of Moher at Hag's Head offer one of the most spectacular views on the Atlantic coast of Ireland. If a boat is 90 meters from the base of the cliff, then it is 150 meters from an observer at the top of the cliff. Solve the equation $a^2 + b^2 = c^2$ for b. Then find the height of the cliff.



29. (77) Solve $\triangle ABC$ given that $m\angle C = 114°$, $a = 8$, and $b = 6$.

30. (81) Rewrite $\ln\left(\frac{e^5}{x^7}\right)^4$ as a sum or difference of terms if possible. Then simplify.

LESSON
90

Graphing the Tangent Function

Warm Up

1. (86) **Vocabulary** The function $y = \cos(x)$ is a trigonometric function that is _____.

2. (86) What is the period of $y = 5\cos(3x)$?

3. (82) What is the period of $y = \sin(4x)$?

New Concepts

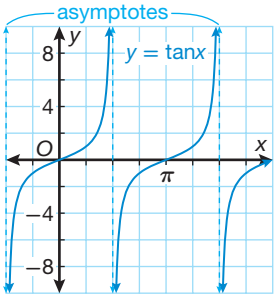
A tangent function is an example of a periodic function. It is based on the trigonometric ratio, $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Therefore when $\cos(x) = 0$, the tangent is undefined. These values are excluded from the domain and represented on the graph by asymptotes. The tangent has no maximum or minimum values so the amplitude is undefined.

Domain: $\{x | x \neq \frac{\pi}{2} + \pi n$ where n is an integer.$\}$

Range: $\{y | -\infty < y < \infty\}$

Period: π

Amplitude: undefined



Properties of Tangent Functions

If $y = a \tan bx$, with $b > 0$

- $\frac{\pi}{b}$ is the period of the function
- One cycle occurs in the interval from $-\frac{\pi}{2b}$ to $\frac{\pi}{2b}$
- Vertical asymptotes occur at each end of the cycle

To graph one cycle of a tangent function, use the asymptotes and three points. These five elements are equally spaced through one cycle. To find the x-coordinates of these points, divide the interval into fourths. The y-coordinates of the three points are $(-a)$, 0, and (a). The pattern of these points is asymptote-$(-a)$-zero-(a).

Math Reasoning

Verify Create a table of values to show that the period of $y = \tan(x)$ is π.

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Example 1 Graphing $y = a \cdot \tan(x)$

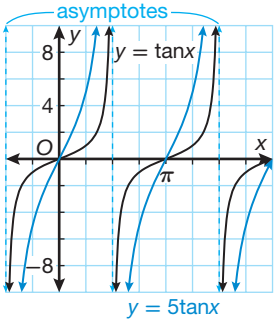
Graph $y = 5\tan(x)$.

SOLUTION The graph is a vertical stretch of the parent function.

Find the period of the function.

$$\frac{\pi}{b} = \frac{\pi}{1} = \pi$$

Find the interval of one cycle to locate asymptotes.



$$-\frac{\pi}{2b} = -\frac{\pi}{2(1)} = -\frac{\pi}{2} \qquad \frac{\pi}{2b} = \frac{\pi}{2(1)} = \frac{\pi}{2}$$

The asymptotes are $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

Determine the three points to use by dividing the interval into fourths to find the x-coordinates. The y-coordinates of the three points are $(-a)$, 0, and (a).

$$\left(-\frac{\pi}{4}, -5\right), (0, 0), \left(\frac{\pi}{4}, 5\right)$$

Use the pattern $(-a)$-zero-(a). Repeat the cycle.

A tangent function can be also be translated vertically. In the function $y = a \tan(bx) + d$, the value of d determines the shift. If $d > 0$, the graph shifts up and if $d < 0$, the graph shifts down. Similar to graphing translations of the sine and cosine functions, a simpler member of the tangent function family can be used.

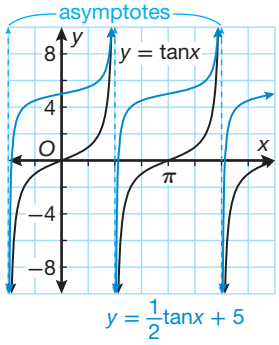
Hint

Recall that $f(x) + k$ is a vertical shift of $f(x)$.

Example 2 Graphing $y = a \cdot \tan(x) + d$

Graph the function $y = \frac{1}{2}\tan(x) + 5$. Determine its period.

SOLUTION



Use $y = \frac{1}{2}\tan(x)$ as the parent function.

Find the period of the function.

$$\frac{\pi}{1} = \frac{\pi}{1} = \pi$$

Find the interval of one cycle to locate asymptotes.

$$-\frac{\pi}{2b} = -\frac{\pi}{2(1)} = -\frac{\pi}{2} \qquad \frac{\pi}{2b} = \frac{\pi}{2(1)} = \frac{\pi}{2}$$

The asymptotes are $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

Determine the three points to use by dividing the interval into fourths. Since $d > 0$ and the graph is shifted up 5, add 5 to the value of the y-coordinates.

$$\left(-\frac{\pi}{4}, -\frac{1}{2} + 5\right), (0, 0 + 5), \left(\frac{\pi}{4}, \frac{1}{2} + 5\right)$$

The three points to graph are $\left(-\frac{\pi}{4}, -\frac{9}{2}\right)$, $(0, 5)$ $\left(\frac{\pi}{4}, \frac{11}{2}\right)$

Use the pattern $(-a)$-zero-(a). Repeat the cycle.

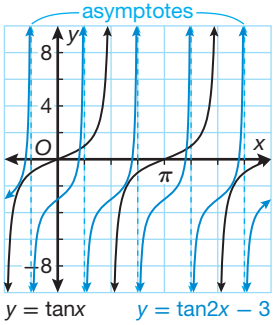
Hint

Be sure to use $\frac{\pi}{b}$ to find the period of a tangent function.

Example 3 Graphing $y = a \cdot \tan(bx) - d$

Graph $y = \tan(2x) - 3$. Identify its period and undefined values.

SOLUTION



Use $y = \tan(2x)$ as the parent function.

Find the period of the function.

$$\frac{\pi}{2} = \frac{\pi}{2}$$

Find the interval of one cycle to locate asymptotes.

$$-\frac{\pi}{2b} = -\frac{\pi}{2(2)} = -\frac{\pi}{4} \qquad \frac{\pi}{2b} = \frac{\pi}{2(2)} = \frac{\pi}{4}$$

The asymptotes are $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$.

Determine the three points to use by dividing the interval into fourths. Since $d < 0$, the graph is shifted down, add -5 to the value of $(-a)$ and (a).

$$\left(-\frac{\pi}{8}, -1 + (-5)\right), (0, 0 + (-5)), \left(\frac{\pi}{8}, 1 + (-5)\right)$$

The three points to graph are $\left(-\frac{\pi}{8}, -6\right)$, $(0, -5)$ $\left(\frac{\pi}{8}, 4\right)$

Use the pattern $(-a)$-zero-(a). Repeat the cycle.

Example 4 Graphing $y = a\tan(b(x - c)) + d$

Graph $y = \tan\left(x - \frac{\pi}{2}\right)$. Identify its period, undefined values, and phase shift.

Hint

The undefined values of the parent tangent function are $x = \frac{\pi}{2} + \pi n$.

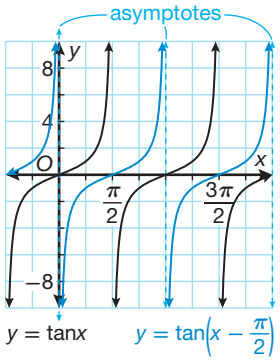
SOLUTION

The graph is a horizontal shift or phase shift of $\frac{\pi}{2}$.

Period: $\frac{\pi}{1} = \pi$

Phase shift: $\frac{\pi}{2}$

Undefined values: $\frac{\pi}{2} + \frac{\pi}{2} + \pi n = \pi n$



Lesson Practice

a. (Ex 1) Graph $y = 10\tan(x)$.

b. (Ex 2) Graph the function $y = 8\tan(x) + 4$. Determine its period.

c. (Ex 3) Graph $y = \tan(6x)$ and compare its period to that of the parent function. Identify values where the function is undefined.

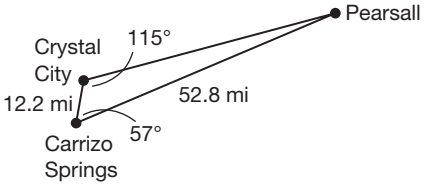
d. (Ex 3) Graph $y = \tan\left(\frac{1}{4}x\right)$ and compare its period to that of the parent function. Identify values where the function is undefined.

e. (Ex 3) Graph $y = \tan(7x) + 5$. Identify its period and undefined values.

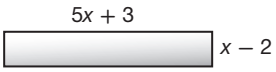
f. (Ex 4) Graph $y = 2\tan\left(x - \frac{\pi}{3}\right)$. Identify its period, undefined values, and phase shift.

Practice Distributed and Integrated

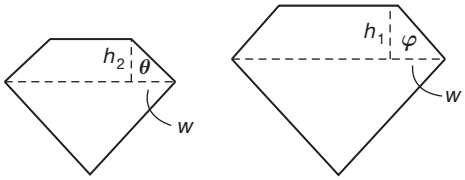
1. (71) **Geography** On a map, line segments connecting three southeast Texas cities form a triangle as shown. About how many square miles are within the triangle?



2. (85) **Construction** The sheet of metal shown is rolled into a cylinder along the longer side. The area of the sheet is 6 square units. Find the radius of the cylinder.



***3.** (90) **Jewelry Making** A jeweler is changing the angle of a facet for a diamond ring from θ to φ. The vertical distances change as shown in the diagram, while the value of w remains constant. Derive the function that the jeweler can use to find values of θ when the other values are known.



4. (55) A bin has 3 apples and 5 oranges. What is the probability of selecting two apples chosen at random without replacement?

Write each expression as a radical.

5. (59) $13^{\frac{3}{11}}$

6. (59) $22^{\frac{5}{9}}$

7. (61) **Geometry** The volume of a rectangular solid with a length of $x + 10$ feet has a volume of $x^3 + 18x^2 + 95x + 150$ cubic feet. Factor the expression for the volume completely.

8. (66) Solve for the roots of the polynomial: $7x(x^2 + 4x - 12) = 0$.

9. (72) **Geology** The Richter magnitude M of an earthquake is expressed by $M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$, where E is the energy released in ergs. In 2007 an earthquake of magnitude 8.0 occurred near the coast of Central Peru. Find the energy released by that earthquake.

10. (73) **Error Analysis** A student said that a systematic sampling method is biased because the first individual on the list will always be chosen. Explain the error in the student's thinking.

11. (74) **Soccer** A player standing on a 15-foot hill kicks a ball straight up at a velocity of 45 feet per second. She wants it to go above the roof of a nearby shed which is about 40 feet above the ground. The function that gives the height of the ball after x seconds is given by $y = -16x^2 + 45x + 15$. Use the discriminant to explain whether the ball will/will not reach the desired height.

12. (73) **Analyze** Why is systematic sampling a probability sample but not a simple random sample?

Solve.

13. $2x^4 - 512 = 0$ (78)

14. $3x^2 + 4x - 4 = 0$ (78)

15. **Multi-Step** The ages of a certain group of people are normally distributed with a mean of 42 years and a standard deviation of 6.5 years. What percent of the ages are between 29 and 55? How you can use that answer to find the percent of scores that are between 42 and 55? Find that percent. (80)

16. **Justify** Explain why $x = e^y$ and $y = \ln x$ are equivalent equations. (81)

17. Using $f(x) = \sin(x)$ as a guide, find the phase shift in $h(x) = \sin\left(x + \frac{\pi}{4}\right)$. (86)

18. **Generalize** Given a quadratic equation, how can you write another quadratic equation with the same roots? (83)

19. Gene can paint a room in 8 hours and Chandra and Gene working together can paint the room in 5 hours. How long would it take Chandra to paint the room by herself? (84)

***20.** **Analyze** Is $y = \tan(\theta)$ an odd or an even function? Explain. (90)

***21.** **Coordinate Geometry** Write the version of the slope formula that can be used to find the tangent function. (90)

***22.** **Write** Describe how to solve a quadratic inequality algebraically. (89)

***23.** **Multiple Choice** Solve $A = \frac{1}{2}bh$ for b. Which equation shows the correct solution? (88)

A $b = \frac{2A}{h}$ **B** $b = \frac{A}{2h}$ **C** $b = A - \frac{1}{2}h$ **D** $b = \frac{1}{2}Ah$

***24.** Use the change of base formula to convert $\log_2(x)^3$ to base e. Then evaluate when $x = 1$. (87)

***25.** **Graphing Calculator** A portion of a roller coaster is modeled by the polynomial function $y = 0.000013x^4 - 0.00255x^3 + 0.13505x^2 - 0.51x + 0.5$. Use a graphing calculator to find the roots of the polynomial. (85)

Solve using a table.

***26.** $x^2 - 2x \geq 8$ (89)

***27.** $x^2 - 15x + 50 < 0$ (89)

***28.** Find the period of $y = \cos\left(\frac{1}{\sqrt{5} - 1}x\right)$. (86)

29. Given $q(x) = -\sin\left(\frac{\pi}{2}x\right)$, identify the amplitude and period. (82)

30. Write a quadratic equation whose roots are $8 \pm 5i$. (83)

INVESTIGATION

9

Understanding Step Functions

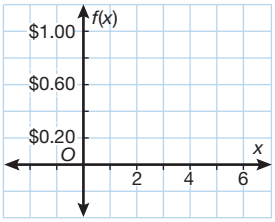
Cell Phone Rates A cell phone company charges customers for each phone call that they make. A call costs \$0.50 for a connection fee plus \$0.10 for each minute of the call. The company rounds the time of the call down to the nearest whole minute.

Graphing Calculator

For help with graphing systems, see the graphing calculator keystrokes on page 19.

1. Find the cost of each phone call:
   - **a.** a 5-minute call
   - **b.** a $5\frac{1}{4}$-minute call
   - **c.** a $\frac{1}{4}$-minute call
2. **Justify** If a phone call lasts for 30 seconds, would the customer be charged for the phone call? Explain.
3. **Analyze** Consider the costs of a 5-minute and a $5\frac{1}{4}$-minute phone call. Would a phone call that lasted $5\frac{1}{4}$ minutes cost more than a 5-minute call? Explain.
4. **Generalize** Complete the table below. Then graph the function as a piecewise function.

x	$f(x)$
$0 < x < 1$	\$0.50
$1 \leq x < 2$	
$2 \leq x < 3$	
$3 \leq x < 4$	
$4 \leq x < 5$	
$5 \leq x < 6$	



Math Reasoning

Generalize What type of function do the values in the table represent?

A step function is a piecewise function that is constant over each interval in its domain. Step functions consist of line segments, rays, or both that resemble steps of a staircase.

One type of step function is the greatest integer function. The following are ways in which the simplest greatest integer function can be written:

$$f(x) = [x] \qquad f(x) = [\![x]\!] \qquad f(x) = \lfloor x \rfloor$$

The cell phone rate problem is represented by a greatest integer function because it rounds the number of minutes down to the nearest integer. The **greatest integer function**, $f(x) = [x]$, is a function where the output, $f(x)$, is the greatest integer that is either less than or equal to the input, x.

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5. Suppose the company uses the exact number of minutes, x, to calculate the total cost of a phone call.

   a. **Multiple Representations** Write a linear function to represent the cost, $f(x)$, of a phone call.

   b. **Discuss** How can the equation be changed to represent the cost of a phone call when the company rounds the length of time down to the nearest minute?

   c. Write the equation to represent the cost, $f(x)$, of a phone call when the company rounds the length of time, x, down to the nearest minute.

   d. **Graphing Calculator** Use a graphing calculator to graph the equation in part **c.**

Another type of step function is the least integer function. The least integer function rounds any number up to the nearest integer. The simplest form of this equation looks like:

$$f(x) = \lceil x \rceil.$$

The least integer function is a function where the output, $f(x)$, is the least integer that is either greater than or equal to the input, x. Use this definition in the following problem.

Calculator Tip

To graph greatest integer functions, click on the **MATH** key and select the **NUM** menu. Then, select ⌈**5: int(**⌉.

Parking Lot Customers are charged by the hour to park in a lot. The length of time that a car is parked is rounded up to the nearest hour. The function for the cost for parking is $f(x) = 6\lceil x \rceil$ where x is the time in hours and $f(x)$ is the cost of parking in dollars.

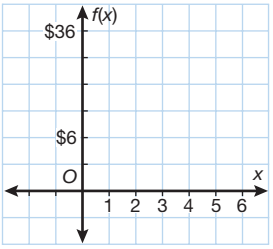
6. Complete the table by filling in the intervals of time, x, and the cost, $f(x)$.

   a. Draw a graph to represent the information in the table.

   b. Use the graph. What is the cost of parking in the lot for 2 hours? 1.5 hours? less than 1 hour?

   c. Use the table. What is the charge per hour of parking?

x	$f(x)$
$0 < x \leq 1$	\$6
$1 < x \leq 2$	
$5 < x \leq 6$	\$36



Investigation Practice

A grocery store charges \$0.03 per ounce for bananas. The store weighs the fruit on a produce scale, and then rounds the weight of the produce up to the nearest ounce.

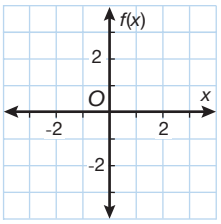
a. Write a function to represent the total cost for x ounces of bananas.

b. What is the domain for this function?

c. Which step function is this? Explain.

d. Complete the table for the following step function. Use the information from your table to graph the function.

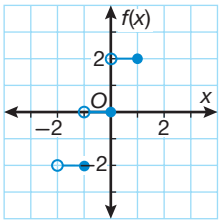
x	$f(x)$
$0 < x \leq 1$	
$1 < x \leq 2$	
$2 < x \leq 3$	
$3 < x \leq 4$	
$4 < x \leq 5$	
$5 < x \leq 6$	



Formulate Below is a graph of a step function.

e. Complete the table, and then write a function that represents the graph.

f. What type of step function is represented by the graph?



x	$f(x)$
$-1 < x \leq 0$	
	2

LAB
13

Graphing a Circle and a Polar Equation

Graphing Calculator Lab (*Use with Lessons 91, 96*)

Graphing a Circle in Function Mode

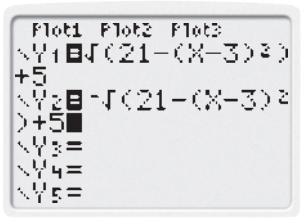
Graphing Calculator

Refer to Calculator Lab 1 on page 19 for graphing a function.

1. Enter the equation of the circle $(x - 3)^2 + (y - 5)^2 = 21$ as the two functions, $y = \pm\sqrt{21 - (x - 3)^2} + 5$.

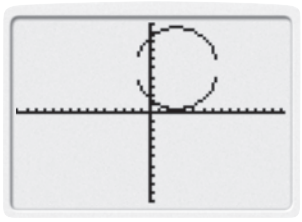
   a. Press Y= to access the Y= equation editor screen.

   b. Type $\sqrt{21 - (x - 3)^2} + 5$ into Y_1 and $-\sqrt{21 - (x - 3)^2} + 5$ into Y_2



   c. Press GRAPH to graph the functions.

   d. Use the ZOOM or WINDOW button to adjust the window. The graph will have a more circular shape using ZOOM 5 rather than ZOOM 6.



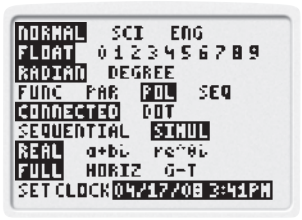
Graphing a Polar Equation

2. Change the mode of the calculator to polar mode.

   a. Press MODE. Press ▼ two times to highlight **RADIAN,** and then press enter.

   b. Press ▼ to **FUNC,** and then ▶ two times to highlight **POL,** and then press ENTER.

   c. Press ▼ two more times to **SEQUENTIAL,** press ▶ to highlight **SIMUL,** and then press ENTER.

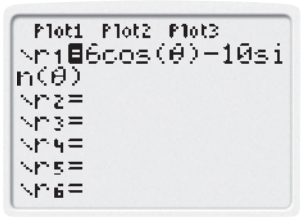


   d. To exit the **MODE** menu, press 2nd MODE.

3. Enter the equation of the circle $(x - 3)^2 + (y - 5)^2 = 34$ as the polar equation $r = 6\cos\theta - 10\sin\theta$.

   a. Press Y= to access the $r_1 =$ equation editor screen.

   b. Type in $6\cos\theta - 10\sin\theta$ into r_1 by pressing 6 COS X,T,θ,n) − 1 0 SIN X,T,θ,n).



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Graphing Calculator

When graphing an equation in **DEGREE** mode, set the θmin value to 0, the θmax value to 360, and the θstep to 1 to graph the entire equation with points at every degree.

4. Adjust the viewing window to polar values.

a. Press WINDOW and type in the following values:

$\theta\text{min} = 0$	$\text{Xmin} = -16$	$\text{Ymin} = -15$
$\theta\text{max} = 2\pi$	$\text{Xmax} = 21$	$\text{Ymax} = 10$
$\theta\text{step} = \pi/16$	$\text{Xscl} = 1$	$\text{Yscl} = 1$

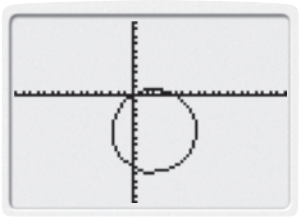
A similar result may be obtained by pressing ZOOM 5.

b. The θmin and θmax values determine the minimum and maximum input for the equation. The θstep value determines the increments of the input.

Graphing Calculator

The points on the graph are formed by the θstep values. Then, the points are connected with lines. For curved graphs choose small θstep values so that a smoother curve is produced.

5. Press GRAPH to graph the equation.



Lab Practice

a. Graph the equation of the circle $(x + 2)^2 + (y - 7)^2 = 14$ in function mode.

b. Graph the equation of the limacon $r = 1 + 3 \cos \theta$ using the given viewing window.

$\theta\text{min} = 0$	$\text{Xmin} = -2$	$\text{Ymin} = -3$
$\theta\text{max} = 2\pi$	$\text{Xmax} = 5$	$\text{Ymax} = 3$
$\theta\text{step} = \pi/16$	$\text{Xscl} = 1$	$\text{Yscl} = 1$

LESSON
91

Making Graphs and Solving Equations of Circles

Warm Up

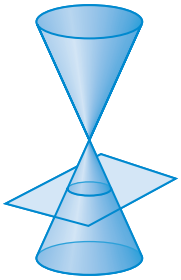
1. (63) **Vocabulary** The _____ of a circle is the distance from the center of the circle to any point on the circle.

2. (41) Find the length of the segment with endpoints at (3, 5) and (−10, 7).

3. (63) True or False: The unit circle passes through the point (1, 1).

New Concepts

A **conic section** is a plane figure formed by the intersection of a double right cone and a plane. The circle is one of the four possible conic sections.



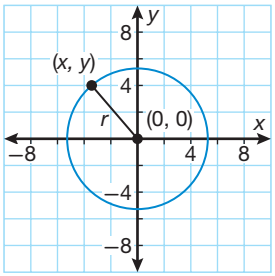
Math Language

A circle is formed by the intersection of right cone and a plane that is parallel to the base of the cone, except where the intersection is at the vertex of the cone.

A circle is not a function because it does not pass the vertical line test. If the center of the circle is located at (0, 0), then, using the Distance Formula, the radius of the circle is given by $r = \sqrt{(x-0)^2 + (y-0)^2}$. Simplifying the radicand and squaring both sides gives $r^2 = x^2 + y^2$.

The equation of a circle with center (0, 0) and radius r is

$$x^2 + y^2 = r^2.$$

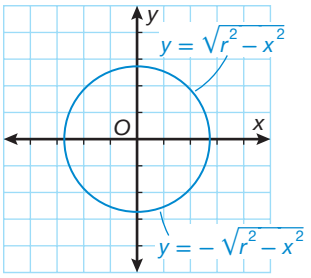


Math Reasoning

Justify A parabola is a conic section. How can a parabola be formed from a plane and a double-right cone?

The equation of a circle must be transformed into two functions in order to graph it on a graphing calculator in function mode. Isolate y and then enter the positive and negative square roots into the equation editor as two functions. The graphs of the two functions are semicircles that, together, form a circle.

$$x^2 + y^2 = r^2$$
$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$



Example 1 Graphing Circles Centered at the Origin

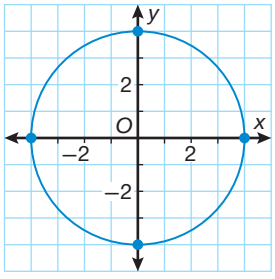
a. Sketch the graph of $x^2 + y^2 = 16$.

SOLUTION 16 equals the value of the radius squared, so the radius is 4 units.

Step 1: Plot the center at (0, 0).

Step 2: Since the radius is 4, plot the points 4 units above, below, left, and right of the center.

Step 3: Sketch a circle that passes through the four points.



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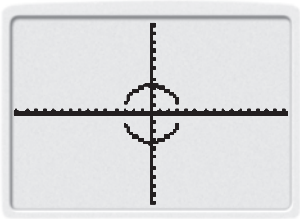
Graphing Calculator

To keep the circle from looking distorted, press **ZOOM** and choose **ZSquare.**

b. Graph $x^2 + y^2 = 10$ on a graphing calculator.

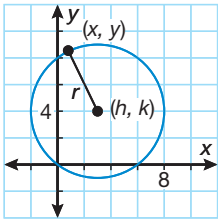
SOLUTION Solve for y. Graph each of two resulting functions.

$$x^2 + y^2 = 10$$
$$y^2 = 10 - x^2$$
$$y = \pm\sqrt{10 - x^2}$$



The center of the circle shown at right is located at (h, k).

Using the Distance Formula, the radius of the circle is given by $r = \sqrt{(x-h)^2 + (y-k)^2}$. Squaring both sides gives $r^2 = (x-h)^2 + (y-k)^2$.



Standard Form of an Equation of a Circle

The equation of a circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2.$$

In order to graph a circle, the center and the radius of the circle must be determined.

Example 2 Graphing Circles Not Centered at the Origin

Hint

Be careful choosing the values of h and k, as the formula involves subtraction. Think:

$(x + 2)^2 = (x - (-2))^2$.

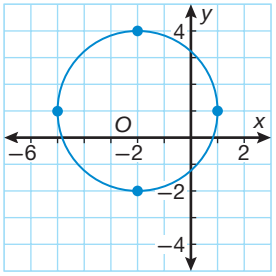
a. Sketch the graph of $(x + 2)^2 + (y - 1)^2 = 9$.

SOLUTION The radius is $\sqrt{9}$, or 3.

Step 1: Plot the center at $(-2, 1)$.

Step 2: Since the radius is 3, plot the points 3 units above, below, left, and right of the center.

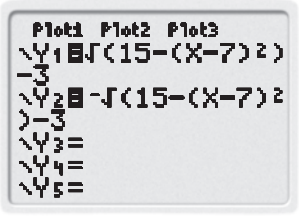
Step 3: Sketch a circle that passes through these four points.

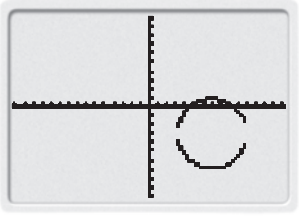


b. Graph $(x - 7)^2 + (y + 3)^2 = 15$ on a graphing calculator.

SOLUTION Solve for y. Graph each of the two resulting functions.

$$(y + 3)^2 = 15 - (x - 7)^2$$
$$y + 3 = \pm\sqrt{15 - (x-7)^2}$$
$$y = \pm\sqrt{15 - (x-7)^2} - 3$$





Sometimes the center and the radius of a circle are not explicitly given, so it is necessary to use the Distance and/or Midpoint Formula to determine these unknown parts.

For a segment whose endpoints are at (x_1, y_1) and (x_2, y_2),

Distance Formula	Midpoint Formula
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Example 3 Writing the Equation of a Circle

a. Write the equation of the circle with center $(-3, -1)$ and radius 7.

SOLUTION Substitute $h = -3$, $k = -1$, and $r = 7$ into $(x - h)^2 + (y - k)^2 = r^2$.

$$(x - (-3))^2 + (y - (-1))^2 = 7^2$$
$$(x + 3)^2 + (y + 1)^2 = 49$$

b. Write the equation of the circle with center $(-2, 4)$ that contains the point $(5, 2)$.

SOLUTION Find the length of the radius by using the Distance Formula.

$$r = \sqrt{(5 - (-2))^2 + (2 - 4)^2}$$
$$= \sqrt{7^2 + (-2)^2}$$
$$= \sqrt{53}$$

Substitute $h = -2$, $k = 4$, and $r = \sqrt{53}$ into $(x - h)^2 + (y - k)^2 = r^2$.

$$(x - (-2))^2 + (y - 4)^2 = \left(\sqrt{53}\right)^2$$
$$(x + 2)^2 + (y - 4)^2 = 53$$

c. Write the equation of the circle that has a diameter whose endpoints are located at $(3, 1)$ and $(6, 3)$.

SOLUTION The center of the circle is midway between the endpoints of a diameter. Use the Midpoint Formula.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{3 + 6}{2}, \frac{1 + 3}{2}\right) = (4.5, 2)$$

Find the distance between the center and either of the points on the circle.

$$r = \sqrt{(3 - 4.5)^2 + (1 - 2)^2}$$
$$= \sqrt{3.25}$$

Substitute $h = 4.5$, $k = 2$, and $r = \sqrt{3.25}$ into $(x - h)^2 + (y - k)^2 = r^2$.

$$(x - 4.5)^2 + (y - 2)^2 = \left(\sqrt{3.25}\right)^2$$
$$(x - 4.5)^2 + (y - 2)^2 = 3.25$$

Hint

The radius will always be greater than 0. A radius less than or equal to 0 indicates an error.

Hint

$C = \pi d$ can also be written as $C = 2\pi r$.

Example 4 Application: Astronomy

The orbit of Earth around the Sun resembles a circle. Given that Earth travels about 584.3 million miles in one orbit, write an equation for the circle that models Earth's orbit around the Sun.

SOLUTION The distance Earth travels in one orbit is the circumference of the circle, and $C = \pi d$. Let the Sun be at (0, 0) on the coordinate plane, where a unit represents one million miles. Find the diameter and divide by 2 to find the radius.

$$C = \pi d$$

$$584.3 \approx \pi d \qquad \text{Substitute 584.3 for } C.$$

$$185.988 \approx d \qquad \text{Divide both sides by } \pi.$$

Divide the diameter by 2 to find the radius: $r \approx 92.994 \approx 93$.
Write the equation of the circle. Substitute $r = 93$ in $x^2 + y^2 = r^2$.

$$x^2 + y^2 = 93^2 \rightarrow x^2 + y^2 = 8649$$

The equation $x^2 + y^2 = 8649$ models Earth's orbit around the Sun, where the Sun is located at (0, 0).

Lesson Practice

(Ex 1) **a.** Sketch the graph of $x^2 + y^2 = 36$.

(Ex 1) **b.** Graph $x^2 + y^2 = 25$ on a graphing calculator.

(Ex 2) **c.** Sketch the graph of $(x - 5)^2 + (y - 3)^2 = 4$.

(Ex 2) **d.** Graph $(x + 8)^2 + (y - 1)^2 = 22$ on a graphing calculator.

(Ex 3) **e.** Write the equation of the circle with center $(6, -7)$ and radius 9.

(Ex 3) **f.** Write the equation of the circle with center $(7, -4)$ that contains the point (8, 1).

(Ex 3) **g.** Write the equation of the circle that has a diameter whose endpoints are located at $(-2, 3)$ and (4, 9).

(Ex 4) **h.** The orbit of Venus around the Sun resembles a circle. Given that Venus travels about 421 million miles in one orbit, write an equation for the circle that models Venus's orbit around the Sun.

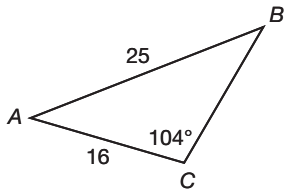
Practice Distributed and Integrated

(70) **1. Geometry** The radius of a sphere is given by $r = \sqrt[3]{\frac{3V}{4\pi}}$, where V is the volume of the sphere. Approximate the volume of a sphere with a radius of 4.5 centimeters.

(91) ***2. Multi-Step** The map of a town is placed on a coordinate grid. A radio station's transmitter is located at (0, 2), and the signal can be picked up by anyone within 95 miles of the transmitter. Write the equation of the circle that represents the broadcast area for the station's signal. Show how to use the equation to prove that a person located at (94, 6) can pick up the signal.

3. (58) Given that $x^2 - 5x + 2 = 0$, what needs to be added to both sides of the equation to get a perfect square on the left side?

4. (71) Solve $\triangle ABC$.



5. (63) A spiral staircase has 15 steps. Each step is a sector with a radius of 42 inches and a central angle of $\frac{\pi}{8}$. What is the length of the arc formed by the outer edge of a step?

***6.** (90) **Error Analysis** Two students are finding the period of $y = \tan\left(\frac{3}{5}x\right)$ but get different results. Which student made the mistake?

Student A	Student B
Period $= \frac{2\pi}{b} = \frac{2\pi}{\frac{3}{5}} = \frac{10\pi}{3}$	Period $= \frac{\pi}{b} = \frac{\pi}{\frac{3}{5}} = \frac{5\pi}{3}$

7. (78) **Multi-Step** For each value of the leading coefficient a, write the quadratic function in the form $y = ax^2 + c$ that has zeros -2 and 2. (Hint: The value of c can be positive or negative.)

a. $a = 1$ **b.** $a = \frac{1}{2}$ **c.** $a = -1$

d. Graph the functions on the same coordinate system.

e. How many quadratic functions exist that have two given zeros?

8. (55) A random number generator will output integers from 1 to 100. What is P(multiple of 5) with the generation of the first number?

9. (63) Find the arc length of a sector with a radius of 8 inches and a central angle of 110°.

***10.** (91) **Generalize** What must be true about the values of h and k for a circle whose center is located in Quadrant II?

11. (75) Explain why $f(x) = \sqrt{3x} + 5$ is a radical function.

***12.** (91) The endpoints of a diameter are located at $(-2, -1)$ and $(4, -1)$. Write an equation for this circle.

13. (84) **Optics** The focal length of a lens, f, the distance from the lens to the object, o, and the distance from the lens to the image, i, are related by the formula $\frac{1}{f} = \frac{1}{o} + \frac{1}{i}$. Find f if o is 12 inches and i is 8 inches.

14. (83) **Analyze** How is deriving a quadratic equation from two identical roots different from deriving a quadratic equation from two unique roots?

15. (73) **Sports** A sports columnist wants to survey college basketball coaches about whether they think women's basketball nets should be raised to the same measure as in men's basketball. The writer mails out surveys to all the coaches and uses only the surveys that are returned. Identify the type of sample described.

***16.** (83) Write a quadratic equation whose roots are $7 + 3i$ and $7 - 3i$.

Find the roots of the polynomial function.

17. (76) $f(x) = 15x^3 + 3x^2 - 6x$

18. (76) $f(x) = (x + 1)(4x^2 + 4) - (x + 1)(2x - 3)$

***19.** (90) **Verify** Let $f(x) = \cos(x)$ and $g(x) = \tan(x)$. Compare the graph of $f(x)\,g(x)$ and $h(x) = \sin(x)$.

***20.** (90) Identify the period and undefined values of $y = \tan(12x) + 3$.

21. (71) **Surveying** A surveyor stands on the east end of a boardwalk and finds that the angle formed by the boardwalk and the line of sight with a lighthouse is 76°. The surveyor then walks to the west end and finds the angle formed by the boardwalk and lighthouse to be 60°. Estimate the closest distance from the boardwalk to the lighthouse, given that the boardwalk has a length of 2700 feet.

***22.** (91) **Graphing Calculator** Graph $(x - 1)^2 + (y + 9)^2 = 7$ on a graphing calculator. Use the Trace feature to find the y-intercepts.

23. (77) **Geography** A map of southeast Illinois shows that Flora is 44.4 miles from Carmi and 42.6 miles from Mt. Carmel. Mt. Carmel is 34.4 miles from Carmi. Find the number of square miles enclosed in the triangle that is formed when lines connecting these three cities are drawn. Round to the nearest square mile.

24. (80) **College Admissions** In 2005, the mean score on the ACT reading test was 21.3 with a standard deviation of 6. The mean score on the mathematics test was 20.7 with a standard deviation of 5. Assume both distributions are normally distributed. Who would you say performed better: a student who got a score of 27 on the reading test or a student who got a score of 24 on the math test? Why?

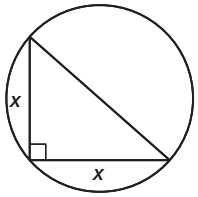
25. (79) Evaluate the piecewise function $f(x) = \begin{cases} 8, & \text{if } x \le 0 \\ 3x - 1, & \text{if } x > 0 \end{cases}$ for $x = -0.4$, $x = 0$, and $x = 6$.

26. (81) **Multiple Choice** Which is equal to $\ln e^{x^2}$?

A $2x$ **B** x^2 **C** $2 \ln x$ **D** $\ln 2e^x$

***27.** (85) **Formulate** Find the roots of $(x^3)^3 - a^3$.

28. (78) **Probability** The right triangle is inscribed in the circle. Suppose a point is chosen at random inside the circle. What is the probability that the chosen point is NOT inside the triangle?



***29.** (88) **Write** Explain why $F = \frac{9}{5}C + 32$ and $C = \frac{5}{9}(F - 32)$ describe the same relationship between F and C.

30. (56) P is a point on the terminal side of θ in standard position. Find the exact value of the six trigonometric functions for θ where $P(-8, -8)$.

LESSON 92

Finding Arithmetic Sequences

Warm Up

1. (4) **Vocabulary** In a function, the _____ **(*domain*|*range*)** is made up of the input values.

2. (7) Solve $18 = 6 + 2(n - 1)$.

3. (SB) True or False: The answer to a subtraction problem is called the difference.

New Concepts

Exploration **Exploring Sequences**

In a DVD club, a member receives 8 DVDs the first month and 3 DVDs every month thereafter.

Step 1: Complete the table.

Month	1	2	3	4	5
Total Number of DVDs	8				

Step 2: Describe a pattern in the total number of DVDs.

Step 3: Find the total number of DVDs for months 6, 7, and 8.

Step 4: Graph the data. What type of function models the data? Why? What is the domain of the function?

A **sequence** is an ordered list of numbers. Each number in the sequence is a **term.** A sequence can be **finite** {2, 4, 6, 8}, or **infinite** {5, 15, 45, 135, …}.

Sequences can be written as functions, where the domain is the set of natural numbers and the range is the set of terms. It is common to use the function name of a for a sequence with the term number, or domain, written as a subscript. Therefore a_1 is the first term of a sequence, a_2 is the second term, and a_n is the nth term.

Math Reasoning

Justify Show that {5, 15, 45, 135, …} is not an arithmetic sequence.

If the differences between the consecutive terms in a sequence are the same, then the sequence is an **arithmetic sequence.** The finite sequence {2, 4, 6, 8} is arithmetic because the difference between any two consecutive terms is 2.

The difference of 2 is called the common difference. The **common difference** of any arithmetic sequence is found by subtracting any term from its successive term. It can be used to extend an infinite sequence.

Example 1 **Identifying and Using the Common Difference**

Find the common difference of the arithmetic sequence and use it to find the next three terms.

$-14, -9, -4, 1, \ldots$

SOLUTION Subtract a term from the term that follows it: $1 - (-4) = 5$. Find the next three terms by adding 5: $1 + 5 = 6$, $6 + 5 = 11$, and $11 + 5 = 16$.

The next three terms are 6, 11, and 16.

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Math Language

To find the *n*th term using a *recursive formula*, the previous term must be known. The recursive formula is $a_n = a_{n-1} + d$.

An **explicit formula** defines the *n*th term of a sequence in terms of *n*, meaning that the *n*th term can be found without knowing the previous term.

For the arithmetic sequence {10, 14, 18, 22, 26, …},

$a_1 = 10 + \mathbf{0}(4)$, $a_2 = 10 + \mathbf{1}(4)$, $a_3 = 10 + \mathbf{2}(4)$, $a_4 = 10 + \mathbf{3}(4)$, $a_5 = 10 + \mathbf{4}(4), \ldots a_n = 10 + (n - 1)4$.

Arithmetic Sequence
The *n*th term of an arithmetic sequence is given by $a_n = a_1 + (n - 1)d$.

Example 2 Finding the *n*th Term of an Arithmetic Sequence

a. Find the 15th term of the arithmetic sequence 4, 15, 26, 37, …

SOLUTION Find d: $37 - 26 = 11$. Then use the formula $a_n = a_1 + (n - 1)d$.

$a_{15} = 4 + (15 - 1)11$ Substitute 15 for n, 4 for a_1, and 11 for d.

$a_{15} = 158$ Simplify.

b. Find the 18th term of the arithmetic sequence 11, 5, −1, −7, …

SOLUTION Find d: $5 - 11 = -6$. Then use the formula $a_n = a_1 + (n - 1)d$.

$a_{18} = 11 + (18 - 1)(-6)$ Substitute 18 for n, 11 for a_1, and −6 for d.

$a_{18} = -91$ Simplify.

The formula $a_n = a_m + (n - m)d$ can be used to find any term when given any two terms of the sequence.

Example 3 Finding the *n*th Term Given Any Two Terms

a. Find a_1 of an arithmetic sequence given that $a_7 = 34$ and $a_{16} = 61$.

SOLUTION In the formula $a_n = a_m + (n - m)d$, replace m with 7 and n with 16.

$a_{16} = a_7 + (16 - 7)d$

$61 = 34 + (16 - 7)d$ Substitute a_{16} with 61 and a_7 with 34.

$61 = 34 + 9d$ Solve for d.

$27 = 9d$

$3 = d$

Now find a_1. Use the formula $a_n = a_1 + (n - 1)d$ and either of the known terms.

$61 = a_1 + (16 - 1)3$ Substitute 61 for a_n and 16 for n.

$61 = a_1 + 45$ Solve for a_1.

$16 = a_1$

The first term of the sequence is 16.

Check by using a_1 to find the 7th term: $a_7 = 16 + (7 - 1)3 = 16 + 18 = 34$.

b. Find a_8 of an arithmetic sequence given that $a_5 = -180$ and $a_{12} = -138$.

SOLUTION In the formula $a_n = a_m + (n - m)d$, replace m with 5 and replace n with 12.

$$a_{12} = a_5 + (12 - 5)d$$

$$-138 = -180 + (12 - 5)d \qquad \text{Substitute } a_{12} \text{ with } -138 \text{ and } a_5 \text{ with } -180.$$

$$-138 = -180 + 7d \qquad \text{Solve for } d.$$

$$42 = 7d$$

$$6 = d$$

Now find a_8. Use $a_n = a_m + (n - m)d$, replacing m with 8, 6 for d, and using one of the given terms.

$$a_{12} = a_8 + (12 - 8)(6) \qquad \text{Use } n = 12.$$

$$-138 = a_8 + (12 - 8)(6) \qquad \text{Substitute } a_{12} \text{ with } -138.$$

$$-138 = a_8 + 24 \qquad \text{Solve for } a_8.$$

$$-162 = a_8$$

The eighth term of the sequence is -162.

Since the eighth and twelfth terms are not too far from the fifth term, the answer could be checked by adding the common difference of 6.

a_5	a_6	a_7	$\boldsymbol{a_8}$	a_9	a_{10}	a_{11}	$\boldsymbol{a_{12}}$
-180	-174	-168	**-162**	-156	-150	-144	**-138**

Math Reasoning

Analyze How can you tell that the common difference will be a positive number before doing any calculations?

Example 4 Application: Salary

Richard's salary is structured so that after the first year, he receives an annual raise of a set amount. What was his starting salary if his salary in his fourth year was \$41,135 and his salary in his fourteenth year was \$50,785?

SOLUTION

Substitute the given information into the formula $a_n = a_m + (n - m)d$. Replace a_m with a_4, or 41,135, m with 4, a_n with a_{14}, or 50,785, and n with 14.

$$50{,}785 = 41{,}135 + (14 - 4)d$$

$$50{,}785 = 41{,}135 + 10d$$

$$9650 = 10d$$

$$965 = d$$

Find a_1.

$$50{,}785 = a_1 + (14 - 1)965$$

$$50{,}785 = a_1 + 12{,}545$$

$$38{,}240 = a_1$$

Richard's starting salary was \$38,240.

Check Use a_1 to find the 4th term: $a_4 = 38{,}240 + (4 - 1)965 = 41{,}135$.

Math Reasoning

Formulate Study what is being done to find d and formulate a way to find it without writing an equation.

Lesson Practice

a. (Ex 1) Find the common difference of the arithmetic sequence $9\frac{1}{2}$, $10\frac{1}{4}$, 11, $11\frac{3}{4}$, …. Then, find the next three terms.

b. (Ex 2) Find the 44th term of the arithmetic sequence −121, −112, −103, −94, …

c. (Ex 2) Find the 33rd term of the arithmetic sequence 0.56, 0.38, 0.2, 0.02, …

d. (Ex 3) Find a_1 of an arithmetic sequence given that $a_3 = 33$ and $a_7 = 63$.

e. (Ex 3) Find a_3 of an arithmetic sequence given that $a_7 = 75$ and $a_{12} = -125$.

f. (Ex 4) Beverly's salary is structured so that after the first year, she receives an annual raise of a set amount. What was her starting salary if her salary in her sixth year was $31,975 and her salary in her thirteenth year was $36,420?

Practice Distributed and Integrated

***1.** (92) **Formulate** Give an example of a sequence that is formed by adding numbers in a pattern but is not arithmetic.

2. (84) **Home Maintenance** A homeowner can rake and bag the leaves in his yard in 5 hours. If he works with his children, the job can be done in 3 hours. How long would it take for just the children to rake and bag the leaves in the yard?

***3.** (92) **Multiple Choice** What is the 40th term of the arithmetic sequence whose first term is 13 and whose second term is 15.5?

A 97.5 **B** 100 **C** 110.5 **D** 123.5

4. (75) **Multi-Step** Find the radical function that is the inverse of $y = x^2 - 6x + 9$. Identify its domain and range.

5. (67) Evaluate the inverse trigonometric function $\cos^{-1}(-1)$. Give your answer in both radians and degrees.

6. (80) **Write** Describe the characteristics of a data set that is normally distributed.

***7.** (91) **Coordinate Geometry** If $A(5, 2)$ and $B(-4, -3)$ are the endpoints of a diameter of a circle, what is the equation of the circle?

***8.** (91) Write the equation for a circle centered at $(2, -9)$ with a radius of 3.

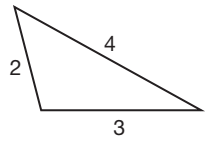
9. (73) **Geometry** A geometry teacher has a database with 400 problems about triangles. Can she choose a sample of questions by using a stratified sampling method where the strata are *right triangles, isosceles triangles,* and *scalene triangles*? Why or why not?

10. (76) **Physics** A satellite in orbit accelerates according to the equation acceleration $= \frac{v^2}{r}$, for orbital speed v and distance from Earth's center r. Two satellites are in orbit, one at a distance of r and the other at a distance of $2r$. How fast should the outer satellite travel to have the same acceleration as the inner satellite?

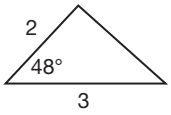
11. (69) Multiply $(3 + i)(1 - 4i)$. Write the result in the form $a + bi$.

12. (77) **Multiple Choice** Which triangle cannot be solved by using the Law of Cosines?

A



B



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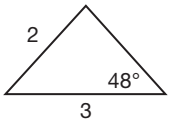

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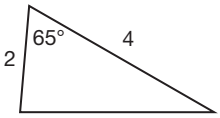

13. (81) **Multi-Step** Technetium-99m, a radioisotope used to image the skeleton and the heart muscle, has a half-life of about 6 hours.

a. How long will it take for 1 mg of technetium-99m to decay to 0.2 mg? (Use the natural decay function $N(t) = N_0 e^{-kt}$.)

b. On a calculator, graph the function $y_1 = 1 \cdot e^{-kx}$, using the value of k you found in part **a.** Also graph the functions $y_2 = 0.5$ and $y_3 = 0.2$. (suggested window values: x $[-1, 20]$ and y $[-0.1, 1.1]$)

c. Describe what is represented by the relationships among the graphs of y_1, y_2, and y_3.

14. (79) **Delivery Charges** A furniture store delivers up to 4 items for a \$75 delivery charge. There is an additional charge of \$25 per item for each additional item delivered. Write a piecewise function for the cost of having x items delivered.

***15.** (92) Find the 17th term of the arithmetic sequence 4, 21, 38, …

Find the roots of the polynomials.

16. (58) $y = 4x^2 + 5x - 1$

17. (76) $y = x^4 - 16$

18. (85) **Geometry** A rectangular prism has a volume of $3x^3 - 8x^2 + 3x + 2$. Find the linear dimensions.

***19.** (92) **Travel** A driver on the Pennsylvania turnpike turned on the cruise control at milepost 30, Warrendale, and kept it on until milepost 359, the Delaware River Bridge, exactly 5 hours later. Show how to use an arithmetic sequence to find the speed at which the car traveled during those 5 hours.

***20.** (86) **Graphing Calculator** Graph $y = 5 \cos 2(x - 3\pi)$ using a graphing calculator. Determine its period and phase shift.

***21.** (87) **Verify** Show that $\log_5 (125x)^2$ equals 8 when $x = 5$.

22. (57) A principal amount of \$1,500 earns interest at an annual rate of 2.7%, compounded monthly. What will its value be after 7 years?

*23. **Write** When solving a quadratic inequality, what is meant by the *critical values*? (89)

24. Solve $6t^2 + 5 = 2t^2 + 1$. Write the solution in terms of i. (62)

25. **Analyze** Which equations have the same solutions as $x^2 - 8x + 7 = 0$? (83)

$-x^2 + 8x - 7 = 0$ $\quad -x^2 + 8x + 7 = 0$ $\quad -2x^2 + 8x - 7 = 0$

$2x^2 - 8x + 14 = 0$ $\quad 2x^2 - 16x + 14 = 0$

Write the exponential equation in logarithmic form or vice versa.

26. $11^4 = 14{,}641$ (64)

27. $\log_9 729 = 3$ (64)

*28. **Basketball** A drafter is creating the plans for a sport and fitness center, which includes an official-size basketball court. On her grid, the units represent feet. The center of the basketball court's center circle is located at $(-14, -1)$, and another point on the center circle is located at $(-14, 2)$. Write an equation for this circle. (91)

29. **Error Analysis** A student found that the discriminant of a quadratic equation was 0 and therefore the number of solutions of the equation was also 0. Correct the student's reasoning. (74)

30. Solve by completing the square: $4x^2 = -12x + 4$. (78)

LESSON 93

Solving Exponential Equations and Inequalities

Warm Up

1. (3) **Vocabulary** In the expression $(1.005)^x$, the _____ is 1.005.
2. (47) What is the base in the expression $240(1.022)^x$?
3. (47) What is the base in the expression $50(1 + 0.04)^x$?

New Concepts

Exponential equations cannot always be solved using the same properties that are used to solve linear equations. One property that can be used to solve simple exponential equations is the Property of Equality.

Math Language

An equation that contains one or more exponential expressions is an exponential equation.

Property of Equality for Exponential Functions
If a is any positive number other than 1 and $a^x = a^y$, then $x = y$.

To apply the property of equality, each side of the equation must be written with the same base, a. If they are not written in the same base, they must be written as a power of the same base.

Hint

Review Lesson 3 to read about the rules of exponents that are necessary to solve exponential equations like these.

Example 1 Using the Property of Equality

a. Solve for x: $2^x = 16$

SOLUTION

Write 16 as an exponent with a base of 2.

$16 = 2^4$

Therefore, $2^x = 2^4$.

By the Property of Equality, $x = 4$.

b. Solve for c: $9^{c+2} = 27^2$

SOLUTION

Both sides of the equation can be written with a base of 3.

$9^{c+2} = (3^2)^{c+2} = 3^{2c+4}$

$27^2 = (3^3)^2 = 3^6$

By the Property of Equality, $2c + 4 = 6$, so $c = 1$.

c. Solve for t: $2^t = \frac{1}{16}$

SOLUTION

It may seem like the right side of this equation cannot be written as a power of 2, but recall that it is in fact a negative power of 2.

$\frac{1}{16} = 2^{-4}$

By the Property of Equality, $t = -4$.

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Another way to solve an exponential function is to apply the Power Property of Logarithms. Recall from Lesson 72 that the Power Property of Logarithms says that $\log a^p = p \log a$.

To apply the Power Property, take the logarithm of both sides of an exponential equation and simplify the result.

Hint

Don't forget the product and quotient rules of logarithms, which may also be useful in simplifying exponential equations as you solve them.

Example 2 Solving by Taking the Logarithm of Each Side

a. Solve for x: $200 = 1.05^x$

SOLUTION

$$200 = 1.05^x$$

$\log 200 = \log 1.05^x$ Take the logarithm of both sides.

$\log 200 = x \log 1.05$ Apply the Power Property.

$x = \dfrac{\log 200}{\log 1.05}$ Divide by log 1.05.

$x \approx 108.59$ Evaluate.

b. Solve for y: $7^{2y+1} = 4^{-y}$

SOLUTION

$4^{-y} = 7^{2y+1}$

$\log 4^{-y} = \log 7^{2y+1}$ Take the logarithm of both sides.

$-y \log 4 = (2y + 1)\log 7$ Apply the Power Property.

$-y \log 4 = 2y \log 7 + \log 7$ Distribute.

$-y \log 4 - 2y \log 7 = \log 7$ Subtract $2y$ log 7 from each side.

$y(-\log 4 - 2 \log 7) = \log 7$ Factor.

$y = \dfrac{\log 7}{(-\log 4 - 2 \log 7)}$ Division Property of Equality.

$y \approx -0.37$ Evaluate.

If e is used in an exponential equation, solve the equation by taking the natural log of both sides. Recall the inverse property of logarithms: $\ln e^x = x$.

Example 3 Solving Exponential Equations with *e*

a. Solve for x: $5 = 6e^x$

SOLUTION

$$6e^x = 5$$

$\ln 6e^x = \ln 5$ Take the logarithm of both sides.

$\ln 6 + \ln e^x = \ln 5$ Apply the Product Property.

$\ln 6 + x = \ln 5$ Apply the Inverse Property.

$x \approx -0.18$ Subtract and evaluate.

Example 4 Solving Exponential Inequalities

a. Suppose you receive a gift of 1 cent on the first day, 2 cents on the second day, and so on, receiving double the amount on each new day. What is the first day on which you will receive more than $1 million?

SOLUTION $1 million = $1,000,000 = 100,000,000 cents = 10^8 cents

On day 1 you receive 2^0 cents, on day 2 you receive 2^1 cents, and so on, receiving 2^{n-1} cents on day n.

$2^{n-1} > 10^8$	Write an inequality.
$\log 2^{n-1} > \log 10^8$	Take the logarithm of both sides.
$\log 2^{n-1} > 8$	Simplify.
$(n-1)\log 2 > 8$	Apply the Power Property of Logarithms.
$n - 1 > \frac{8}{\log 2}$	Divide both sides by log 2.
$n > \frac{8}{\log 2} + 1 \approx 27.6$	Use a calculator to approximate.

The first day on which you will receive more than $1 million is day 28.

Check Day 27: 2^{26} cents = 67,108,864 cents = $671,088.64

Day 28: 2^{27} cents = 134,217,728 cents = $1,342,177.28

b. A merchant is having trouble selling a jacket and decides to mark down the price 10% at the beginning of every day. The regular price of the jacket is $199.50. He marks the price down for the first time on day 1. On what day will the price first be below $50?

SOLUTION The price on day n is $199.50(1 - 0.10)^n$, or $199.50(0.90)^n$.

$199.50(0.90)^n < 50$	Write an inequality.
$\log [199.50(0.90)^n] < \log 50$	Take the common logarithm of both sides.
$\log 199.50 + \log (0.90)^n < \log 50$	Apply the Product Property of Logarithms.
$\log (0.90)^n < \log 50 - \log 199.50$	Subtract log 199.50 from both sides.
$n \log (0.90) < \log 50 - \log 199.50$	Apply the Power Property of Logarithms.
$n > \frac{\log 50 - \log 199.50}{\log 0.90}$	Divide both sides by log 0.90 and reverse the inequality symbol.
$n > 13.13$	Use a calculator to approximate.

The price will first be below $50 on day 14.

Caution

Know the sign of a logarithm if you divide both sides of an inequality by that logarithm. log m is negative if $0 < m < 1$. log m is positive if $m > 1$.

Graphing Calculator Tip

The expression log (2) ÷ 4 log (1.0225) is NOT correct for solving Example 4 because it equals $\left(\frac{\log 2}{4}\right)$ (log 1.0225). A correct expression is log (2) ÷ (4 log (1.0225)).

Example 5 Application: Compound Interest

How long will it take an investment to double in value at 9% interest compounded quarterly (4 times per year)?

SOLUTION $A = P\left(1 + \frac{r}{n}\right)^{nt}$ Write the compound interest formula.

$2P = P\left(1 + \frac{0.09}{4}\right)^{4t}$ Substitute $2P$ for A because P will double. Substitute 0.09 for r and 4 for n.

$2 = (1.0225)^{4t}$ Divide both sides by P and simplify.

$\log 2 = \log (1.0225)^{4t}$ Take the common logarithm of both sides.

$\log 2 = 4t \log (1.0225)$ Apply the Power Property of Logarithms.

$\frac{\log 2}{4 \log (1.0225)} = t$ Divide both sides by 4 log 1.0225.

$t \approx 7.79$ years Use a calculator to approximate.

The interest is compounded quarterly, so the compounding will likely occur when t has these values: 7, 7.25, 7.50, 7.75, and 8. The value of the investment will not have doubled when $t = 7.75$, but it will have more than doubled when $t = 8$.

Check When $t = 7.75$: $A = P\left(1 + \frac{0.09}{4}\right)^{(4)(7.75)} = P(1.0225)^{31} \approx 1.993P$

When $t = 8$: $A = P\left(1 + \frac{0.09}{4}\right)^{(4)(8)} = P(1.0225)^{32} \approx 2.038P$

Lesson Practice

Solve for the variable in each equation.

a. *(Ex 1)* $49^{2y} = 343^{2y+3}$

b. *(Ex 1)* $3^r = 9^5$

c. *(Ex 1)* $25^t = \frac{1}{25}$

d. *(Ex 2)* $45 = 2.3^{2x}$

e. *(Ex 3)* $3e^{2c+1} = 432$

f. *(Ex 4)* Stock in a certain company was purchased on December 31, 2007, for \$1.50 per share. Suppose the price of a share doubles every year. What is the first year in which the price of a share will be greater than \$90?

g. *(Ex 4)* The regular price of a rug is \$325. The price is marked down 6% at the beginning of each week, beginning with week 1. In what week will the price first be below \$200?

h. *(Ex 5)* How long will it take an investment to increase 50% in value at 4.8% interest compounded monthly (12 times per year)?

Practice Distributed and Integrated

1. (73) **Fast Food** At a fast food restaurant, each take-out order has a ticket in the bottom of the bag asking customers to call an 800 number to register for a prize drawing. Describe the individuals in the population and in the sample.

***2.** (93) **Multiple Choice** Which equation is equivalent to $1.4 = 1.06^x$?

A $x = 1.4 - 1.06$ **B** $x = \log 1.4 - \log 1.06$ **C** $x = \frac{1.4}{1.06}$ **D** $x = \frac{\log 1.4}{\log 1.06}$

3. (78) **Gravity** The height of a free-falling object (neglecting air resistance) is given by the function $h(t) = -4.9t^2 + v_0t + h_0$, where h_0 is the initial height in meters, v_0 is the initial velocity in meters per second, and $h(t)$ is the height in meters at time t seconds. How long does it take for an object to hit the ground after it is dropped from a height of 14.4 meters?

***4.** (92) **Verify** The first term of an arithmetic sequence is 19 and the common difference is 9. Show that the 13th term is 127.

***5.** (92) **Geometry** A cube has 6 sides. When two cubes are joined together, the figure has 10 sides. When three cubes are joined together, to form a row, there are 14 sides. Show how to use an arithmetic sequence to find the number of sides in a row of 20 cubes joined together.

Determine the domain and range for each of the following.

6. (75) $y = -3\sqrt[3]{x}$

7. (75) $y = (x + 1)^{\frac{1}{2}} + 8$

***8.** (92) Find the common difference and use it to find the next three terms. 4, 6.5, 9, …

***9.** (91) **Science** Vredefort is an impact crater located in South Africa. Given that its circumference is about 942.5 kilometers, write an equation for the circle that represents the rim of the crater, assuming the center is located at the origin of a coordinate plane.

***10.** (93) Solve for x: $4^{3x-6} = 32^{x+4}$

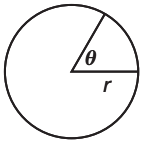
11. (84) **Generalize** Describe two ways to find the solution(s) of $\frac{3}{3x + 9} + \frac{2}{3} = x - 1$ on a graphing calculator.

***12.** (90) **Graphing Calculator** Graph $y = \tan(2x + 3)$ on your graphing calculator. Identify its period, undefined values, and phase shift.

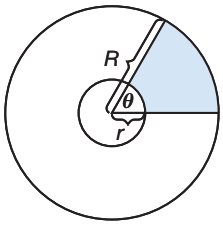
***13.** (89) **Tourist Attractions** The world's largest pair of cowboy boots are located outside of a mall in San Antonio, Texas. The boots are 40 feet tall. Suppose a painter at the top of a boot drops a paintbrush. The height of the brush, in feet, after x seconds, is modeled by $y = -16x^2 + 40$. Write and solve a quadratic inequality that can be solved to find the time span for which the brush is above a second painter located 20 feet beneath the first painter. Round to the nearest hundredth of a second.

***14.** (93) **Depreciation** A truck is purchased for $18,500 on January 2, 2007. Its value decreases 16% each year. In what year will its value first be less than $2,000?

15. (88) **Multi-Step** The formula for the area of a sector of a circle is $A = \frac{\theta r^2}{2}$, where θ is the central angle measure in radians and r is the radius.



a. A landscape architect wants to create a flower bed in the shape of a sector of a circular ring, as shown by the shaded region. Write a formula for the area S of a sector of a circular ring.



b. Solve the formula you wrote in part **a** for θ.

c. The landscape architect wants the area S of the flower bed to be 200 square feet, with $R = 25$ feet and $r = 10$ feet. Determine the necessary angle measure θ and then convert it to the nearest degree.

16. (87) **Error Analysis** A student rewrote the expression $\log_8 (3x)$ as $\frac{\ln 8}{\ln 3 + \ln x}$.

a. Describe the student's error.

b. Rewrite the original expression correctly.

17. (85) **Analyze** Consider the polynomial function $f(x) = ax^2 + bx + cx$. Suppose that the polynomial is divisible by $\left(x - \frac{c}{a}\right)$. Use synthetic division to show that $a + b + c = 0$.

18. (82) Which of the following has a period of 2π?

A $y = \sin(2\pi x)$ **B** $y = \sin(\pi x)$ **C** $y = \sin(x)$ **D** $y = \sin\left(\frac{x}{2}\right)$

***19.** (93) **Estimate** Thallium has a half-life of 73.1 hours. Estimate how long it will take for 1 gram of thallium to decay to 0.26 gram. Explain your method.

Solve to the nearest tenth using the given restrictions.

20. (67) $\sin\theta = 0.95$, for $-90° \le \theta \le 90°$

21. (67) $\cos\theta = -0.181$, for $180° \le \theta \le 360°$

Determine the domain and range of each function.

22. (79) $f(x) = \begin{matrix} 0, & \text{if } x < 0 \\ 3, & \text{if } x \ge 0 \end{matrix}$

23. (79) $f(x) = \begin{matrix} x, & \text{if } 0 < x < 10 \\ 10, & \text{if } x \ge 10 \end{matrix}$

Solve the following equations.

24. (76) $3x^4 = 12x^3 - 9x^2 + 6x$

25. (76) $x^3 = 2x^4 - 5x^2 + 4x$

26. (83) **Generalize** Given that a, b, c, and d are real numbers, explain why $(x - a)(x - b) = 0$ and $(cx - ca)(dx - db) = 0$ have the same roots. What are the roots?

27. (81) Given $N(t) = 0.6$, $N_o = 1$, $k = 0.6372$, write the decay function and solve for t.

28. (83) Write a quadratic equation whose roots are -5 and 4.

29. (78) **Error Analysis** A student says that the roots of the equation $x^4 - 81 = 0$ are -3 and 3. What complex roots were left out?

30. (80) **Probability** The area under a normal curve is the probability that a randomly selected data value from that data set falls within that interval. Suppose the weights of cereal boxes have a mean of 15.5 ounces with a standard deviation of 0.4 ounces. What is the probability that a randomly selected box of cereal weighs less than 14.7 ounces?

LESSON
94

Solving Rational Inequalities

Warm Up

1. (84) **Vocabulary** To clear the fractions in a rational equation, multiply both sides of the equation by the _____.

2. (84) Solve $\frac{2x}{x+5} = 14$ by using a graphing calculator.

3. (89) Solve $x^2 + 10x < -16$ algebraically.

New Concepts

A **rational inequality** is an inequality that contains at least one rational expression. They can be solved algebraically or graphically. When a rational inequality is solved by using the LCD, the LCD could be positive or negative.

Example 1 Solving Rational Inequalities Using the LCD

Solve $\frac{6}{x+2} < 8$ by using the LCD.

SOLUTION **Case 1:** The LCD is positive.

Step 1: The LCD is $x + 2$. Since it is positive, $x + 2 > 0$, so $x > -2$.

Step 2: Multiply both sides of the inequality by $x + 2$.

$$(x+2)\frac{6}{x+2} < 8(x+2)$$

$6 < 8x + 16$ Simplify.

$-10 < 8x$ Solve for x.

$x > -1.25$

The solution must satisfy both $x > -2$ from Step 1 and $x > -1.25$ from Step 2, therefore, $x > -1.25$.

Case 2: The LCD is negative.

Step 1: The LCD is $x + 2$. Since it is negative, $x + 2 < 0$, so $x < -2$.

Step 2: Multiply both sides of the inequality by $x + 2$.

$$(x+2)\frac{6}{x+2} > 8(x+2)$$ Reverse the inequality sign.

$6 > 8x + 16$ Simplify.

$-10 > 8x$ Solve for x.

$x < -1.25$

The solution must satisfy both $x < -2$ from Step 1 and $x < -1.25$ from Step 2, therefore, $x < -2$.

Combine the solutions from both cases: $x < -2$ or $x > -1.25$.

Math Reasoning

Analyze What is the restricted value of the inequality in Example 1?

Math Reasoning

Justify Why did the direction of the sign change in Case 2?

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As with solving quadratic inequalities, rational inequalities can be solved by finding critical values, or boundary points, and testing a point in each interval they form. Recall that one side must be equal to 0 to find the critical values.

Example 2 Solving Rational Inequalities by Finding the Values of Test Points

Solve $\frac{x+4}{x^2+x-6} \leq 0$ by finding the values of test points.

SOLUTION The right side is already 0. Find the boundary points.

Factor. Find the values that make either the numerator or denominator 0.

$$\frac{x+4}{(x-2)(x+3)} \leq 0$$

The numerator is 0 when $x = -4$. The denominator is zero when $x = 2$ or $x = -3$.

The boundary points are -4, -3, and 2. This gives four intervals to test. Use $<$ or $>$ for the asymptotes of 2 and -3. These values are restricted because they make the fraction undefined.

Math Reasoning

Analyze Why is $\leq$ instead of $<$ used in $x \leq 4$?

Interval	Test Value	Is the test value a solution?
$x \leq -4$	-5	$\frac{-5+4}{(-5)^2+(-5)-6} \stackrel{?}{\leq} 0$ $-\frac{1}{14} \leq 0$ Yes
$-4 \leq x < -3$	-3.5	$\frac{-3.5+4}{(-3.5)^2+(-3.5)-6} \stackrel{?}{\leq} 0$ $\frac{2}{11} \leq 0$ No
$-3 < x < 2$	0	$\frac{0+4}{0^2+0-6} \stackrel{?}{\leq} 0$ $-\frac{2}{3} \leq 0$ Yes
$x > 2$	5	$\frac{5+4}{(5)^2+(5)-6} \stackrel{?}{\leq} 0$ $\frac{9}{24} \leq 0$ No

The value of x is a solution when $x \leq -4$ or $-3 < x < 2$.

Finding the actual value of a test point can become cumbersome. Notice that all that is really required from the test point is its sign. The intervals that contain solutions to the inequality have a negative sign because negative values are less than 0.

The sign of the rational expression can be determined by finding the sign of the numerator and denominator, or by finding the sign of each factor.

Example 3 Solving Rational Inequalities Using Sign Tables

a. Solve $\frac{x-6}{x+2} > 0$ by finding the sign of the numerator and denominator of the rational expression.

SOLUTION The right side is already 0. Find the boundary points.

The numerator is 0 when $x = 6$. The denominator is zero when $x = -2$.

The boundary points are -2 and 6. This gives three intervals to test.

They are listed in the top row with a test value below them.

For each interval, find the signs of the numerator and denominator. Find the sign of the value of the rational expression for that interval by analyzing the signs of the numerator and denominator.

Hint

The quotient is negative when the signs are different and positive when the signs are the same.

	$x < -2$	$-2 < x < 6$	$x > 6$
Test value for x	-3	0	9
Numerator: $x - 6$	$-$	$-$	$+$
Denominator: $x + 2$	$-$	$+$	$+$
Value of rational expression	$+$	$-$	$+$

The value of x is a solution when $x < -2$ or $x > 6$.

b. Solve $\frac{(x-5)}{(x+1)(x-4)(x-7)} \geq 0$ by finding the sign of each factor of the rational expression.

SOLUTION The right side is already 0. Find the boundary points.

The numerator is 0 when $x = 5$. The denominator is zero when $x = -1$, $x = 4$, or $x = 7$.

The boundary points are -1, 4, 5, and 7. This gives five intervals to test. Use $<$ or $>$ for the undefined values (asymptotes) of -1, 4, and 7.

Find the product of the "signs" of the factors to find the sign of the expression.

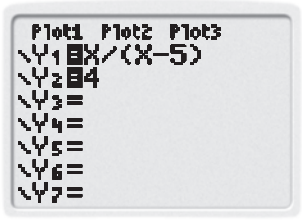
	$x < -1$	$-1 < x < 4$	$4 < x \leq 5$	$5 \leq x < 7$	$x > 7$
Test value for x	-2	0	4.5	6	10
$x - 5$	$-$	$-$	$-$	$+$	$+$
$x + 1$	$-$	$+$	$+$	$+$	$+$
$x - 4$	$-$	$-$	$+$	$+$	$+$
$x - 7$	$-$	$-$	$-$	$-$	$+$
Value of rational expression	$+$	$-$	$+$	$-$	$+$

The value of x is a solution when $x < -1$ or $4 < x \leq 5$ or $x > 7$.

Example 4 Solving Rational Inequalities by Using a Graphing Calculator

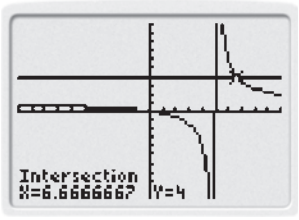
Solve $\frac{x}{x-5} \leq 4$ by using a graphing calculator.

SOLUTION Graph Y1 $= \frac{x}{x-5}$ and Y2 $= 4$.



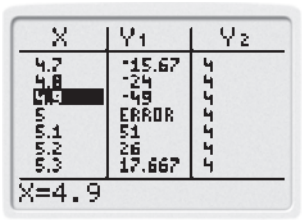
Look for where the first graph intersects or is below the line. The entire left part of the rational expression is below the line. This occurs for all x-values to the left of the asymptote of $x = 5$, so one part of the solution is $x < 5$.

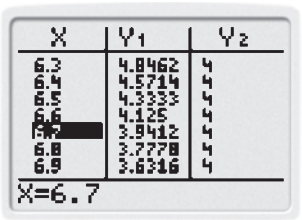
Use the intersect command to find where the right part of the graph is below the line. The intersection is at $x = 6.\overline{6}$, so the second part of the solution is $x \geq 6.\overline{6}$.



The solution of the inequality is $x < 5$ or $x \geq 6.\overline{6}$.

Check the solutions by viewing the table.





Graphing Calculator Tip

Press 2nd Window (Tblset) to change the increments at which the x-values increase in the table.

Example 5 Application: Cost per Person

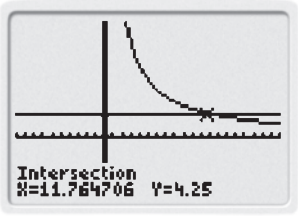
A group of students are sharing the cost of a \$50 gift for their teacher. Write and solve an inequality to show the numbers of students whose participation would bring the cost per student to less than \$4.25.

SOLUTION

1. Understand The cost per student is $\frac{50}{x}$ where x is the number of students.

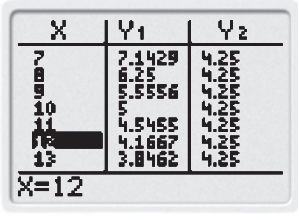
2. Plan Solve $\frac{50}{x} < 4.25$.

3. Solve Graph Y1 $= \frac{50}{x}$ and Y2 $= 4.25$.



Find the point of intersection. The first graph is below the line for all x-values greater than the x-value of the point of intersection. Since x represents numbers of people, round up to 12 people. Since the number of people must be positive, disregard the negative solutions.

Twelve or more students must participate to have a per person cost of less than \$4.25.



4. Check Look at the values in the table.

Lesson Practice

a. (Ex 1) Solve $\frac{7}{x+4} \leq 3$ by using the LCD.

b. (Ex 2) Solve $\frac{x-8}{x^2+7x+10} \geq 0$ by finding the values of test points.

c. (Ex 3) Solve $\frac{x+1}{x-4} < 0$ by finding the sign of the numerator and denominator of the rational expression.

d. (Ex 3) Solve $\frac{(x+8)}{(x-2)(x-5)(x+1)} > 0$ by finding the sign of each factor of the rational expression.

e. (Ex 4) Solve $\frac{x}{x+6} \geq -2$ by using a graphing calculator.

f. (Ex 5) A group of students are sharing the cost of a $75 gift for their teacher. Write and solve an inequality to show the number of students whose participation would bring the cost per student to less than or equal to $6.25.

Practice Distributed and Integrated

Solve using the quadratic formula. Write the solution as complex numbers in standard form.

1. (65) $f(x) = x^2 + 2x + 4$

2. (65) $f(x) = x^2 - x + 12$

3. (88) Given $P = 2l + 2w$, find l.

***4.** (94) **Multi-Step** **a.** Factor both the numerator and denominator of $\frac{x^2 - 3x - 18}{x^2 - 16} \leq 0$.

b. Determine the intervals created by the critical values.

c. The intervals for which test values are negative contain the solutions. What is the solution?

5. (80) The average height of giant sunflowers is 6 feet tall and the standard deviation is 3 inches. Find the z-score for a giant sunflower that is 5 feet 6 inches tall.

6. (78) **Multiple Choice** What are the roots of the equation $(4x^2 - 4x + 1) - (16x^2 - 24x + 9) = 0$?

A $\frac{2}{3}$ and -2 **B** $\frac{2}{3}$ and 1 **C** $\frac{1}{2}$ and $-\frac{3}{4}$ **D** $\frac{1}{2}$ and $\frac{3}{4}$

***7.** (94) **Government** Suppose the citizens of counties in the state of Texas had the option to donate an equal amount of money toward a one million dollar donation to be given to a charity. Write and solve an inequality to show the number of counties whose citizen participation would bring the cost per county to less than $6500. There are 254 counties in Texas.

8. (80) **Error Analysis** A student said if a number cube is rolled and the results are recorded in a histogram, a smooth curve drawn through the tops of the histogram would resemble a bell curve. Explain the error in the student's thinking.

*9. (94) **Justify** Explain why there are four intervals when solving $\frac{(x+2)}{(x-3)(x+1)}$ by using test points.

10. (77) **Geometry** The perimeter of an isosceles triangle is 200 inches. The base has a length of 92 inches. Find the measure of the vertex angle.

*11. (93) **Analyze** If \$500 is invested at 4% compounded monthly, then at the end of t years the investment will have the value A, where A is given by the expression $500\left(1+\frac{0.04}{12}\right)^{12t}$. What expression gives the value A if the compounding is daily instead of monthly?

12. (71) What is the side length c in $\triangle ABC$ if $A = 32°$, $C = 67°$, and $b = 31$ ft?

*13. (93) **Error Analysis** While attempting to solve an inequality, a student performed the step shown below.

$$n \log (0.5) < \log 2 - \log 20$$

$$n < \frac{\log 2 - \log 20}{\log 0.5}$$

What is the error? What is the correct step?

14. (74) An object is thrown upward from a height of 15 feet at an initial velocity of 35 feet per second. How long will it take the object to hit the ground?

15. (89) **Business** The profit earned from selling tickets to a certain event is modeled by $y = -1.25x^2 + 66x - 720$. If the cost per ticket will be a whole number amount, what range of ticket prices will earn a profit of at least \$100?

*16. (93) A school's enrollment increases 6.5% a year and is currently 1100. How long will it take to reach 1500 students?

17. (68) **Probability** A student has a spinner with four equal sections labeled 2, 4, 6, and 8. If he spins the spinner twice, find the probability that the first spin lands on 6 and the sum of the results is less than or equal to 10.

*18. (90) **Multiple Choice** Which of the following has a period of 2π?

A $y = \tan(2x)$ **B** $y = \tan(2\pi x)$ **C** $y = 2\tan(x)$ **D** $y = \tan\left(\frac{x}{2}\right)$

*19. (92) **Graphing Calculator** With the Table set to increase by 1, enter $15 + (x-1)(-4)$ for Y1 to represent an arithmetic sequence whose first term is 15 and whose common difference is -4. Use the table to find the 6th and 15th terms.

20. (83) **Formulate** Write a quadratic equation with roots of -1 and 5 so that the parabola of the related function opens up. Then write one so that the parabola opens down.

*21. (92) **Analyze** When finding the nth term of an arithmetic sequence, when will the value added to the first term be positive and when will it be negative?

22. (82) Given $k(x) = 3\sin(x)$, identify the amplitude and period.

*23. (91) **Drafting** A computer-aided drafting technician is drawing a bicycle wheel such that the center is located at (6, 2) on the coordinate grid. Each unit on the grid represents one inch. What is the equation of the wheel given that it has a diameter of 28 inches?

24. (73) **Data Analysis** Twelve names already numbered are listed below. Three must be randomly chosen for a sample. What three names will be picked if numbers are chosen from the string of random numbers below?

15 Ann	16 Luke	17 Cassie	18 Finn	19 Jared	20 Tanela
21 Jude	22 Gunner	23 Drew	24 Halle	25 Colby	26 Anton

60201 60482 89824 08705 17913 46815 76221 25650 89337 53603 73597 04633

25. (88) Given $f = \frac{1}{\frac{1}{d_1} + \frac{1}{d_2}}$, find d_2.

26. (75) **Physics** The equation for finding the period of a pendulum is a radical function given by $T = 2\pi\sqrt{\frac{L}{9.8}}$. The period is the amount of time it takes for the pendulum to complete a back-and-forth sweep. How much shorter does a pendulum need to be to have half the period?

27. (81) **Chemistry** Uranium-238 is a naturally abundant isotope because of its long half-life. Its half-life is about 4.5 billion years. About how long would it take for just 1% of an initial amount to decay? (Use the natural decay function $N(t) = N_0e^{-kt}$.)

28. (87) Using the change of base formula, solve for y when $x = 2$ and $x = 4$ given that $y = \frac{\log x}{\log 2}$.

Find the zeros of the polynomial function.

29. (76) $f(x) = 4x(7x - 3)(6x + 1)$

30. (76) $f(x) = 16x^3 - 24x^2 + 6x$

LESSON
95

Factoring Higher-Order Polynomials

Warm Up

(35) **1. Vocabulary** If $f(x) = ax^2 + bx + c$ is divisible by $(x - d)$, then $(x - d)$ is a _____ of $f(x)$.

(51) **2.** Use synthetic division to test if $6x^3 - 5x^2 - 34x + 40$ is divisible by $(x - 2)$.

(51) **3.** Use synthetic division to test if $20x^3 - 53x^2 - 122x + 56$ is divisible by $(x - 2)$.

New Concepts

A constant a is a root of polynomial $P(x)$ if $P(a) = 0$.

You can use the Reminder Theorem and the Factor Theorem to test for roots of a polynomial.

Remainder Theorem
The **Remainder Theorem** states:
If the polynomial function $P(x)$ is divided by $x - a$, then the remainder r is $P(a)$.

Math Reasoning

Generalize Suppose a polynomial $P(x)$ has no real roots. What does the Factor Theorem state about such a polynomial?

Factor Theorem
The **Factor Theorem** states:
For any polynomial $P(x)$,$(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.

Use the Factor Theorem to test if a value is a root of a polynomial.

Example 1 Using the Factor Theorem to Test for Roots

a. Determine if $x = 5$ is a root of $P(x)$.

$$P(x) = 2x^6 - 43x^5 + 75x^4 + 1765x^3 - 857x^2 - 22{,}542x - 30{,}240$$

SOLUTION

Use synthetic division to test if $r(5) = 0$. Divide $P(x)$ by $(x - 5)$.

5	2	−43	75	1765	−857	−22,542	−30,240
		10	−165	−450	6,575	28,590	30,240
	2	−33	−90	1,315	5,718	6,048	0

Since $P(5) = r(5) = 0$, 5 is a root of the polynomial.

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b. Determine if $x = -7$ is a root of $P(x)$.

$$P(x) = 9x^8 - 101x^7 + 195x^6 + 622x^5 - 621x^4 - 1565x^3 - 1947x^2 + 2772x + 2940$$

SOLUTION

Use synthetic division to test if $r(-7) = 0$. Divide $P(x)$ by $(x + 7)$.

-7	9	-101	195	622	-621	-1565	-1947	2772	2940
		-63	1148	-9401	61,453	$-425,824$	2,991,723	$-20,928,432$	146,479,620
	9	-164	1343	-8779	60,832	$-427,389$	2,989,776	$-20,925,660$	146,482,560

Since $P(-7) = r(-7) \neq 0$, -7 is not a root of the polynomial.

Example 2 Finding Roots of *P(x)*

Find all the rational roots of $P(x) = x^5 + 4x^4 - 10x^2 - x + 6$.

SOLUTION

By the Rational Root Theorem, the possible rational roots are ± 1, ± 2, ± 3, ± 6.

Use synthetic division to test the roots.

1	1	4	0	-10	-1	6
		1	5	5	-5	-6
	1	5	5	-5	-6	0

Since $P(1) = 0$, 1 is a root of the polynomial.

$P(x) = (x - 1)(x^4 + 5x^3 + 5x^2 - 5x - 6)$.

By the Rational Root Theorem, the possible rational roots of the quartic quotient are ± 1, ± 2, ± 3, ± 6. Use synthetic division to test the roots.

1	1	5	5	-5	-6
		1	6	11	6
	1	6	11	6	0

Since $P(1) = 0$, 1 is a root of the polynomial with multiplicity 2.

$P(x) = (x - 1)(x - 1)(x^3 + 6x^2 + 11x + 6)$.

By the Rational Root Theorem, the possible rational roots of the cubic quotient are ± 1, ± 2, ± 3, ± 6. Use synthetic division to test the roots.

1	1	6	11	6
		1	7	18
	1	7	18	24

-1	1	6	11	6
		-1	-5	-6
	1	5	6	0

Since $P(-1) = 0$, -1 is a root of the polynomial.

$P(x) = (x - 1)(x - 1)(x + 1)(x^2 + 5x + 6)$

The last quotient is a quadratic that can be factored.

$P(x) = (x - 1)(x - 1)(x + 1)(x + 2)(x + 3)$

The roots are **1**, **-1**, **-2**, and **-3**. The root 1 has a multiplicity 2.

Math Reasoning

Analyze Suppose that a polynomial $P(x)$ has two real roots a and b. Could $P(x)$ be of degree 3?

Example 3 Application: Braking Distance

When a car needs to be brought to an immediate stop, it takes time for someone to step on the brake and time for the car to come to a complete stop. During these times, the car is still moving. The stopping distance is how far the car travels during this time frame. A reasonable mathematical model for the stopping distance d, in feet, based on the car's speed s, in miles per hour, is shown below. Use the Remainder Theorem to evaluate $d(20)$, $d(30)$, and $d(55)$.

$d(s) = 0.05s^2 + 2.2s$

SOLUTION Use synthetic division.

20	0.05	2.2	0
		1	64
	0.05	3.2	64

30	0.05	2.2	0
		1.5	111
	0.05	3.7	111

55	0.05	2.2	0
		2.75	272.25
	0.05	4.95	272.25

The stopping distances are $d(20) = 64$ ft, $d(30) = 111$ ft

$d(55) = 272.25$ ft.

Lesson Practice

Determine if $x = 7$ is a root of $P(x)$.

(Ex 1)

a. $P(x) = 2x^6 - 43x^5 + 75x^4 + 1765x^3 - 857x^2 - 22{,}542x - 30{,}240$

b. $P(x) = 9x^8 - 101x^7 + 195x^6 + 622x^5 - 621x^4 - 1565x^3 - 1947x^2 + 2772x + 2940$

(Ex 2)

c. Find all the rational roots of $P(x) = x^2 + 4x^2 + 1x - 6$.

(Ex 3)

d. The height of an object thrown at 40 feet per second can be modeled by $h(t) = -16t^2 + 40t$ where h is the height in feet and t is time in seconds. Use the remainder theorem to find the height of the object after 2 seconds.

Practice Distributed and Integrated

Solve.

1. (84) $x - \frac{6}{x} = 1$

2. (84) $\frac{x^2 + x - 6}{x + 1} = 0$

3. (84) $\frac{7x}{3x + 2} = 2$

4. (57) Does the graph of $y = b^x + k$, through the points $(3, -3)$ and $(5, 0)$, model exponential growth or decay?

***5.** (94) Solve $\frac{1}{x^3} \geq 5$. Round to the nearest thousandth.

6. (8) In a certain recipe, the amount of sugar is directly proportional to the amount of flour. If 3 cups of sugar are used with 8 cups of flour, how many cups of sugar are used with 12 cups of flour?

***7.** (95) **Error Analysis** Two students were evaluating $P(-4)$ for the polynomial $P(x)$, but they got different results. Which student made the mistake?

$$P(x) = x^6 + 4x^5 - x^4 - 6x^3 - 8x^2 + 2x + 8$$

Student A

$$\begin{array}{r|rrrrrrr} -4 & 1 & 4 & -1 & -6 & -8 & 2 & 8 \\ & & -4 & 0 & 4 & 8 & 0 & -8 \\ \hline & 1 & 0 & -1 & -2 & 0 & 2 & 0 \end{array}$$

$P(-4) = 0$

Student B

$$\begin{array}{r|rrrrrrr} 4 & 1 & 4 & -1 & -6 & -8 & 2 & 8 \\ & & 4 & 32 & 124 & 472 & 1856 & 7432 \\ \hline & 1 & 8 & 31 & 118 & 464 & 1858 & 7440 \end{array}$$

$P(-4) = 7440$

***8.** (94) **Analyze** When solving $\frac{(x - 4)(x + 6)}{(x + 2)(x - 3)} \geq 0$, which intervals will use strict inequalities ($<$ or $>$) and why?

***9.** (64) **Multi-Step** As of the 2000 U.S. census, Nevada had the fastest growing population of all the states, and Connecticut had nearly the slowest (47th out of 50). The table below is based on U.S. census statistics.

	Nevada	Connecticut
Population in 2000 (to nearest thousand)	1,998,000	3,406,000
Average annual increase (1990–2000)	5.21%	0.36%

a. Write a function ($y_1 = ab^t$), where y_1 represents Nevada's population t years after 2000.

b. Write a function ($y_2 = ab^t$), where y_2 represents Connecticut's population t years after 2000.

c. Write and solve an equation to predict the year in which Nevada's population will overtake Connecticut's population.

Simplify each of the following.

10. (81) $2 \ln e^x$

11. (81) $x \cdot \ln e^3$

12. (73) **Estimate** A researcher visits a wildlife preserve and marks endangered African elephants. He marks 6 elephants and releases them. On a return visit several months later, he comes across 16 elephants, and 4 of them are marked from the previous visit. Estimate the elephant population in the preserve.

13. (77) **Surveying** A surveyor finds that the lengths between three stakes on the ground are 125 feet, 182 feet, and 211 feet. He connects the stakes with string, forming a triangle. To the nearest tenth, what is the measure of the largest angle in this triangle?

14. (79) Determine the domain and range of the function $f(x) = \begin{cases} -x, & \text{if } x < 0 \\ 4, & \text{if } 0 \le x < 10. \\ 18, & \text{if } x \ge 10 \end{cases}$

15. (80) **Research** In a study of 413 men, the mean height was 174.1 centimeters and the standard deviation was 7 centimeters. Assuming that the distribution is normal, approximate the z-score of a man whose height is 182 centimeters.

16. (80) A set of values has a mean of 13 and a standard deviation of 0.625. Find the z-score of a value of 10.25.

17. (11) **Justify** Formulate a conjecture (statement) about the degree of a polynomial that is a sum or difference of polynomials compared to the degree of the polynomials that are added or subtracted to get that sum or difference. Give examples to justify your conjecture. (Hint: Name the polynomials P_1, P_2, P_3, etc. to make it easy to refer to them.)

18. (23) **Geometry** The area of a rectangular field is 100 meters. The field is 15 meters longer than it is wide. Find the length and width of the field.

***19.** (95) **Multiple Choice** Which of the following polynomials has $P(5) = 0$?

A $P(x) = 48x^6 + 212x^5 - 1098x^4 - 5298x^3 + 1174x^2 + 22{,}350x + 18{,}900$

B $P(x) = 48x^6 + 644x^5 + 2754x^4 + 2154x^3 - 12{,}974x^2 - 30{,}750x - 18{,}900$

C $P(x) = 48x^6 + 868x^5 + 6282x^4 + 23{,}238x^3 + 46{,}274x^2 + 46{,}950x + 18{,}900$

D $P(x) = 48x^6 - 268x^5 - 818x^4 + 4282x^3 + 6254x^2 - 14{,}790x - 18{,}900$

20. (1) **Multiple Choice** Identify which property or properties of real numbers are being demonstrated.

$$(2 \cdot 9) \cdot 5 = (2 \cdot 5) \cdot 9$$

A Commutative Property of Multiplication

B Associative Property of Multiplication

C Distributive Property

D Both A and B

***21.** (95) Write a possible polynomial $P(x)$ that fits the parameters $P(0) = 0$, $P(2) = 20$, and $P(-4) = 0$.

***22.** (87) Use the properties of logarithms to expand the expression $\ln (8x)^3 + \ln e\left(\frac{2x}{3}\right)$.

23. (13) **Predict** The table below shows the sales in dollars, y, made by a gift shop x years after it opened.

Years	2	5	8
Sales	$30,000	$75,000	$120,000

a. Plot the data on a coordinate grid and draw a line that fits the data.

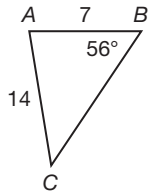
b. Use the line to predict the total sales 15 years after the shop opened.

24. (76) **Error Analysis** A student performs the following steps while finding the roots of $y = (x^2 - 7)(3x^2 + 4) - (x^2 - 7)(8x - 3)$. What is his mistake?

$$(x^2 - 7)(3x^2 + 4) - (x^2 - 7)(8x - 3) = (x^2 - 7)(3x^2 - 8x + 1)$$

***25.** (92) **Travel** A driver on the Pennsylvania turnpike turned on the cruise control at milepost 30, Warrendale, and kept it on until milepost 242, Harrisburg West, exactly 4 hours later. Show how to use an arithmetic sequence to find the speed at which the car traveled during those 4 hours.

26. (71) Solve $\triangle ABC$.



***27.** (91) Write the equation for the circle centered at (3, 8) with a radius of 4.

28. (1) John saves $125 each month. Use the Distributive Property to mentally calculate the amount of money that he will save in 18 months.

***29.** (90) **Graphing Calculator** Use a graphing calculator to graph the function $y = 3 \tan(2x - 2\pi) - 4$. Identify its period, undefined values, and phase shift.

***30.** (83) Write a quadratic equation whose root is -3.

LESSON
96

Using Polar Coordinates

Warm Up

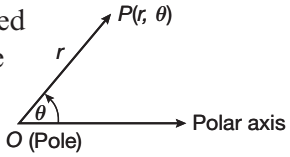
1. (46) **Vocabulary** The trigonometric ratio $\frac{\text{opposite side}}{\text{hypotenuse}}$ is called the _____ ratio.

2. (46) $\cos \pi =$ _____

3. (46) $\sin \pi =$ _____

New Concepts

A polar coordinate system in a plane is formed by a fixed point O, called the pole (or origin), and a ray, called the polar axis, whose endpoint is O. Each point in the plane can be assigned polar coordinates as follows.

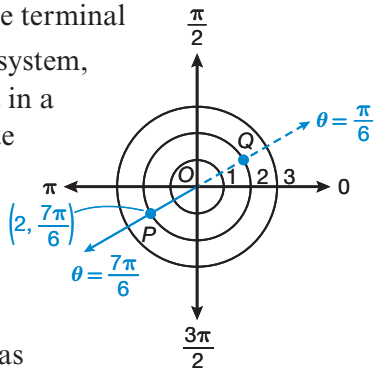


Polar Coordinates

Every point P in the polar coordinate system has an ordered pair of **polar coordinates** (r, θ), where

- r is the directed distance from O to P, and
- θ is the directed angle measure counterclockwise from the polar axis to $\overrightarrow{OP}$.

The point $P(2, \frac{7\pi}{6})$ lies 2 units from the pole on the terminal side of the angles $\frac{7\pi}{6}$. In a rectangular coordinate system, every point has exactly one ordered pair (x,y). But in a polar coordinate system, every point has an infinite number of ordered pairs. The point P shown at the right has coordinates $\left(2, \frac{7\pi}{6}\right)$. But the angle with measure $\frac{7\pi}{6} + 2\pi = \frac{19\pi}{6}$ also has terminal ray $\overrightarrow{OP}$, so point P also has coordinates $\left(2, \frac{19\pi}{6}\right)$.



Some of the coordinates of P: $\left(2, \frac{7\pi}{6}\right)$, $\left(2, \frac{19\pi}{6}\right)$, and $\left(-2, \frac{\pi}{6}\right)$

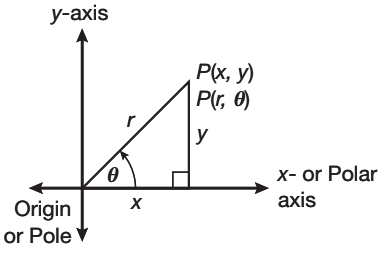
And because r is a *directed* distance, point P also has coordinates $\left(-2, \frac{\pi}{6}\right)$. To understand this, notice that point Q has coordinates $\left(2, \frac{\pi}{6}\right)$; the point with coordinates $\left(-2, \frac{\pi}{6}\right)$ is point P, which is 2 units from the pole, on the ray *opposite* $\overrightarrow{OQ}$.

Math Language

A **rectangular coordinate system** is also called a **Cartesian coordinate system.**

In general, the point (r, θ) can be represented as $(r, \theta \pm 2n\pi)$ or $(-r, \theta \pm (2n + 1)\pi)$, where n is any integer.

To relate the Cartesian and Polar coordinate systems, let the pole system coincide with the origin of a Cartesian (rectangular) coordinate system and let the polar axis coincide with the positive x-axis. Then $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, and $\tan\theta = \frac{y}{x}$.



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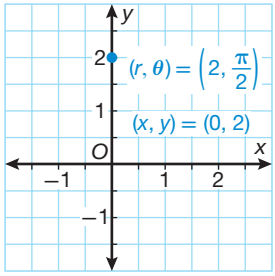
To convert coordinates, use the following equations.

Converting Coordinates	
Polar to Cartesian	**Cartesian to Polar**
$x = r\cos\theta$ $y = r\sin\theta$	$\tan\theta = \frac{y}{x}$ $r^2 = x^2 + y^2$

Example 1 Converting Polar Coordinates to Cartesian Coordinates

a. Convert $\left(2, \frac{\pi}{2}\right)$ to Cartesian coordinates.



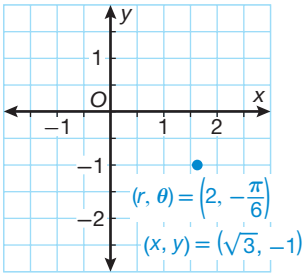
SOLUTION

$$x = r\cos\theta = 2\cos\frac{\pi}{2} = 2(0) = 0$$

$$y = r\sin\theta = 2\sin\frac{\pi}{2} = 2(1) = 2$$

The Cartesian coordinates are $(0, 2)$.

b. Convert $\left(2, -\frac{\pi}{6}\right)$ to Cartesian coordinates.



SOLUTION

$-\frac{\pi}{6}$ is equivalent to $-30°$.

$$x = r\cos\theta = 2\cos\left(-\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$y = r\sin\theta = 2\sin\left(-\frac{\pi}{6}\right) = 2\left(-\frac{1}{2}\right) = -1$$

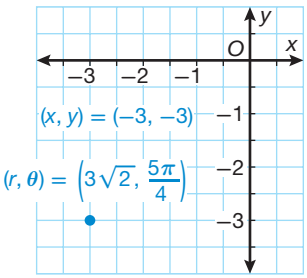
The Cartesian coordinates are $\left(\sqrt{3}, -1\right)$.

Example 2 Converting Cartesian Coordinates to Polar Coordinates

a. Convert $(-3, -3)$ to polar coordinates.

Hint

$\text{Tan}^{-1}\,1 = \frac{\pi}{4}$, and θ terminates in quadrant III, so one value of θ is $\frac{\pi}{4} + \pi = \frac{5\pi}{4}$.



SOLUTION

$\tan\theta = \frac{y}{x} = \frac{-3}{-3} = 1$

Since $(-3, -3)$ lies in quadrant III, θ terminates in quadrant III. Therefore, one value of θ is $\frac{5\pi}{4}$.

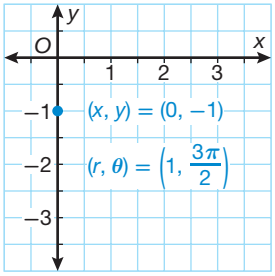
$$r^2 = x^2 + y^2 = (-3)^2 + (-3)^2 = 18$$

$$r = \sqrt{18} = 3\sqrt{2}$$

One ordered pair of polar coordinates is $\left(3\sqrt{2}, \frac{5\pi}{4}\right)$.

b. Convert $(0, -1)$ to polar coordinates.

SOLUTION



$\tan\theta = \frac{y}{x} = \frac{-1}{0}$, which is undefined. Since θ terminates on the negative y-axis, one value of θ is $\frac{3\pi}{2}$.

$r^2 = x^2 + y^2 = (0)^2 + (-1)^2 = 1$

$r = 1$

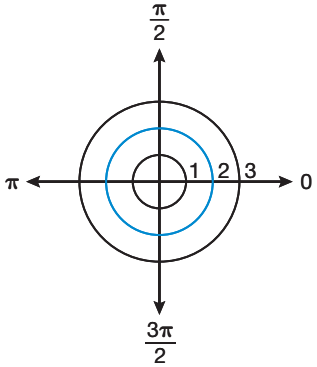
One ordered pair of polar coordinates is $\left(1, \frac{3\pi}{2}\right)$.

A **polar equation** is an equation for a curve with coordinates r and θ. There are two basic polar equations $r = k$ and $\theta = \alpha$, where k is a positive constant and α is a constant. The graph of $r = k$ forms a circle and the graph of $\theta = \alpha$ forms a line through the origin. These graphs can be identified by converting these types of basic polar equations into the Cartesian coordinate system.

Example 3 Graphing Polar Equations

a. Graph $r = 2$.

SOLUTION



A circle with all points 2 units from the pole.

Check Convert to a Cartesian equation.

$r = 2$

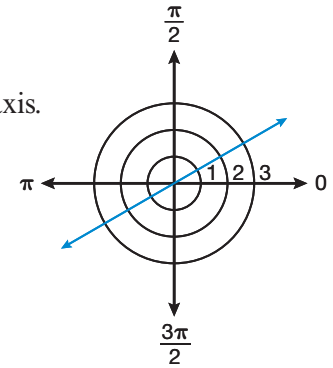
$r^2 = 4$ Square both sides.

$x^2 + y^2 = 4$ $r^2 = x^2 + y^2$

This is the Cartesian equation of the same circle.

b. Graph $\theta = \frac{\pi}{6}$.

SOLUTION



The graph consists of all points on the line that makes an angle of $\frac{\pi}{6}$ radian with the positive x-axis.

Check Convert to a Cartesian equation.

$\tan\theta = \frac{y}{x}$ Definition of $\tan\theta$.

$\tan\frac{\pi}{6} = \frac{y}{x}$ Substitute $\theta = \frac{\pi}{6}$.

$\frac{\sqrt{3}}{3} = \frac{y}{x}$ Evaluate $\tan\frac{\pi}{6}$.

$y = \frac{\sqrt{3}}{3}x$ Solve for y.

Math Reasoning

Analyze If a is a positive constant, describe the graphs of polar equations of the following forms.

$r = a$ and $\theta = a$

To graph a polar equation that has two variables, make a table and plot points. Below is a table of values that could be used to begin the graph of the polar equation $r = 2\theta$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$r(\theta) = 2\theta$	0	$\frac{\pi}{3} \approx 1.05$	$\frac{2\pi}{3} \approx 2.09$	$\pi \approx 3.14$

Graphing polar equations by plotting points is often a long process. The equation in the table above is graphed by calculator in Example 4a.

Graphing Calculator Tip

Make sure your calculator is in polar mode by pressing the mode key.

Example 4 Graphing Polar Equations on a Graphing Calculator

Graph each polar equation on a graphing calculator.

a. $r = 2\theta$

SOLUTION

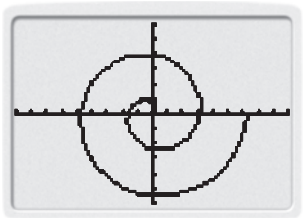
Use the window indicated below.

$\theta\text{min} = 0$ $\quad X\text{ min} = -36$

$\theta\text{max} = 4\pi$ $\quad X\text{ max} = 36$

$\theta\text{step} = \frac{\pi}{24}$ $\quad Y\text{ min} = -24$

$Y\text{ max} = 24$



b. $r = 1 - \sin\theta$

SOLUTION

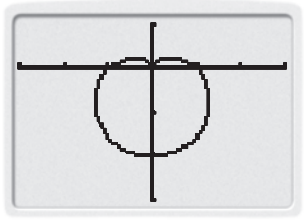
Use the window indicated below.

$\theta\text{min} = 0$ $\quad X\text{ min} = -3$

$\theta\text{max} = 2\pi$ $\quad X\text{ max} = 3$

$\theta\text{step} = \frac{\pi}{24}$ $\quad Y\text{ min} = -3$

$Y\text{ max} = 1$



c. $r = 5\cos 2\theta$

SOLUTION

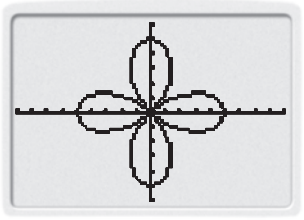
Use the window indicated below.

$\theta\text{min} = 0$ $\quad X\text{ min} = -9$

$\theta\text{max} = 2\pi$ $\quad X\text{ max} = 9$

$\theta\text{step} = \frac{\pi}{24}$ $\quad Y\text{ min} = -6$

$Y\text{ max} = 6$

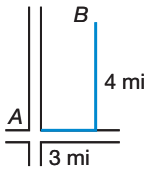


Math Language

The graphs in Example 4 have these names:
a. Archimedean spiral
b. cardioid
c. rose

Example 5 Application: Cell Phone Tower

A planned cell phone tower will service an area with a 5-mile radius. It is to be located 3 miles east and 4 miles north of a highway intersection. Write and graph a polar equation to show the boundary of the region that will be serviced. Place the highway intersection at the pole.



SOLUTION

The boundary is a circle with radius 5 and center (3, 4) on a Cartesian coordinate system, so the Cartesian equation is $(x-3)^2+(y-4)^2=5^2$.

$$(x-3)^2+(y-4)^2=25$$

$$x^2-6x+9+y^2-8y+16=25 \quad \text{Expand the binomials.}$$

$$x^2+y^2-6x-8y=0 \quad \text{Simplify.}$$

$$r^2-6r\cos\theta-8r\sin\theta=0 \quad x^2+y^2=r^2,\ x=r\cos\theta, \text{ and } y=r\sin\theta.$$

$$r(r-6\cos\theta-8\sin\theta)=0 \quad \text{Factor.}$$

$$r-6\cos\theta-8\sin\theta=0 \quad r\neq 0, \text{ so } r-6\cos\theta-8\sin\theta=0.$$

$$r=6\cos\theta+8\sin\theta \quad \text{Add } 6\cos\theta+8\sin\theta \text{ to both sides.}$$

The polar equation is $r=6\cos\theta+8\sin\theta$.
Graph the polar equation on a calculator.

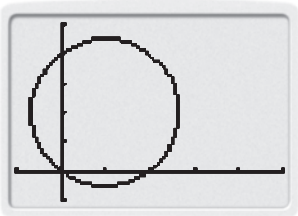
Use the window indicated below.

$\theta\text{min}=0$ $\quad X\text{min}=-3$

$\theta\text{max}=2\pi$ $\quad X\text{max}=15$

$\theta\text{step}=\frac{\pi}{24}$ $\quad Y\text{min}=-2$

$Y\text{max}=10$



Lesson Practice

a. (Ex 1) Convert $(3, \pi)$ to Cartesian coordinates.

b. (Ex 1) Convert $\left(1, \frac{2\pi}{3}\right)$ to Cartesian coordinates.

c. (Ex 2) Convert $\left(2\sqrt{3}, -2\right)$ to polar coordinates.

d. (Ex 2) Convert $(-4, 0)$ to polar coordinates.

e. (Ex 3) Graph $r=3$.

f. (Ex 3) Graph $\theta=\frac{3\pi}{4}$.

g. (Ex 4) Graph $r=\theta$ on a graphing calculator.

h. (Ex 4) Graph $r=1+\cos\theta$ on a graphing calculator.

i. (Ex 4) Graph $r=4\sin 3\theta$ on a graphing calculator.

j. (Ex 5) A planned cell phone tower will service an area with a 2-mile radius. It is to be located 2 miles north of an office building. Write and graph a polar equation to show the boundary of the region that will be serviced. Place the location of the building at the pole.

Practice Distributed and Integrated

1. (87) **Multi-Step** **a.** Use the change of base formula to convert $\log_{1000}(10x)^2$ to base 10.

b. Evaluate when $x = 100$.

2. (52) **Geography** The city of Madera, California, is 95 miles due southeast of Stockton, which is 95 miles due southwest of South Lake Tahoe. How far would you have to fly from Madera to South Lake Tahoe, and in what direction?

***3.** (96) Convert $\left(-3, \frac{\pi}{2}\right)$ to Cartesian coordinates.

4. (73) A taxicab company that wants to know if customers are satisfied has each driver survey three customers during the day. Is the sample biased? Explain.

***5.** (95) **Justify** Suppose that for polynomials $P_1(x)$ and $P_2(x)$, there are constants a and b such that $P_1(a) = P_2(b) = 0$. Is it correct to conclude that b is a root of $P_1(x)$ and a is a root of $P_2(x)$? Explain.

Solve and graph the following compound inequalities.

6. (10) $3t > 18$ or $t - 3 < 0$

7. (10) $-(h - 2) > 7$ or $-8 \geq -2h$

8. (1) **Statistics** The table shows the number of home runs hit by a softball player over a three-year period. Use properties of real numbers to mentally calculate the average annual home runs that she hit during this period.

	Year 1	Year 2	Year 3
Home runs	8	15	17

Determine the domain and range.

9. (75) $y = \frac{3}{4}x^{\frac{1}{3}} - 1$

10. (75) $f(x) = \sqrt{x + 2} + \sqrt[3]{x - 1}$

***10.** (93) If \$1,000 is invested at 7% compounded continuously, how long would it take for the value of the investment to reach \$3,000?

11. (74) **Juggling** A juggler tosses a ball into the air. The ball leaves the juggler's hand 4 feet above the ground and has an initial velocity of 40 feet per second. The juggler catches the ball when it falls back to a height of 3 feet. How long is the ball in the air?

12. (77) In $\triangle LMN$, $LM = 14$, $MN = 8$, and $m\angle M = 84°$. Find LN.

13. (88) **Simple Interest** The simple interest formula is $A = P + Prt$, where P is the principal, or original amount invested, r is the annual simple interest rate, and A is the value of the investment at the end of t years. Solve the formula for t. Then determine how many years it will take for an investment to double in value at a 5% annual simple interest rate.

14. (82) **Geometry** Derive the function that can be used to find the area of an isosceles triangle with base a and congruent sides measuring b. Each base angle measures θ. Express in simplified form.

*15. (91) **Multiple Choice** Which equation represents a circle centered at (–5, 4) with a radius of 3?

A $(x-5)^2+(y+4)^2=3$ **B** $(x-5)^2+(y+4)^2=9$

C $(x+5)^2+(y-4)^2=3$ **D** $(x+5)^2+(y-4)^2=9$

16. (83) **National Parks** The Surprise Arch, in Arches National Park, has a span of about 60 feet and a height of about 50 feet. Write a quadratic function to approximate the arch.

*17. (95) Find the roots of $y=x^9-x^7-8x^6-16x^5+8x^4+16x^3+128x^2-128$.

18. (84) **Formulate** Solve $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$ for c.

19. (85) Find the roots of $f(x)=0.15x^3-2.85x^2+14.85x-12.15$ using the Rational Root Theorem.

*20. (86) **Graphing Calculator** Use a graphing calculator to graph $y=\cos(7x)$ and compare its period to that of the parent function.

21. (89) **Multiple Choice** Which is not a solution of $\begin{array}{l} y \le -x^2+4x+8 \\ y < -3x+1 \end{array}$?

A (0, 0) **B** (0, −3) **C** (1, −6) **D** (2, −2)

22. (90) Let $f(x)=\tan(5x+5)$ and $g(x)=3x+5$. Find the period of $f(g(x))$.

*23. (84) A kayaker paddles with the current for 7 miles, turns around, and paddles against the current for 7 miles. The average paddling rate is 3 miles per hour. Write an inequality to represent the speed of the current if the total time is less than 7 hours.

24. (81) **Error Analysis** To rewrite $\ln(ex)^3$, a student wrote $\ln(ex)^3=\ln(3\cdot(ex))=\ln 3+\ln(ex)=\ln 3+\ln e+\ln x=\ln 3+\ln x+1$. What is the error? Rewrite $\ln(ex)^3$ as a sum or difference of terms correctly.

*25. (96) **Write** Explain why every point in a polar coordinate system has more than one set of polar coordinates.

*26. (92) **Savings** During week 1, a customer opened a bank account with $450. Each week thereafter, the customer deposits $50. How much will the customer have in the account after the deposit is made during week 52?

27. (78) **Error Analysis** A student says that the roots of the equation $4x^3-36x=0$ are -3 and 0. What is the error?

28. (23) **Verify** Use FOIL to verify the difference of two squares formula $a^2-b^2=(a+b)(a-b)$.

29. (80) **Analyze** How are z-scores of –0.5 and 0.5 alike? How are they different?

*30. (96) **Cell Phone Tower** A planned cell phone tower will service an area with a 6 mile radius. It is to be located 6 miles west of a small town. Write a polar equation of the boundary of the region that will be serviced. Place the location of the town at the pole. Graph the equation on a calculator.

LESSON
97

Finding Geometric Sequences

Warm Up

1. (92) **Vocabulary** Each number in a sequence is called a _____.
2. (92) True or False: The sequence {2, 4, 8, 16} is arithmetic.
3. (2) Simplify $5(-3)^4$.

New Concepts

The sequence {3, 6, 12, 24, 48} is not arithmetic because the differences between consecutive terms are not the same. However, the ratio of any two consecutive terms is the same.

$$\frac{48}{24} = 2, \frac{24}{12} = 2, \frac{12}{6} = 2, \frac{6}{3} = 2$$

This makes the sequence geometric. In a **geometric sequence,** the ratio of successive terms is a constant other than 1. This constant is called the **common ratio.** It is found by dividing a term by its previous term and can be used to extend an infinite sequence.

Example 1 Identifying and Using the Common Ratio

Find the common ratio of each geometric sequence and use it to find the next three terms.

a. $6, 3, \frac{3}{2}, \frac{3}{4}, \ldots$

SOLUTION Divide a term by a previous term: $3 \div 6 = \frac{1}{2}$.

Find the next three terms by multiplying by $\frac{1}{2}$:
$\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}, \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16}$, and $\frac{3}{16} \cdot \frac{1}{2} = \frac{3}{32}$.
The next three terms are $\frac{3}{8}, \frac{3}{16}$, and $\frac{3}{32}$.

b. $-4, 8, -16, 32, \ldots$

SOLUTION Divide a term by a previous term: $32 \div -16 = -2$.

Find the next three terms by multiplying by -2:
$32 \cdot -2 = -64, -64 \cdot -2 = 128$, and $128 \cdot -2 = -256$.

The next three terms are -64, 128, and -256.

Hint

For $a \neq 0$, $a^0 = 1$.

An explicit formula can be determined to find the nth term of a geometric sequence.

For the geometric sequence {2, 6, 18, 54, 162, …},

$a_1 = 2(3)^0, a_2 = 2(3)^1, a_3 = 2(3)^2, a_4 = 2(3)^3, a_5 = 2(3)^4$.

Math Reasoning

Formulate What is a recursive formula for a geometric sequence?

Geometric Sequences

The nth term of a geometric sequence is given by

$$a_n = a_1 r^{n-1},$$

where r is the common ratio.

Example 2 Finding the nth Term of a Geometric Sequence

a. Find the tenth term of the geometric sequence 3, 12, 48, 192, …

SOLUTION Find r: $12 \div 3 = 4$. Then use the formula $a_n = a_1 r^{n-1}$.

$a_{10} = 3(4)^{10-1}$ Substitute 10 for n, 3 for a_1, and 4 for r.

$a_{10} = 3(262{,}144)$ Simplify.

$a_{10} = 786{,}432$

b. Find the 8th term of the geometric sequence 810, −270, 90, −30, ...

SOLUTION Find r: $-30 \div 90 = -\frac{1}{3}$. Then use the formula $a_n = a_1 r^{n-1}$.

$a_8 = 810\left(-\frac{1}{3}\right)^{8-1}$ Substitute 8 for n, 810 for a_1, and $-\frac{1}{3}$ for r.

$a_8 = 810\left(-\frac{1}{2187}\right)$ Simplify.

$a_8 = -\frac{810}{2187} = -\frac{10}{27}$

The formula $a_n = a_m r^{n-m}$ can be used to find a term when given either a term and the common ratio or any two terms of the sequence.

Example 3 Finding the nth Term Given a Term and r

The fifth term of a geometric sequence is 567. The common ratio is 3. Find the ninth term.

SOLUTION In the formula $a_n = a_m r^{n-m}$, replace both instances of m with 5, replace n with 9, and replace r with 3.

$a_9 = a_5 r^{9-5}$

$a_9 = 567(3)^4$

$a_9 = 567(81)$

$a_9 = 45{,}927$

When r must first be found, there may be two cases, depending on whether an even root or an odd root must be taken. This could lead to two possible values for the nth term.

Online Connection
www.SaxonMathResources.com

Example 4 Finding the *n*th Term Given Any Two Terms

a. Find a_1 of a geometric sequence given that $a_4 = 32$ and $a_9 = 1024$.

SOLUTION In the formula $a_n = a_m r^{n-m}$, replace m with 4 and replace n with 9.

$$a_9 = a_4 r^{9-4}$$

$1024 = 32r^5$ Substitute a_9 with 1024 and a_4 with 32.

$32 = r^5$ Solve for r.

$2 = r$ Take the fifth root of each side.

Now find a_1. Use either of the known terms.

$32 = a_1(2)^{4-1}$ Substitute 32 for a_n and 4 for n.

$32 = a_1(8)$ Solve for a_1.

$4 = a_1$

The first term of the sequence is 4.

Check

Check by using a_1 to find the ninth term: $a_9 = 4(2)^8 = 4(256) = 1024$.

b. Find a_7 of a geometric sequence given that $a_3 = -36$ and $a_5 = -324$.

SOLUTION In the formula $a_n = a_m r^{n-m}$, replace m with 3 and replace n with 5.

$$a_5 = a_3 r^{5-3}$$

$-324 = -36r^2$ Substitute a_5 with -324 and a_3 with -36.

$9 = r^2$ Solve for r.

$\pm 3 = r$ Take the square root of each side.

Now find a_7. Consider both $r = 3$ and $r = -3$.

Case 1: $r = 3$	Case 2: $r = -3$
$a_7 = a_3(3)^{7-3}$	$a_7 = a_3(-3)^{7-3}$
$a_7 = -36(3)^4$	$a_7 = -36(-3)^4$
$a_7 = -36(81)$	$a_7 = -36(81)$
$a_7 = -2916$	$a_7 = -2916$

In this example, both possible values of r give the same value for a_7. The seventh term of the sequence is -2916.

Check

	a_3	a_4	a_5	a_6	a_7
$r = 3$	-36	-108	-324	-972	-2916
$r = -3$	-36	108	-324	972	-2916

Math Reasoning

Verify Use the formula to show that a_4 has two possible values.

Example 5 Application: Salary

An employee's salary is structured so that he earns $43,400 in the first year with a 3.5% raise each year thereafter. How much can the employee expect to earn in his fifteenth year?

SOLUTION The salaries each year form a geometric sequence where $r = 1.035$. Substitute 15 for n, 43,400 for a_1, and 1.035 for r in $a_n = a_1 r^{n-1}$.

$$a_{15} = 43{,}400(1.035)^{15-1}$$

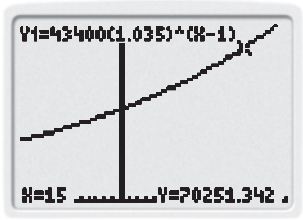
$$a_{15} = 70{,}251.34 \qquad \text{Round to the hundredths place.}$$

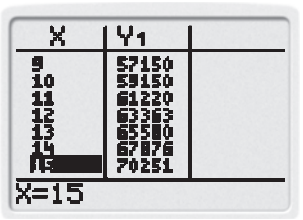
The employee's salary in the fifteenth year will be $70,251.34.

Math Reasoning

Justify Why is r equal to 1.035?

Check Use a graphing calculator.





Lesson Practice

Find the common ratio of each geometric sequence and use it to find the next three terms.

a. (Ex 1) 2, 0.2, 0.02, 0.002, …

b. (Ex 1) $-200, -40, -8, -\frac{8}{5}\ldots$

c. (Ex 2) Find the ninth term of the geometric sequence $-1, 5, -25, 125, \ldots$

d. (Ex 2) Find the eleventh term of the geometric sequence $16, 4, 1, \frac{1}{4}, \ldots$

e. (Ex 3) The fourth term of a geometric sequence is $1\frac{1}{4}$. The common ratio is $-\frac{1}{2}$. Find the eighth term.

f. (Ex 4) Find a_1 of a geometric sequence given that $a_3 = 128$ and $a_6 = 8192$.

g. (Ex 4) Find a_9 of a geometric sequence given that $a_4 = 24$ and $a_6 = 96$.

h. (Ex 5) An employee's salary is structured so that he earns $35,890 in the first year with a 4.25% raise each year thereafter. How much can the employee expect to earn in his tenth year?

Practice Distributed and Integrated

***1.** (88) Given $V = \pi r^2 h$, find h.

2. (73) **Write** A survey asks students, "Who would you vote for in the next school election, the relatively unknown sophomore Eileen Johnson, or the popular Zack Jennings?" Explain the bias in the question.

Simplify the following expressions.

3. (72) $8^{\log_8 x}$

4. (72) $\log_3 81^x$

*5. (97) **Multi-Step** The value of a car in its first year was $18,900. Each year thereafter, the value was 85% of what it was the previous year. The values each year form a geometric sequence.

a. Write a formula to find the value of the car in the nth year.

b. Find the value of the car in year 6.

6. (Inv 6) What number can be added to both sides of the equation $x^2 + 18x + 36 = 0$ to make it a perfect square?

7. (70) **Analyze** How can you tell that $4 + \sqrt{x} = 1$ will have no solutions?

Find each of the following matrix operations for $A = \begin{bmatrix} -6 & 2 \\ -1 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 11 & 2 \\ 8 & 3 \end{bmatrix}$.

8. (5) Find $A + D$.

9. (5) Find $D - 3A$.

*10. (97) Find the fifteenth term of the geometric sequence 2, 6, 18, …

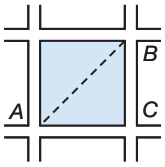
11. (74) **Error Analysis** Which student made an error in determining the number of solutions of $-3x^2 + 6x = 4$? What was the error?

Student A:		**Student B:**	
	$b^2 - 4ac$		$b^2 - 4ac$
	$6^2 - 4(-3)(4)$		$6^2 - 4(-3)(-4)$
	$36 + 48$		$36 - 48$
	84		-12
	2 real solutions		2 complex solutions

*12. (96) **Estimate** Write approximate polar coordinates for the point with Cartesian coordinates (6, 6.2). Explain your method.

13. (62) **Geometry** A student is told to draw a rectangle so that the perimeter is 48 units and the area is 150 square units. The equation $24x - x^2 = 150$ gives the values of x that meet these criteria where x is the length of the rectangle. Solve the equation. Describe your findings.

*14. (96) **City Planning** A park lawn is a square. A diagonal sidewalk is planned, shown as $\overline{AB}$ in the diagram. Using point A as the pole and $\overrightarrow{AC}$ as the polar axis, write a polar equation of the line that contains $\overline{AB}$.



15. (22) **Coordinate Geometry** Graph {(2, 4), (5, 6), (–1, 0), (7, 10)}. Determine if this relation is a function. Determine if the graph is continuous, discontinuous, and/or discrete.

16. (75) **Model** Graph the square root function and its inverse. $y = \frac{1}{3}\sqrt{x} - 4$

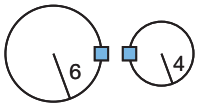
17. (84) **National Highways** In Texas, the speed limit on state highways at night is 5 miles per hour less than what it is during the day. A driver drove 350 miles during the day and 195 miles at night, with the cruise control set to the appropriate speed limit. The total driving time was 8 hours. Write and solve a rational equation to find the speed limit during the day.

*18. (97) **Multiple Choice** The second term of a geometric sequence is 16 and the fourth term is 1. What is the common ratio?

A $\frac{1}{16}$ **B** $\pm\frac{1}{16}$ **C** $\frac{1}{4}$ **D** $\pm\frac{1}{4}$

19. (85) **Analyze** Consider the polynomial function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. Suppose that the polynomial is divisible by $(x - 1)$. Use synthetic division to show that $a + b + c + d + e = 0$.

20. (86) **Physics** Two wheels spin in the same direction. The larger wheel completes a cycle in 3 seconds and the smaller wheel completes a cycle in 5 seconds. The two squares attached to the wheels are facing each other at time $t = 0$. How many seconds elapse before they are back in sync?



21. (92) Find the common difference of the arithmetic sequence $-62, -67, -72, -77, \ldots$

*22. (89) **Graphing Calculator** Graph $\begin{cases} 4y + 12 > x^2 \\ -\frac{1}{2}x \geq y \\ y \geq -2 \end{cases}$ on your graphing calculator. Name two points in the solution set.

23. (82) **Multiple Choice** Which of the following has a period of 7π?

A $y = \cos(7x)$ **B** $y = \cos\left(\frac{2}{7}x\right)$ **C** $y = \cos(3.5x)$ **D** $y = \cos\left(\frac{x}{7}\right)$

24. (82) Given $g(x) = 0.5\sin(2x)$, identify the amplitude and period.

25. (80) A set of values has a mean of 15 and a standard deviation of 1.5. Find the percent of values above 18.

26. (79) Evaluate the piecewise function $f(x) = \begin{cases} 9 - 5x, \text{ if } x < -0.1 \\ 2, \quad \text{if } -0.1 \leq x \leq 2 \\ x^3 - x, \text{ if } 2 < x \leq 6 \\ 3 - x^2, \text{ if } x > 6 \end{cases}$ for $x = -0.4$, $x = 0$, and $x = 6$.

*27. (93) **Medicine** Cobalt-60 is an isotope that is used in radiation therapy for cancer patients. It has a half-life of about 5.3 years. How long will it take for 1 gram of cobalt-60 to decay to 0.9 gram? (Use the natural decay function $N(t) = N_0e^{-kt}$.)

28. (77) **Boating** A boat leaves a dock and travels in a straight path for 6 miles at an angle 15° north of east. Another boat leaves the same dock and travels in a straight path for 11 miles at an angle 65° south of east. If the second boat puts down an anchor and the first boat travels directly toward the second, estimate the time it will take the first boat to reach the second boat if the first boat travels at a constant rate of 8 miles per hour.

*29. (91) The endpoints of a diameter of a circle are located at $(-10, -2)$ and $(2, -2)$. Write an equation for this circle.

30. (87) Convert $\log_6 (5x)^4$ to base e. Then evaluate when $x = 3$.

LESSON
98

Making Graphs and Using Equations of Ellipses

Warm Up

1. (91) **Vocabulary** The graph of $x^2 + y^2 = r^2$ is a _____.

2. (91) What figure does the equation $(x - 1)^2 + (y - 1)^2 = 5^2$ form?

3. (91) Where does $x^2 + y^2 = 7^2$ touch the x-axis?

4. (2) If $c^2 = a^2 - b^2$, which of these is the value of c if $a = 13$ and $b = 5$?

A $c = \pm 12$ **B** $c = \pm\sqrt{194}$

C $c = \pm 144$ **D** $c = \pm 194$

5. (2) If $c^2 = a^2 - b^2$, find c if $a = 4$ and $b = 3$.

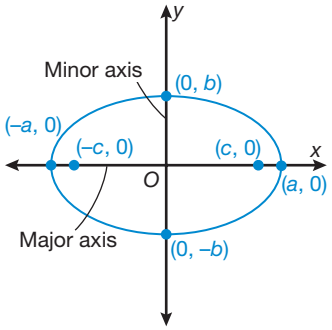
New Concepts

An **ellipse** is the set of all points P in a plane such that the sum of the distance from P to two fixed points F_1 and F_2 is constant. The two fixed points, F_1 and F_2, are called the **foci**. An ellipse has two axes. The **major axis** is the longer axis of the ellipse and passes through the foci. The endpoints of the major axis are the **vertices** of the ellipse. The **minor axis** is the shorter axis of the ellipse. and its endpoints are the **co-vertices**. The major and minor axes are perpendicular and their point of intersection is the center of the ellipse.

Math Reasoning

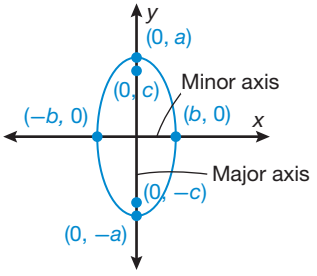
Model Take a length of string and thumbtack both ends to a sheet of paper so that the string is not taut. Take a pencil and use the string as a guide to drawing an ellipse. What role do the thumbtacks play?

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Vertices: (±a, 0)
Covertices: (0, ±b)
Foci: (±c, 0)

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$



Vertices: (0, ±a)
Covertices: (±b, 0)
Foci: (0, ±c)

Standard Form of the Equation of an Ellipse: Center at (0, 0)	
Major Axis Horizontal	Major Axis Vertical
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
Vertices: $(a, 0), (-a, 0)$	Vertices: $(0, a), (0, -a)$
Foci: $(c, 0), (-c, 0)$	Foci: $(0\ c\ ), (0, -c)$
Co-vertices: $(0, b), (0, -b)$	Co-vertices: $(b, 0), (-b, 0)$

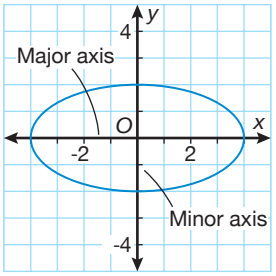
Online Connection
www.SaxonMathResources.com

Hint

a always corresponds to the major (longer) axis of the ellipse.

Example 1 Writing an Equation of an Ellipse

Write an equation in standard form for the ellipse with center (0, 0).



SOLUTION

Step 1: Choose the appropriate standard form.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ The horizontal axis is longer.

Step 2: Identify the values of a and b.

$a = 4$ The vertex (4,0) gives the value of a.

$b = 2$ The co-vertex (0,2) gives the value of b.

Step 3: Write the equation.

$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$ Substitute the values of a and b into the equation.

The equation of the ellipse is $\mathbf{\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1.}$

There is an important relationship among a, b, and c: $c^2 = a^2 - b^2$. This relationship can be used to find the foci of an ellipse.

The eccentricity, e, of an ellipse is a measure of its curvature. Eccentricity is defined as $\frac{c}{a}$, where c is the distance from the center to a focus and a is the distance from the center to a vertex.

Example 2 Graphing an Ellipse Centered at the Origin

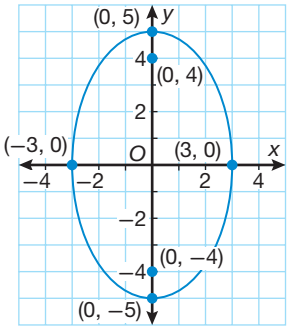
Graph the following equation. Find the vertices, co-vertices and the foci. Calculate the eccentricity e.

$25x^2 + 9y^2 = 225$

SOLUTION

Step 1: Write the equation in standard form by dividing both sides by the constant term.

$\frac{x^2}{9} + \frac{y^2}{25} = 1$ $\frac{y^2}{5^2} + \frac{x^2}{3^2} = 1$



Step 2: Find the values of a, b, and c.

From the equation, $a = 5$ and $b = 3$.

Use the equation $c^2 = a^2 - b^2$ to find c.

$c^2 = 5^2 - 3^2$

$c = 4$

The vertices are $\mathbf{(0, \pm 5)}$, the co-vertices are $\mathbf{(\pm 3, 0)}$, and the foci are $\mathbf{(0, \pm 4)}$.

The eccentricity is $\frac{c}{a} = \mathbf{\frac{4}{5}}$.

Ellipses can be translated so that the center is not the origin.

Standard Form of the Equation of an Ellipse: Center at (h, k)	
Major Axis Horizontal	Major Axis Vertical
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$
Vertices: $(h+a, k), (h-a, k)$	Vertices: $(h, k+a), (h, k-a)$
Foci: $(h+c, k), (h-c, k)$	Foci: $(h, k+c), (h, k-c)$
Co-vertices: $(h, k+b), (h, k-b)$	Co-vertices: $(h+b, k), (h-b, k)$

Example 3 Graphing Ellipses Not Centered at the Origin

Graph the following equation. Find the center, vertices, co-vertices and the foci. Calculate the eccentricity e.

$$9(x-2)^2+64(y-5)^2=576$$

Math Reasoning

Analyze Why is the equation of an ellipse not a function?

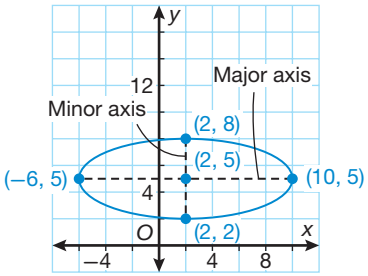
SOLUTION

Step 1: Write the equation in standard form by dividing both sides by the constant term.

$$\frac{(x-2)^2}{64}+\frac{(y-5)^2}{9}=1 \qquad \frac{(x-2)^2}{8^2}+\frac{(y-5)^2}{3^2}=1$$

Step 2: Find the values of a, b, c, h, and k.

The graph has a horizontal axis so, from the equation, $a = 8$, $b = 3$, $h = 2$, and $k = 5$.



$a = 8$
$b = 3$
$c = \sqrt{8^2 - 3^2} = 7.416$
$e = \sqrt{1 - \frac{3^2}{8^2}} = 0.927$

Use the equation $c^2 = a^2 - b^2$ to find c.

$c^2 = 8^2 - 3^2$

$c = \sqrt{55} \approx 7.416$

The eccentricity is $\frac{c}{a} = \frac{\sqrt{55}}{8} \approx \mathbf{0.927}$.

Step 3: Find the translated center, vertices, co-vertices, and foci.

Center: **(2, 5)**

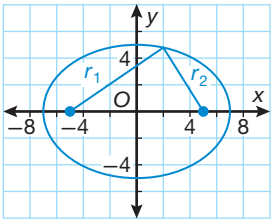
Vertices: $(2 + 8, 5), (2 - 8, 5) \longrightarrow$ **(10, 5), (−6, 5)**

Foci: $\mathbf{(2 + \sqrt{55}, 5), (2 - \sqrt{55}, 5)}$

Co-vertices: $(2, 5 + 3), (2, 5 - 3) \longrightarrow$ **(2, 8), (2, − 2)**

The geometric definition of an **ellipse** is the locus of points such that the sum of distances from any point on the ellipse to the foci is constant.

$$r_1 + r_2 = 2a$$



Example 4 Application: Planetary Orbits

All the planets have elliptical orbits, but each has a different eccentricity. The table lists each planet's orbital eccentricity. Write the equation of an ellipse centered at the origin that can be used as a model of Earth's orbit.

Planet	e
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.056
Uranus	0.047
Neptune	0.009

SOLUTION Use the eccentricity to find the relationship between a and b.

$$e = 0.017 = \sqrt{1 - \frac{b^2}{a^2}}$$

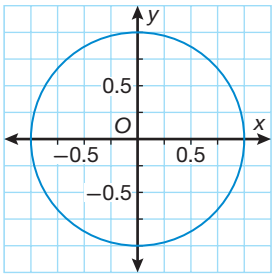
$$0.0003 = 1 - \frac{b^2}{a^2}$$

$$\frac{b^2}{a^2} = 0.9997$$

$$b = 0.9998a$$

A scale model of Earth's orbit maintains this relationship between a and b. So, let $a = 1$ to get this equation and graph.

$$\frac{x^2}{1^2} + \frac{y^2}{0.9998^2} = 1$$

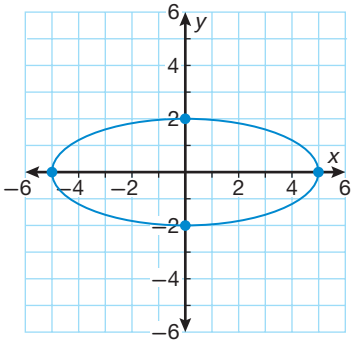


Math Reasoning

Analyze Why does the Earth's orbit look more circular than oval?

Lesson Practice

a. *(Ex 1)* Write an equation in standard form for the ellipse with center (0, 0).



Graph the following equations. Find the center, vertices, co-vertices and the foci. Calculate the eccentricity e.

b. *(Ex 2)* $16x^2 + 36y^2 = 576$

c. *(Ex 3)* $9(x - 2)^2 + 64(x - 5)^2 = 576$

d. *(Ex 4)* Write the equation of an ellipse centered at the origin that can be used as a model of Mars's orbits.

Practice Distributed and Integrated

*1. (14) **Graphing Calculator** Calculate the determinant of $\begin{bmatrix} -5 & 2 & 11 \\ 4 & -1 & 0 \\ 19 & 1 & 1 \end{bmatrix}$ using a graphing calculator.

*2. (98) **Generalize** How many ellipses have an eccentricity of 0.5 and cross the minor axis at (0, 2)?

*3. (98) **Optics** An elliptical mirror is modeled by the equation $\frac{x^2}{15^2} + \frac{y^2}{10^2} = 1$. The property of an elliptical mirror is that if you shine a beam of light from a focal point to the mirror, the light is reflected toward the other focal point. What is the length of the two light beams?

4. (71) Find the missing side lengths of $\triangle ABC$. $A = 73°$; $a = 18$; $b = 11$

*5. (98) Write the equation of the ellipse given by $2x^2 + 3y^2 = 6$ in standard form.

6. (66) **Packaging** The manager of a company that sells pasta is looking into a new container shape of a rectangular prism whose base is square and whose height is 5 inches greater than the length of one side of the base. If the volume is 28 cubic inches, what is the length of one side of the base? What is the height?

7. (2) **Geometry** A rectangle has length $4c$ and width d. Another rectangle has length 16 and width $(2 + 5c)$. Write a simplified expression of their combined area.

*8. (97) **Population** The estimated population of the District of Columbia from 2005 to 2006 decreased by about -0.089%. Assuming that the percent of decrease stays the same, the populations each year form a geometric sequence. Show how to use the formula for the nth term of a geometric sequence to find the population of DC in 2012 given that the estimated population in 2005 was 582,049. (Hint: Call 2005 year 1.)

9. (87) **The Ehrenberg Relation** The Ehrenberg relation $\log(w) = 0.8h + 0.4 \pm 0.04$ shows the relationship between height h and weight w in young children ages 5 through 13. Solve for h and determine the height, in meters, of a child who weighs 40 kilograms.

10. (13) **Write** Explain how to graph $3x + 5y = -35$ using slope and intercept. Be sure to show each step.

*11. (97) **Error Analysis** Explain and correct the error a student made in finding the common ratio of the geometric sequence 270, 90, 30, 10, …

$$r = \frac{30}{10} = 3$$

12. (82) Given $h(x) = 6\sin(8x - 9)$, identify the amplitude and period.

13. (76) Find the roots of the polynomial function.
$f(x) = (x + 8)(3x^2 - 6) - (x + 8)(11x - 4)$

*14. (95) **Justify** Suppose that for polynomials $P_1(x)$ and $P_2(x)$ there are constants a and b such that $P_1(a) \neq 0$ and $P_2(b) \neq 0$. Is it correct to conclude that b is *not* a root of $P_1(x)$ and a is *not* a root of $P_2(x)$? Explain.

15. (86) **Acoustics** A guitar string is strummed twice, which creates two waves along the string. The first wave is reflected back and meets the second wave. If the waves are modeled by the equations below, what happens when the waves meet? (Hint: When one wave's crest coincides with another wave's trough, they tend to cancel each other out. This called destructive interference.)

$y = \cos\left(x - \frac{\pi}{2}\right)$ $\qquad$ $y = \cos\left(x + \frac{\pi}{2}\right)$

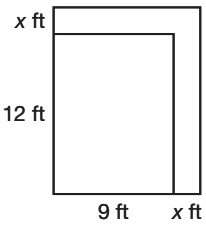
16. (88) **Error Analysis** A student attempted to solve the formula $V = \frac{1}{3}\pi r^3$ for r and wrote $r = \sqrt[3]{\frac{V}{3\pi}}$. What is the error? Solve the formula for r correctly.

Simplify.

17. (3) $\dfrac{(x^{-2}yp)^{-3}(x^0yp)^2}{(2x^2)^{-2}}$

18. (3) $-\dfrac{3x^2y}{xx} + \dfrac{2x^{-2}x^4}{y^{-1}x^2} - \dfrac{5xy^2}{xy}$

19. (89) **Multi-Step** A rectangular window has dimensions of 9 feet by 12 feet. The homeowner wants to increase the window size by lengthening two sides by the same amount as shown. The area of the window is to remain less than 300 square feet.



a. Write an inequality to find the range of the amounts that can be added to the two sides.

b. Solve the inequality.

20. (90) **Model** For what values of θ is $y = \dfrac{\sin(x)}{\cos(x - \theta)}$ the equivalent of $y = \tan(\theta)$?

Evaluate each piecewise function for $x = 3$ and $x = 0$.

21. (79) $f(x) = \begin{cases} x^2 - 5, & \text{if } x \le 0 \\ x^2, & \text{if } 0 < x < 3 \end{cases}$

22. (79) $f(x) = \begin{cases} \frac{1}{2}, & \text{if } x < 1 \\ x, & \text{if } 1 \le x < 10 \\ -x, & \text{if } x \ge 10 \end{cases}$

23. (83) Write a quadratic equation whose roots are $5i$ and $-5i$.

Identify all the real roots.

24. (85) $x^3 + 10x^2 + 17x = 28$

25. (85) $x^3 - 343 = 0$

***26.** (94) **Multiple Choice** When solving $\frac{x^2 + 5x - 6}{x^2 - 12x + 32} > 0$ by using a sign table, for how many intervals will a test point need to be chosen?

A 3 $\qquad$ **B** 4 $\qquad$ **C** 5 $\qquad$ **D** 6

Simplify.

27. (48) $\dfrac{\frac{1}{2} - \frac{4}{15}}{\frac{5}{6} + \frac{1}{9}}$

28. (48) $\dfrac{\frac{1}{x+1} - \frac{1}{x-1}}{\frac{x}{x^2 - 1}}$

29. (84) **Write** How and why is the LCD used to solve a rational equation?

***30.** (96) **Probability** A point is randomly chosen inside the graph of the polar equation $r = 3$. What is the probability that the chosen point is at least 1 unit away from the pole?

LESSON
99

Using Vectors

Warm Up

1. (62) **Vocabulary** For the complex number $a + bi$, the a term is known as the _____.

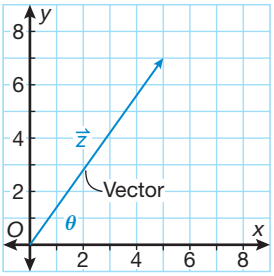
2. (9) True or False: A 2×3 matrix can be multiplied by another 2×3 matrix.

3. (41) What is the distance between the points $(3, 7)$ and $(-2, 3)$?

New Concepts

A **vector** is a quantity that has magnitude, orientation, and direction.

There are several ways of expressing vector quantities, including complex numbers and matrices. Vectors can be graphed on Cartesian coordinate systems as well as complex coordinate systems.



For vector $\vec{z}$:
$|\vec{z}| \rightarrow$ magnitude
$\theta \rightarrow$ orientation

Example 1 Graphing Vectors

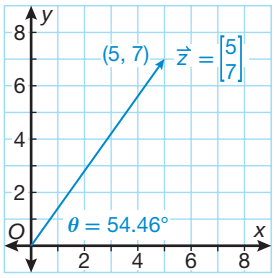
a. Graph vector $\vec{z}$ whose endpoints are the origin and $(5, 7)$ on the Cartesian and complex planes. Express the Cartesian version as a coordinate matrix and the complex version as a complex number. Find the orientation of the vector.

Math Reasoning

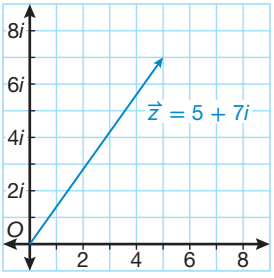
Verify How do you know that the angle of orientation is 54.46°?

SOLUTION

Cartesian Coordinate System



Complex Coordinate System



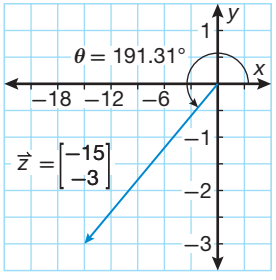
b. Graph vector $\vec{z}$ whose endpoints are the origin and $(-15, -3)$ on the Cartesian and complex planes. Express the Cartesian version as a coordinate matrix and the complex version as a complex number. Find the orientation of the vector.

Math Reasoning

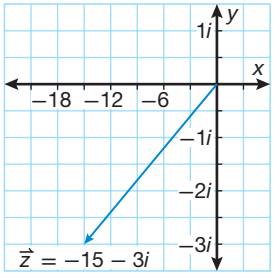
Verify How do you know that the angle of orientation is 191.31°?

SOLUTION

Cartesian Coordinate System



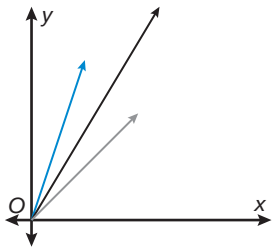
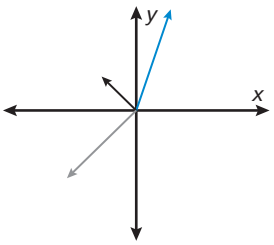
Complex Coordinate System



Hint

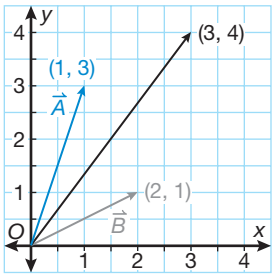
When adding terms in matrices, the terms are added according to their positions. Vector addition and subtraction is similar to matrix addition and subtraction.

Vectors can be added and subtracted.

Vector Addition	Vector Subtraction
$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$	$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}$
$(a + bi) + (c + di) = (a + c) + (b + d)i$	$(a + bi) - (c + di) = (a - c) + (b - d)i$
Vector Addition	**Vector Subtraction**
	

Example 2 Adding and Subtracting Vectors

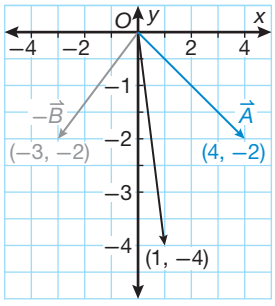
a. Find the vector sum $\vec{A} + \vec{B}$ using matrix and complex addition.



SOLUTION

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad (1 + 3i) + (2 + i) = 3 + 4i$$

b. Find the vector difference $\vec{A} - \vec{B}$ using matrix and complex subtraction.



SOLUTION

$$\begin{bmatrix} 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \qquad (4 - 2i) - (3 + 2i) = 1 - 4i$$

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Vectors can be multiplied by a process known as finding the **dot product.** The dot product is a scalar quantity, not a vector.

The dot product can be found by using the coordinate vectors.

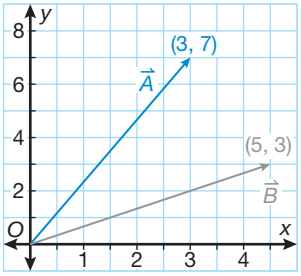
$$[x_1 y_1] \times \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1 x_2 + y_1 y_2$$

Example 3 The Dot Product of Two Vectors

Find the dot product $\vec{A} \cdot \vec{B}$.

SOLUTION

$$[3 \quad 7] \times \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 15 + 21 = 36$$



The magnitude of a vector, represented by $|\vec{A}|$, in standard position can be determined by using the distance formula with the endpoints of the vector, the origin (0, 0) and (x_1, y_1).

$$|\vec{A}| = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2} = \sqrt{(x_1)^2 + (y_1)^2}$$

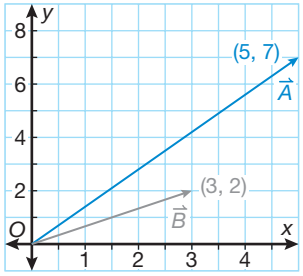
The angle between two nonzero vectors can be found using the dot product of the two vectors and the magnitude of each vector.

$$\cos\theta = \frac{x_1 x_2 + y_1 y_2}{\sqrt{(x_1)^2 + (y_1)^2}\sqrt{(x_2)^2 + (y_2)^2}}$$

$$\theta = \arccos \frac{x_1 x_2 + y_1 y_2}{\sqrt{(x_1)^2 + (y_1)^2}\sqrt{(x_2)^2 + (y_2)^2}}$$

Example 4 Finding the Angle between Two Vectors

Find the angle between the vectors $\vec{A}$ and $\vec{B}$.



SOLUTION

Use the coordinates to find the angle.

$$\theta = \arccos\left(\frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}\right)$$

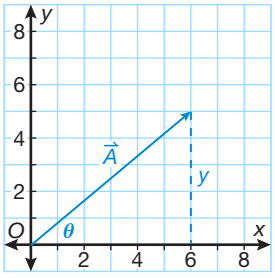
$$\theta = \arccos\left(\frac{5 \cdot 3 + 7 \cdot 2}{\sqrt{5^2 + 7^2}\sqrt{3^2 + 2^2}}\right) = \arccos\left(\frac{29}{\sqrt{25 + 49}\sqrt{9 + 4}}\right)$$

$$= \arccos(0.935) = 20.77°$$

If you know the magnitude and orientation of a vector, you can derive the horizontal and vertical components by using the properties of a right triangle.

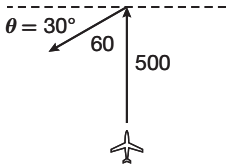
$$\cos\theta = \frac{x}{|\vec{A}|} \qquad x = |\vec{A}|\cos\theta$$

$$\sin\theta = \frac{y}{|\vec{A}|} \qquad y = |\vec{A}|\sin\theta$$



Example 5 Application: Navigation

An airplane traveling north at 500 mph hits a 60 mph headwind blowing 30° south of due west. What is the actual speed and direction of the plane?



SOLUTION

Step 1: Find the horizontal and vertical components.

$$\begin{bmatrix} 0 \\ 500 \end{bmatrix} + \begin{bmatrix} -60\cos(30°) \\ -60\sin(30°) \end{bmatrix} = \begin{bmatrix} -51.96 \\ 470 \end{bmatrix}$$

Step 2: Find the magnitude and direction.

Magnitude	Direction
$\sqrt{(-51.96)^2 + 470^2} = 472.86$	$\theta = \arccos\left(\frac{-51.96}{472.86}\right) = 96.31°$ NNW

The plane is heading 96.31° NNW at 472.86 mph.

Lesson Practice

(Ex 1) **a.** Express a vector whose endpoint is at (10, 5) as a coordinate matrix and a complex number.

(Ex 1) **b.** Express a vector whose endpoint is at (−12, −36) as a coordinate matrix and a complex number.

(Ex 2) **c.** Add the vectors $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ 4 \end{bmatrix}$.

(Ex 2) **d.** Subtract the vectors $\begin{bmatrix} -10 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 9 \end{bmatrix}$.

(Ex 3) **e.** Find the dot product between the vectors $\begin{bmatrix} -10 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 9 \end{bmatrix}$.

(Ex 4) **f.** Find the angle between the vectors $\begin{bmatrix} -8 \\ 12 \end{bmatrix}$ and $\begin{bmatrix} 9 \\ -12 \end{bmatrix}$.

(Ex 5) **g.** An airplane traveling north at 600 mph hits a 30 mph headwind blowing east at a 40° angle from north. What is the actual speed and direction of the plane?

Practice Distributed and Integrated

(70) **1. Error Analysis** Explain the error a student made below.

$5 + \sqrt{x} = 3 \rightarrow \sqrt{x} = -2 \rightarrow (\sqrt{x})^2 = (-2)^2 \rightarrow x = 4$ The solution is 4.

(98) ***2. Generalize** How many ellipses have an eccentricity of 0? Explain.

(81) **3. Write** Explain why $\ln e^{x^2+3} = x^2 + 3$.

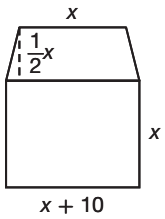
(8) **4. Engines** Two-stroke engines require a premix of gas and oil. The amount of gas used varies directly with the amount of oil used. In some two-stroke engines, 5 gal of gas requires 20 oz of oil.

a. Write the proportion that can be used to find how many ounces of oil to use with 9 gal of gas.

b. Solve the proportion in part **a.**

c. Check the answer from part **b** with the proportion from part **a.**

5. *(11)* **Geometry** Write a polynomial in standard form that represents the total area formed by the rectangle and trapezoid. (Hint: The formula for the area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$).



6. *(68)* **Probability** A student has a spinner with four equal sections labeled 2, 4, 6, and 8. If the student spins the spinner two times, find the probability that the first spin lands on a 4 and the sum of the results is greater than or equal to 10.

7. *(79)* Evaluate the piecewise function $f(x) = \begin{cases} x - x^2, \text{ if } x \le 1 \\ 12, \quad \text{if } 1 < x \le 2 \\ -6, \quad \text{if } 2 < x < 4 \\ -x - 8, \text{ if } x \ge 4 \end{cases}$ for $x = -0.4$, $x = 0$, and $x = 6$.

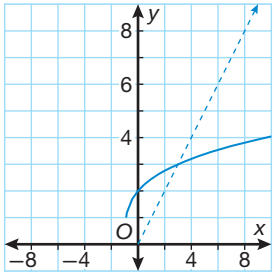
8. *(80)* **Error Analysis** Explain and correct the error a student made in finding the z-score for a data value of 15, taken from a normally distributed set of data, where the mean is 20 and the standard deviation is 4.

$$z = \frac{20 - 15}{4} = \frac{5}{4} = 1.25$$

9. *(77)* In $\triangle RST$, $ST = 17$, $RT = 20$, and $m\angle T = 38°$. Find $m\angle S$.

10. *(11)* **Write** If two quintic polynomials are added, what are all the possible types of the resulting polynomial (classified by degree)?

11. *(75)* The graph of $y = \sqrt[4]{x + 1} + \sqrt[3]{x + 1}$ is shown to the right. Use the axis of symmetry to draw the inverse function.



12. *(91)* **Seismology** To locate the epicenter of an earthquake, a seismologist graphs the circles created from three seismographs in different locations and finds their point of intersection.

a. The equations below were created from the readings of three seismographs where each unit represents one mile. Sketch the circles on the same coordinate plane.

$$(x + 4)^2 + (y - 4)^2 = 25$$
$$(x - 1)^2 + (y + 3)^2 = 49$$
$$(x - 5)^2 + (y - 4)^2 = 16$$

b. Estimate the location of the epicenter.

13. *(73)* **Prom** A member of the prom committee polls 40 random seniors from the senior class meeting to ask about possible locations for the prom. Describe the individuals in the population and in the sample.

14. (83) **National Parks** Kolob Arch in Zion National Park is one of the longest arches in the world with a span of about 288 feet and a height of about 105 feet. Write a quadratic function to approximate the arch.

15. (87) **Sound Intensity** The loudness of a given sound is represented by $d = 10\log\frac{I}{I_o}$ where d is decibels, I is intensity of the sound, and I_o is the intensity of a sound that can just be heard by humans. Determine the decibel level of a sound with an intensity of $(2.3 \times 10^9)I_o$.

16. (84) Use cross multiplying to show that $\frac{3}{x+5} = \frac{2}{x+5}$ has no solution.

17. (73) **Write** An English teacher needs to pick 5 students to present book reports to the class. The teacher writes the names of all students in the class on pieces of paper, puts the pieces in a hat, and chooses 5 names without looking. Determine what type of sample is used and whether it is biased or unbiased. Explain your answer.

***18.** (89) **Analyze** Can a quadratic inequality have no critical values? Explain.

***19.** (92) **Coordinate Geometry** The points (1, −5), (2, −11), (3, −17), (4, −23), and (5, −29) represent the first five terms of an arithmetic sequence. What is the common difference? What is the y-coordinate when the x-coordinate is 12?

***20.** (32) **Graphing Calculator** Use a graphing calculator to find the inverse of the matrix, if it exists. $H = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & 4 & 6 \end{bmatrix}$

***21.** (99) Find the dot product between the vectors $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

***22.** (96) **Multiple Choice** Which are polar coordinates for the point with Cartesian coordinates $\left(-3, -3\sqrt{3}\right)$?

A $\left(3, \frac{7\pi}{6}\right)$ **B** $\left(3, \frac{4\pi}{3}\right)$ **C** $\left(6, \frac{7\pi}{6}\right)$ **D** $\left(6, \frac{4\pi}{3}\right)$

***23.** (98) Write an equation in standard form for the ellipse with center (0, 0), vertex (6, 0), and covertex (0, −5).

Solve.

24. (2) $3\frac{2}{5}x - 4\frac{1}{10}x = 2\frac{1}{4}$

25. (2) $0.02(p - 2) = 0.03(2p - 6)$

***26.** (85) Use the Rational Root Theorem to find the roots of $y = 2x^3 - 3x^2 - 8x + 12$.

Add.

27. (37) $3xy^2m + \frac{4}{x}$

28. (37) $\frac{5x^2}{pm} - 4 + \frac{c}{p^2m}$

***29.** (99) **Multi-Step** Find the magnitude of vectors $\begin{bmatrix} -3 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$. Calculate the angle between the two vectors. Then calculate the dot product.

30. (41) Find the distance between (−3, 7) and (4, −2).

LESSON
100

Graphing Rational Functions I

Warm Up

1. (28) **Vocabulary** In order for $\frac{1}{g(x)}$ to be a rational expression, $g(x)$ must be a _____ not equal to zero.

2. (23) Write $\frac{4x^2 - 11x - 20}{3x^2 + 21x - 54}$ in factored form.

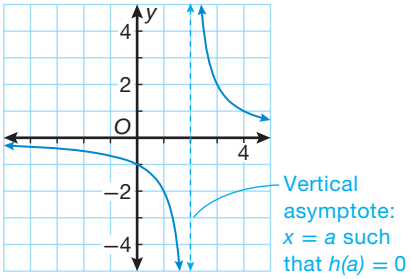
3. (23) What values of x are not allowed for this expression? $\frac{1}{7x^2 + 3x - 4}$

New Concepts

A rational function is of the form

$f(x) = \frac{g(x)}{h(x)}$ for polynomials $g(x)$ and $h(x)$. The domain of $f(x)$ is all real numbers except those for which $h(x) = 0$.

A real number a that makes the denominator of a rational function equal to zero is not in the domain of the function. The graph of the function is discontinuous at $x = a$.



Example 1 Finding Points of Discontinuity

Find any points of discontinuity for each rational function.

a. $y = \frac{x + 1}{x^2 + 5x + 6}$

SOLUTION

The function is undefined when the denominator equals zero. Set the denominator equal to zero and solve.

$$x^2 + 5x + 6 = 0$$
$$(x + 2)(x + 3) = 0$$
$$(x + 2) = 0 \qquad (x + 3) = 0$$
$$x = -2 \qquad x = -3$$

The points of discontinuity are **−2** and **−3**.

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b. $y = \dfrac{x+5}{x^2+6}$

SOLUTION

The function is undefined when the denominator equals zero. Set the denominator equal to zero and solve.

$$x^2+6 = 0$$
$$x^2 = -6$$
$$x = \pm\sqrt{-6} = \pm i\sqrt{6}$$

Since $i\sqrt{6}$ is not a real number, there is no point of discontinuity.

A vertical asymptote or a hole in the graph occurs at a point of discontinuity. If there are no common factors in the numerator and denominator, there are vertical asymptotes at the points of discontinuity. If there are common factors in the numerator and denominator, there are either holes or vertical asymptotes at the points of discontinuity.

Example 2 Finding Vertical Asymptotes and Holes

In each rational function, describe the vertical asymptotes and holes for the graph.

a. $y = \dfrac{(x+1)}{(x+4)(x-2)}$

SOLUTION

The points of discontinuity are -4 and 2 and since these are no common factors of the numerator and denominator, $x = -4$ and $x = 2$ are vertical asymptotes.

Hint

When the factors in the numerator and denominator are the same and thus cancel, a hole will occur at the x-value that causes the factor to equal zero.

b. $y = \dfrac{(x+5)(x+6)}{(x+6)(x+5)(x+6)}$

SOLUTION

Since there are common factors in the numerator and denominator, the graph of this rational function is the same as $y = \frac{1}{x+6}$, except it has a hole at $x = -5$. The vertical asymptote is at $x = -6$.

The graph of a rational function has at most one horizontal asymptote. If the degree of the denominator is greater than the degree of the numerator the graph has a horizontal asymptote at $y = 0$. If the degree of the numerator is greater than the degree of the denominator, then the graph has no horizontal asymptote $y = 0$. If the degrees of the numerator and denominator are equal, then the graph has a horizontal asymptote at $y = \frac{a}{b}$, where a is the coefficient of the term with the highest degree in the numerator and b is the coefficient of the term with the highest degree in the denominator.

Example 3 Finding Horizontal Asymptotes

Find the horizontal asymptotes of each rational function.

a. $y = \frac{3x^2 + 2}{x^2 - 5}$

SOLUTION

Since the degree of the numerator and denominator are equal, the equation of the vertical asymptotes is $\boldsymbol{y = \frac{3}{1} = 3.}$

b. $y = \frac{2x^2 + 3x + 7}{x - 5}$

SOLUTION

Since the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

Example 4 Graphing Using Asymptotes

Graph the following function. Identify any asymptotes.

a. $y = \frac{2}{x - 1}$

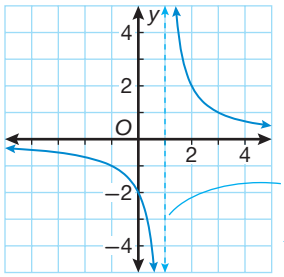
SOLUTION

Determine any points of discontinuity.

$x - 1 = 0$

$x = 0$

There is a point of discontinuity at $x = 1$. Since the numerator and denominator have no common factors, $x = 1$ is a vertical asymptote.



Since the degree of the denominator is greater than the degree of the numerator, the graph has a horizontal asymptote at $y = 0$.

Calculate the values of y for values of x near the asymptotes. Plot these points and sketch the graph.

Math Reasoning

Verify Show that as x gets infinitely large, there is a horizontal asymptote at $y = 0$.

b. $y = \frac{x^2 + 2x - 8}{x^3 - 4x}$

SOLUTION

$$y = \frac{x^2 + 2x - 8}{x^3 - 4x} = \frac{(x - 2)(x + 4)}{x(x^2 - 4)} = \frac{(x - 2)(x + 4)}{x(x + 2)(x - 2)}$$

$$= \frac{(x + 4)}{x(x + 2)}$$

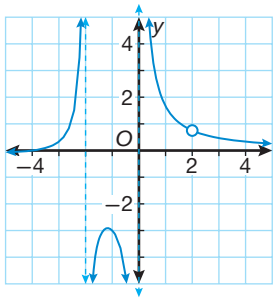
A common factor divides out of both the numerator and denominator, leaving vertical asymptotes $x = 0$ and $x = -2$. However, $x = 2$, while not an asymptote, is still not part of the function and is graphed as a "hole."

Math Reasoning

Generalize Identify the holes and vertical asymptotes for rational functions of the form $y = \frac{x - a}{x^2 - a^2}$.

The x-intercepts of a rational function occur when the value of the function is zero for any value in the domain. The x-intercept is -4.

Calculate the values of y for values of x near the asymptotes. Plot these points and the x-intercepts. Sketch the graph.



Lesson Practice

Find any points of discontinuity for each rational function.
(Ex 1)

a. $y = \dfrac{x+2}{x^2-4x-5}$

b. $y = \dfrac{(2x+7)(x+3)}{(x+3)}$

In each rational function, describe the vertical asymptotes and holes for the graph.
(Ex 2)

c. $y = \dfrac{(x+7)}{(x+3)(x-5)}$

d. $y = \dfrac{(x+5)(x+6)}{(x+6)(x+5)(x+6)}$

Find the horizontal asymptotes of each rational function.
(Ex 3)

e. $y = \dfrac{6x^2+3}{2x^2+4}$

f. $y = \dfrac{2x^2+3x+7}{x-5}$

Graph the following functions. Identify any asymptotes or holes.
(Ex 4)

g. $y = \dfrac{4}{x-2}$

h. $y = \dfrac{2}{x^3-x}$

Graph the following functions. Identify any asymptotes, holes, and x-intercepts.
(Ex 5)

i. $y = \dfrac{x^2-2x-15}{x^2+5x+6}$

j. $y = \dfrac{x^2-2x-15}{x^3-9x}$

Practice Distributed and Integrated

Solve.

1. (84) $x = 4 + \frac{5}{x}$

2. (84) $-x^2 - 2x - 1 > 2$

***3.** (99) **Error Analysis** Two students calculated the angle between vectors $\vec{A} = \begin{bmatrix} 12 \\ 9 \end{bmatrix}$ and $\vec{B} = \begin{bmatrix} 17 \\ 8 \end{bmatrix}$ but their results were different. Which student made the mistake?

Student A	Student B
$\theta = \arccos\left(\frac{(12 \cdot 17) + (9 \cdot 8)}{\sqrt{12^2 + 9^2}\sqrt{17^2 + 8^2}}\right)$ $= 11.67°$	$\theta = \arccos\left(\frac{(12 \cdot 9) + (17 \cdot 8)}{\sqrt{12^2 + 9^2}\sqrt{17^2 + 8^2}}\right)$ $= 30.03°$

4. (36) Find the equation of the line parallel to $y = 7x + 6$ that crosses (6, 10).

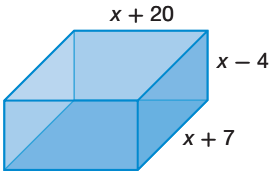
***5.** (96) **Phone Service** A planned cell phone tower will service an area with a 13-mile radius. It is to be located 5 miles east and 12 miles south of a radio tower. Write a polar equation of the boundary of the region that will be serviced. Place the location of the radio tower at the pole. Graph the equation on a calculator.

6. (56) **Multi-Step** The formula $\theta = \frac{180°(n-2)}{n}$ gives the measure of each interior angle of an n-sided regular polygon.

a. Use the formula to find the measure of an interior angle of a regular decagon.

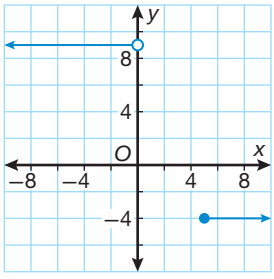
b. Find the reference angle for the interior angle.

***7.** (95) **Geometry** A rectangular prism has the dimensions shown. The volume $V(x)$ is the product of the linear dimensions. Use the Remainder Theorem to evaluate $V(20)$.

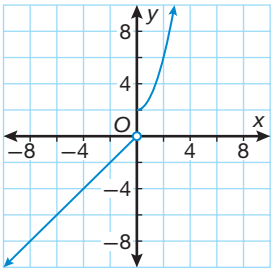


Write a piecewise function rule for each graph.

8. (79)



9. (79)



***10.** (100) **Multiple Choice** Which of the following rational functions includes a vertical asymptote at $x = 7$?

A $y = \frac{x - 7}{x^2 - 49}$

B $y = \frac{x - 7}{x^2 + 12x + 35}$

C $y = \frac{x^2 - 16}{x^2 + 11x + 28}$

D $y = \frac{x^2 - 16}{x^2 - 11x + 28}$

11. (87) **Analyze** Use the table of values for $y = \log_2 x^2$ to answer the questions.

x	−4	−3	−2	−1	0	1	2	3	4
y	4	3.17	2	0	–	0	2	3.17	4

a. As the negative x values increase to 0, what happens to the y values?

b. As the positive x values increase, what happens to the y values?

12. (11) **Justify** Can the difference of two polynomials have a greater degree than the sum of the same two polynomials? Justify your answer with an example.

***13.** (22) **Graphing Calculator** Use a graphing calculator to evaluate $y = 2x + 3$ for the given domain: −2, −1, 0, 1, 2. Identify the range of the function and classify the function.

14. (79) **Parking** An airport parking garage costs $38 per day for the first week. After that, the cost decreases to $35 per day. Write a piecewise function for the cost of parking x days.

15. (95) **Analyze** Suppose that for polynomials $P_1(x)$ and $P_2(x)$ there is a constant a such that $P_1(a) = P_2(a) = 0$. What can you conclude about the graphs of $P_1(x)$ and $P_2(x)$?

16. (80) **Research** In a study of 413 men and 335 women, the mean height of men was 174.1 centimeters with a standard deviation of 7 centimeters. The mean height of women was 162.3 centimeters with a standard deviation of 6.2 centimeters. Assume both distributions are normal. Compare the actual height of a man with a z-score of 2 to the actual height of a woman whose z-score is 2.

17. (82) **Data Analysis** The number of days it takes to complete a year for different planets is shown in the table. Let Earth's yearly motion be modeled by $y = \sin(0.017x)$. How would you model Jupiter's motion? Hint: The days in a year corresponds to the period of the function.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
Days in a Year	88.03	224.63	365.25	686.67	4331.87	10,760.27	30,681.00	60,193.20

***18.** (94) **Error Analysis** Which student made an error in determining intervals to check for solutions and nonsolutions of $\frac{x-2}{(x+7)(x-5)} \geq 0$? What is the error?

Student A
$x \leq -7$
$-7 \leq x \leq 2$
$2 \leq x \leq 5$
$x \geq 5$

Student B
$x < -7$
$-7 < x \leq 2$
$2 \leq x < 5$
$x > 5$

19. (1) **Analyze** Identify which property of real numbers is being demonstrated by $(4 \cdot 6) \cdot 5 = 4 \cdot (6 \cdot 5) = 4 \cdot 30 = 120$. Explain why this property might be helpful in solving this problem without a calculator.

20. *(89)* **Multi-Step** The perimeter of a rectangular garden will be 100 feet. Write an inequality to find the widths that will give an area of at least 450 square feet. Solve the inequality and round to the nearest tenth.

21. *(90)* For what values of θ is $y = \frac{\sin(x)}{\cos(x - \theta)}$ no longer a periodic function?

22. *(82)* **Physics** Every millisecond a radio beacon emits a signal. If the magnitude of the signal is 100, write the periodic function that models the signal using the form $y = a\cos(bx)$.

Simplify.

23. *(40)* $2\sqrt{3}\left(5\sqrt{3} - 2\sqrt{6}\right)$

24. *(40)* $2\sqrt{3} \cdot 3\sqrt{6} \cdot 5\sqrt{12}$

25. *(91)* **Write** A student wants to graph a circle whose equation is $(x - 5)^2 + (y - 5)^2 = 2$ on a graphing calculator. How would you explain to the student how this can be done?

26. *(100)* **Physics** In a parallel circuit, the total resistance, R_T, is found using $R_T = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \ldots$. Suppose that a parallel circuit has two resistors of magnitude R and $2R + 1$. Solve for R_T.

Divide.

27. *(38)* $x^4 + 5x^3 - 6x^2$ by $(x - 1)$

28. *(38)* $x^4 - 20x^3 + 123x^2 - 216x$ by $(x - 3)$

29. *(99)* Add vectors $\begin{bmatrix}10\\6\end{bmatrix}$ and $\begin{bmatrix}4\\5\end{bmatrix}$.

***30.** *(99)* Subtract vectors $\begin{bmatrix}10\\4\end{bmatrix}$ and $\begin{bmatrix}3\\14\end{bmatrix}$.

INVESTIGATION

10

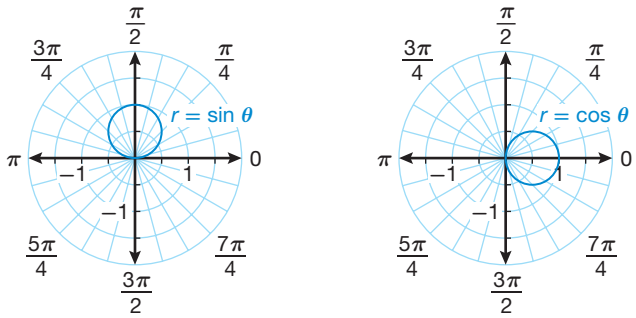
Graphing Polar Models

An equation in polar coordinates uses the variables r and θ.

Polar Equations for Circles

A circle can be graphed using the polar equations:

$r = a \sin \theta$ $\qquad$ $r = a \cos \theta$



1. How do the graphs of $r = \sin \theta$ and $r = \cos \theta$ compare to the unit circle?

2. Graph $r = \sin \theta$ and then graph $r = a \sin \theta$ for three values of a where $0 < a < 1$. How did the value of a change the graph from the parent graph?

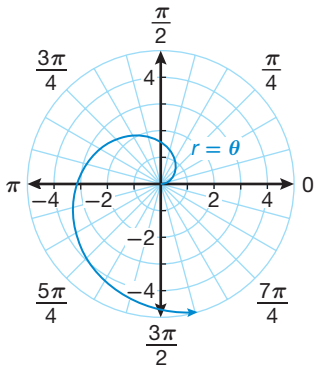
3. Graph $r = \cos \theta$ and then graph $r = a \cos \theta$ for three values of a where $a > 1$. How did the value of a change the graph from the parent graph?

Graphing Calculator

To graph polar equations with your graphing calculator, press the MODE key and select the POL option. Once in this mode, pressing the Y= key allows you to input an equation in $r = f(\theta)$ mode.

Polar Equations for Spirals

One of the simplest polar equations that uses the variable θ is $r = \theta$. Its graph is a spiral.



4. Make a table of values for $r = \theta$ on a graphing calculator. How does the value of r change compared to changes in θ?

5. Graph $r = \theta$ and then graph $r = c\theta$ for three values of c where $0 < c < 1$. How did the value of c change the graph from the parent graph?

6. Graph $r = \theta$ and then graph $r = c\theta$ for three values of c where $c > 1$. How did the value of a change the graph from the parent graph?

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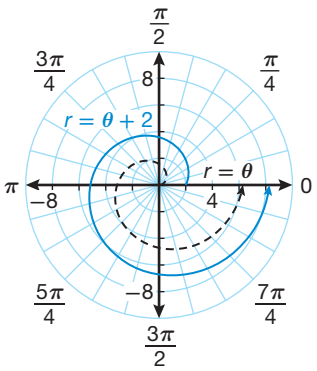
Graphing Calculator

The graph on the right shows values of θ ranging from 0 to 2π. With a graphing calculator, you can set θ to higher values. Press on the WINDOW key and set θmax to a greater multiple of π.

For linear equations of the form $r = a\theta + b$, the graph is also a spiral.

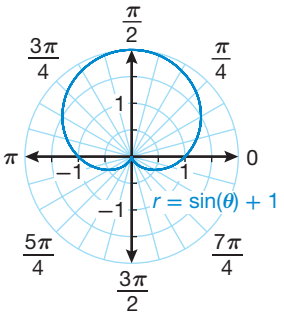


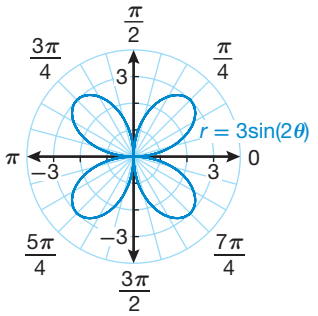
7. Graphing Calculator Describe the role of a and b in the graph of $r = a\theta + b$.

Math-Reasoning

Verify Use a graphing calculator to show that the general form of the of the equation, $r = a(\sin(\theta) + 1)$, always yields the graph of a cardioid.

A more general form of the equation $r = \theta$ is $r = a\sin(b\theta) + c$. Below are some of the interesting graphs that result from different values of a, b, and c. The graph of $r = \sin(\theta) + 1$ is a heart-shaped curve known as a cardioid. The graph of $r = 3\sin(2\theta)$ is called a rose curve.





8. Graph 3 rose curves given by the general equation $y = \sin(b\theta)$ where b is an even integer. How does the number of petals relate to the value of b?

9. Graph 3 rose curves given by the general equation $y = \sin(b\theta)$ where b is an odd integer and not equal to 1. How does the number of petals relate to the value of b?

10. What would be the value of b in the equation of a rose curve with 20 petals?

11. How many petals are in a rose curve given by the equation $y = \sin(19\theta)$?

12. Graph the rose curve $y = a\sin(2\theta)$ where $a = 1$. Graph 3 more rose curves by choosing the value of a to be greater than 1. How did the graph change?

13. Graph the rose curve $y = a\sin(2\theta)$ where $a = 1$. Graph 3 more rose curves by choosing the value $0 < a < 1$. How did the graph change?

The Polar Equations for Conic Sections

The general form of the polar equation for conic sections is shown below. By varying the value of the constant e, the graph will be a circle, ellipse, parabola, or hyperbola.

$$r = \frac{l}{1 + e\sin(\theta)}$$

Caution

The variable *e* used in polar equations determines the eccentricity of the conic section. It is not the constant *e*, which equals approximately 2.718.

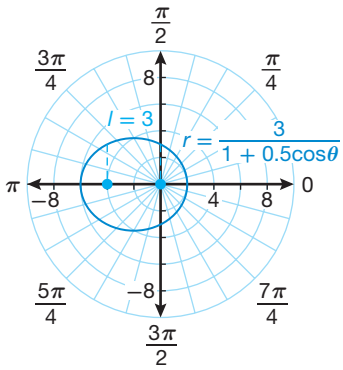
Graphing Calculator The variable e is the eccentricity, or a measure of the curvature of a conic section. For each of the following graphs, let the value of l remain constant, while varying the value of e.

14. Choose a value for l and let $e = 0$. What kind of conic section is this graph?

15. Choose a value for l. Choose values of e from the range $0 < e < 1$. What kind of conic section is this graph?

16. Choose a value for l and let $e = 1$. What kind of conic section is this graph?

The variable l represents the vertical distance from the conic section's focus or center and the curve itself.



17. In the case of a circle, what does the value of l represent?

Graphing Calculator Write and graph the equation of a conic section based on the requirements below.

18. A parabola with $l = 3$

19. A circle of radius 7

20. An ellipse with $l = 10$

Investigation Practice

a. Give a parameter change for the circle equation $r = 2 \sin \theta$ that will increase the size of the circle and translate up.

b. Describe the change of the graph $r = 3\theta$ from the parent function $r = \theta$.

c. Write an equation for a rose curve with 14 petals.

d. Write an equation for a rose curve with 9 petals.

e. Write an equation for a rose curve that has the same number of petals but lengthened in the equation $y = 2 \sin(3\theta)$.

f. What kind of conic section is given by the equation $r = \frac{2}{1 + 0.5 \sin(\theta)}$?

g. What kind of conic section is given by the equation $r = \frac{4}{1 + 1 \sin(\theta)}$?

LESSON
101

Making Graphs of Polynomial Functions

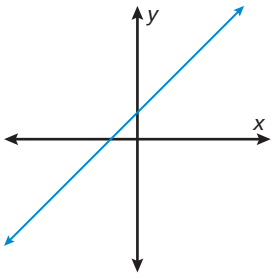
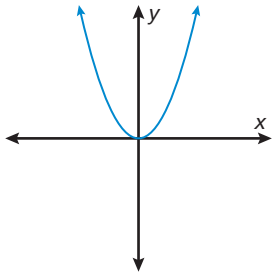
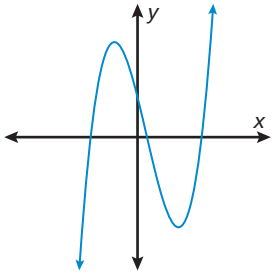
Warm Up

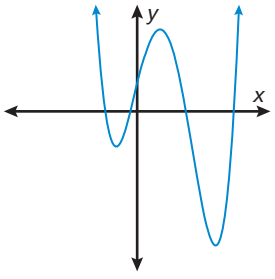
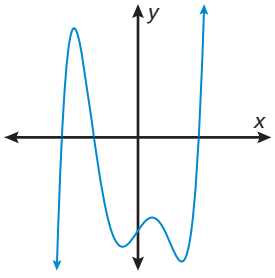
1. (11) **Vocabulary** For the polynomial $2x^3 - x + 5$, the leading _____ is 2.
2. (11) If $f(x) = 2x^3 - x + 5$, then $f(-2) =$ _____.
3. (66) Find the real zeros of $g(x) = x^3 + 3x^2 + 2x$ by factoring.
4. (19) Simplify $x - 2(3x - 1)$.
5. (19) Multiply $3y^2(-3y)$.

New Concepts

Polynomial functions are classified by their degree. Some characteristics of graphs of polynomial functions are shown below.

Graphs of Polynomial Functions

Linear function Degree 1	Quadratic function Degree 2	Cubic function Degree 3
		

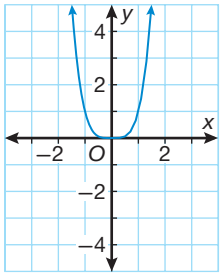
Quartic function Degree 4	Quintic function Degree 5
	

All linear function graphs are lines. All quadratic function graphs are parabolas. The shapes of cubic, quartic, and quintic function graphs have more variety than shown above.

End behavior is a description of the y-values of a function as x approaches positive infinity ($x \to +\infty$) or negative infinity ($x \to -\infty$). End behavior of a polynomial function is determined by its degree and leading coefficient. Knowing about end behavior is helpful when graphing a polynomial function.

Hint

Below is the graph of $f(x) = x^4$, the parent quartic function.



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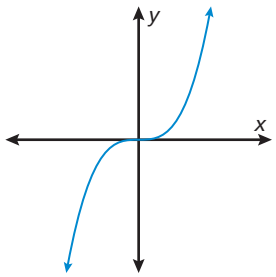
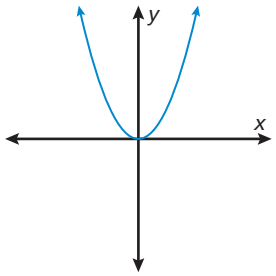
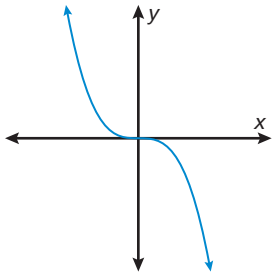
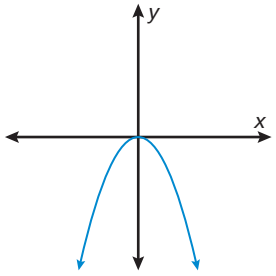
Hint

Notice that if the polynomial has an even degree: both ends head upward ($a > 0$) or downward ($a < 0$). If the polynomial has an odd degree: the ends head in opposite directions, one upward and the other downward, with the direction determined by whether $a > 0$ or $a < 0$.

Polynomial End Behavior		
$P(x)$ has …	Odd Degree	Even Degree
Leading coefficient $a > 0$	As $x \to -\infty$, $P(x) \to -\infty$. As $x \to +\infty$, $P(x) \to +\infty$. 	As $x \to -\infty$, $P(x) \to +\infty$. As $x \to +\infty$, $P(x) \to +\infty$. 
Leading coefficient $a < 0$	As $x \to -\infty$, $P(x) \to +\infty$. As $x \to +\infty$, $P(x) \to -\infty$. 	As $x \to -\infty$, $P(x) \to -\infty$. As $x \to +\infty$, $P(x) \to -\infty$. 

Example 1 Determining End Behavior of Polynomial Functions

Identify the leading coefficient and degree, and describe the end behavior.

a. $P(x) = 2x^3 + 3x^2 - 4x + 1$

SOLUTION

The leading coefficient is 2, which is positive.

The degree is 3, which is odd.

As $x \to +\infty$, $P(x) \to +\infty$, and as $x \to -\infty$, $P(x) \to -\infty$.

Hint

Both the leading coefficient and the degree of the polynomial are contained in the term of the greatest degree. When determining end behavior, you can ignore all other terms.

b. $R(x) = x^2 - x^6 - 5x$

SOLUTION

Rewrite the function in standard form: $R(x) = -x^6 + x^2 - 5x$.

The leading coefficient is -1, which is negative.

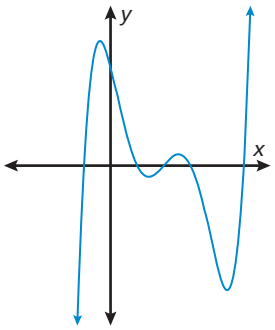
The degree is 6, which is even.

As $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to -\infty$.

Example 2 Using Graphs to Analyze Polynomial Functions

Determine whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

a.

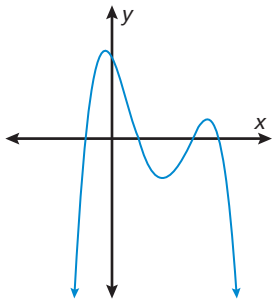


SOLUTION

As $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to +\infty$.

$P(x)$ is of odd degree with a positive leading coefficient.

b.



SOLUTION

As $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to -\infty$.

$P(x)$ is of even degree with a negative leading coefficient.

Use the following steps to graph a polynomial function.

Steps for Graphing a Polynominal Function

1. Identify the real zeros of the function.
2. Make a table of ordered pairs. Include the x-intercept(s), x-values that lie between the x-intercepts, and the y-intercept.
3. Plot the points from your table.
4. Determine the end behavior.
5. Sketch the graph.

Example 3 Graphing a Polynomial Function

Graph the function.

$f(x) = x^3 + 2x^2 - 5x - 6$

SOLUTION

Step 1: To identify the real zeros, begin by identifying the possible rational roots of the equation $1x^3 + 2x^2 - 5x - 6 = 0$. Use the Rational Root Theorem.

Since $p = -6$ and $q = 1$, the possible rational roots are ± 1, ± 2, ± 3, and ± 6.

Next, test possible rational roots until a zero is identified.

Test $x = 1$.

1	1	2	−5	−6
		1	3	−2
	1	3	−2	−8

Test $x = -1$.

−1	1	2	−5	−6
		−1	−1	6
	1	1	−6	0

$x = -1$ is a zero, and $f(x) = (x + 1)(x^2 + x - 6)$.

Factor to identify the other zeros. $f(x) = (x + 1)(x + 3)(x - 2)$

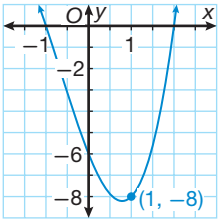
The real zeros are -1, -3, and 2. These are the x-intercepts.

Hint

The zeros of $f(x)$, the roots of $y = 0$, and the x-intercepts of the graph of $f(x)$ are all different names for the same numbers.

Caution

Do not assume that (1, −8) is lower than all other nearby points. See the graph below.



If more exact values for the turning points are required, beyond what a sketch can provide, a graphing calculator can be used.

Step 2: Make a table of ordered pairs. Include the x-intercepts, x-values that lie between x-intercepts, and the y-intercept.

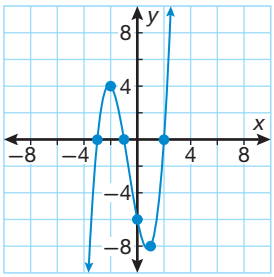
x	−3	−2	−1	0	1	2
y	0	4	0	−6	−8	0

The x-intercepts are −3, −1, and 2. The y-intercept is −6.

Step 3: Plot the points from the table.

Step 4: Determine the end behavior. The degree is odd and the leading coefficient is positive, so as $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to +\infty$.

Step 5: Sketch the graph.



A **turning point** is where a graph changes from increasing to decreasing or from decreasing to increasing values of y. The function value at a turning point is either a *local maximum* or *local minimum*.

Local Maxima and Minima

For a function $f(x)$, $f(a)$ is a **local maximum** if there is an interval around a such that $f(x) < f(a)$ for every x-value in the interval except a.

For a function $f(x)$, $f(a)$ is a **local minimum** if there is an interval around a such that $f(x) > f(a)$ for every x-value in the interval except a.

A polynomial function of degree n has at most n x-intercepts and at most $n - 1$ turning points. If the function has exactly n distinct real zeros, then it has exactly n x-intercepts and exactly $n - 1$ turning points. You can use a graphing calculator to find local maxima and minima. In some cases, they will be approximate values.

Reading Math

Maxima is the plural form of maximum. **Minima** is the plural form of *minimum*.

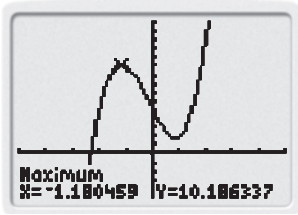
Example 4 Determining Local Maxima and Minima with a Calculator

Graph $g(x) = 2x^3 + x^2 - 6x + 5$ on a calculator, and then determine the local maxima and local minima.

SOLUTION

Step 1: Graph.

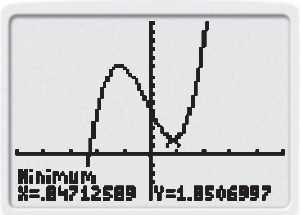
The graph appears to have one local minimum and one local maximum.



Step 2: Determine the local maximum.

Press 2nd CALC TRACE to access the **CALC** menu. Choose **4:maximum**.

The local maximum is approximately 10.1863.



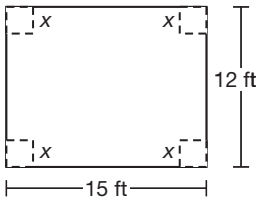
Step 3: Determine the local minimum.

Press 2nd CALC TRACE to access the **CALC** menu. Choose **3:minimum**.

The local minimum is approximately 1.8507.

Example 5 Application: Sheet Metal Fabrication

A sheet metal mechanic plans to construct an open box from a 12 ft by 15 ft rectangular sheet of metal by cutting squares from the corners and folding up the flaps. Find the maximum possible volume of the box.

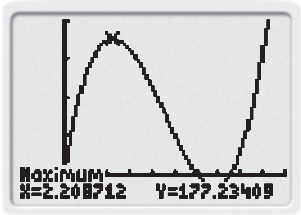


SOLUTION

Write a function to represent volume.

$V(x) = x(15 - 2x)(12 - 2x)$

Graph the function. Note that values of x greater than 6 or less than 0 do not make sense for this problem.



The graph has a local maximum of approximately 177.2 when $x \approx 2.21$. So, the maximum possible volume is approximately 177.2 ft^3.

Math Reasoning

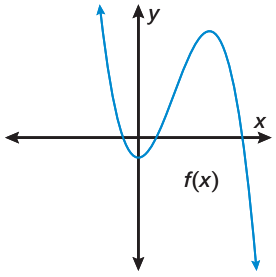
Write Explain why values of x greater than 6 or less than 0 do not make sense for this problem.

Lesson Practice

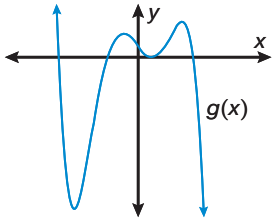
(Ex 1) **a.** Identify the leading coefficient and degree, and describe the end behavior of $f(x) = x^4 + 6x - 8$.

(Ex 1) **b.** Identify the leading coefficient and degree, and describe the end behavior of $g(x) = -2x^5 + 3x^4 - x$.

(Ex 2) **c.** The function $f(x)$ is graphed. Determine whether $f(x)$ has an odd or even degree and a positive or negative leading coefficient.



(Ex 2) **d.** The function $g(x)$ is graphed. Determine whether $g(x)$ has an odd or even degree and a positive or negative leading coefficient.



(Ex 3) **e.** Graph the function $f(x) = x^3 - 2x^2 - 5x + 6$.

(Ex 4) **f.** Graph $g(x) = x^4 - 4x^3 + 3x + 8$ on a calculator, and then determine the local maxima and local minima.

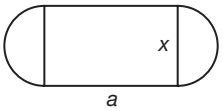
(Ex 5) **g.** An open box is to be made from an 8 ft by 10 ft rectangular sheet of metal by cutting squares from the corners and folding up the flaps. Find the maximum possible volume of the box.

Practice Distributed and Integrated

1. (32) Determine whether the matrices are inverses.

$$A = \begin{bmatrix} 2 & 7 \\ -1 & -2 \end{bmatrix} \text{ and } L = \begin{bmatrix} -\frac{2}{3} & -\frac{7}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

***2.** (101) **Furniture Design** A table top is to consist of a rectangular region with a semicircle on each end. The perimeter is to be 200 inches. Find the maximum possible area of the rectangular region.



***3.** (93) **Graphing Calculator** Use a graphing calculator to determine how long it will take an investment of $1000 to grow to $1800 at 6% interest compounded monthly.

4. (89) **Error Analysis** Explain and correct the error a student made in solving $x^2 + 4x > 12$ algebraically.

$$x^2 + 4x - 12 > 0$$
$$(x - 2)(x + 6) > 0$$

critical values: 2, −6
solutions: $-6 < x < 2$

5. (85) Find the roots of $f(x) = 0.1x^3 - 1.4x^2 + 6.0x - 7.2$ using the Rational Root Theorem.

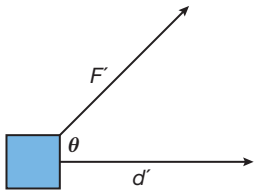
***6.** (100) The product of two variables is a constant. Show that this describes a rational function.

Factor.

7. (23) $6x^3 - 12x^2 - 18x + 6$

8. (23) $x^2 - 64$

***9.** (99) **Physics** Work is defined as the dot product of the force, $\vec{F}$, on an object and the displacement, $\vec{d}$. If $\theta = 45°$, $|\vec{F}| = 25$, and $|\vec{d}| = 10$, how much work is exerted on the wooden block?



10. (5) Solve for x, y, and z: $\begin{bmatrix} 1 & x^2 \\ y + 4 & \frac{y}{2} \end{bmatrix} = \begin{bmatrix} z + 2 & 25 \\ 16 & x + 1 \end{bmatrix}$.

11. *(79)* Evaluate the piecewise function $f(x) = \begin{cases} -12 \text{ if } x < 1 \\ 7x \text{ if } 1 \le x \le 4 \\ 2x - x^2 \text{ if } 4 < x \end{cases}$ for $x = -3$ and $x = 4$.

12. *(33)* **Ice Cream** A store has 31 flavors of ice cream. They also have a choice of chocolate or rainbow sprinkles and chocolate or caramel syrup. Assume a child picks one flavor ice cream, one type of sprinkle, and one type of syrup. How many different desserts could the child make?

***13.** *(97)* **Predict** Without using a formula, how can you find the 100th term of the geometric sequence, 3, −3, 3, −3, …?

14. *(83)* **National Parks** The Black Arch, in Arches National Park, has a span of about 65 feet and a height of about 45 feet. Write a quadratic function to approximate the arch.

15. *(80)* A set of data is normally distributed with a mean of 34 and a standard deviation of 2.7. Find the percent of data that is less than 25.9.

16. *(77)* **Geometry** A triangle has side lengths of 14 meters, 19 meters, and 21 meters. Use Heron's formula to find the area of the triangle.

***17.** *(96)* **Multi-Step** Complete parts a–d to graph a polar equation and analyze the graph.

a. Graph the polar equation $r = \frac{\cos\theta}{\sin^2\theta}$ on a calculator, using these window settings: $\theta\,\text{min} = 0$, $\theta\,\text{max} = \pi$, $\theta\,\text{step} = \frac{\pi}{24}$, $X\,\text{min} = -1$, $X\,\text{max} = 11$, $Y\,\text{min} = -4$, $Y\,\text{max} = 4$.

b. What shape does the graph appear to be?

c. Convert the polar equation to a Cartesian equation. (Hint: Multiply both sides by $\sin^2\theta$, then multiply both sides by r, and then substitute x for $r\cos\theta$ and y for $r\sin\theta$.)

d. The equation obtained in part c is the inverse of what function?

18. *(9)* **Football** The first table below shows the scoring plays made in three games by the Philadelphia Eagles. The second table shows the point value of each play. Use matrix multiplication to find out in which game the team scored the most points.

Plays Made

Game	Touchdown	Field Goal	Extra Point
#9	3	2	3
#15	2	3	2
#16	3	1	3

Point Value

Touchdown	6
Field Goal	3
Extra Point	1

19. *(72)* Expand the expression $\log 3x^4$.

20. (12) **Multiple Choice** Based on the data set below, what is the relationship between x and y?

x	1	2	3	4
y	4	2	$1\frac{1}{3}$	1

A Joint Variation **B** Direct Variation

C Inverse Variation **D** None of the above

21. (98) **Multiple Choice** Which of the following points is not on the ellipse $\frac{x^2}{4^2} + \frac{y^2}{7^2} = 1$?

A $(-7, 0)$ **B** $(4, 0)$ **C** $(0, -7)$ **D** $(-4, 0)$

22. (82) **Error Analysis** Two students are finding the period of $y = 5 \sin\left(\frac{1}{\sqrt{2}+1}\, x\right)$ but get different results. Which student made the mistake?

Student A	Student B
The period is $\frac{2\pi}{b}$.	The period is $\frac{2\pi}{b}$.
Period $= \frac{2\pi}{\sqrt{2}+1}$	Period $= 2\pi(\sqrt{2}+1)$

23. (84) **Analyze** How can a rational equation with all linear polynomials form a quadratic equation after the fractions are cleared? Give a simple example.

24. (87) **Write** Explain the steps used to apply the change of base formula.

25. (90) **Generalize** Find the period and phase shift of $y = a \tan(bx + c)$.

***26.** (100) Let $f(x) = x - 1$ and $g(x) = x^2 - 1$. Find the vertical asymptotes and holes for $\frac{f(x)}{f(g(x))}$.

Use $P(x) = x^3 - 10x^2 + 19x + 30$.

***27.** (95) Determine if $x - 6$ is a factor of $P(x)$.

***28.** (95) Factor $P(x)$.

29. (39) Determine if (1, 1) is a solution of the inequality $y < -5x + 3$.

***30.** (101) **Analyze** Which function has end behavior that is different from the end behavior of the others? Explain.

$f(x) = 4x^3 - x^2 - 2x$ $g(x) = x^7 - 1$ $h(x) = -3x^3 - 3x^4 + x^5$ $k(x) = -5x^3 + x^2 + x$

LESSON 102

Solving Logarithmic Equations and Inequalities

Warm Up

1. (81) **Vocabulary** In the expression ln t, the _____ is e.
2. (87) What is the value of $\log_3 9$?
3. (72) True or False: $\log m + \log(m + 4) = \log[m(m + 4)]$
4. (72) True or False: $\log a - \log(a - 6) = \log[a - (a - 6)]$
5. (72) Write $\log_x x = 1$ in exponential form.

New Concepts

A **logarithmic equation** is an equation with a logarithmic expression that contains a variable. A **logarithmic inequality** is an inequality with a logarithmic expression that contains a variable. You can use the following properties of logarithms to solve logarithmic equations and inequalities.

If $\log_b x = \log_b y$, then $x = y$.	Definition of logarithm
$b^{\log_b x} = x$ and $\log_b b^x = x$.	Inverse Properties of Logarithms and Exponents
$\log_b mn = \log_b m + \log_b n$	Product Property of Logarithms
$\log_b \frac{m}{n} = \log_b m - \log_b n$	Quotient Property of Logarithms
$\log_b a^p = p \log_b a$	Power Property of Logarithms

Hint

In the properties, the values of b, m, n, and a must be positive, and $b \neq 1$.

Example 1 Solving Logarithmic Equations

a. Solve $\log_5(x + 2) = 3$.

SOLUTION

$\log_5(x + 2) = 3$

$5^{\log_5(x+2)} = 5^3$	Use 5 as the base for both sides.
$x + 2 = 125$	Use an inverse property of logarithms and exponents.
$x = 123$	Subtract 2 from both sides.

Hint

Using the definition of logarithm, Example 1a can also be solved by expressing the logarithm in exponential form:

$5^3 = x + 2$

b. Solve $\log_8 x^3 = 2$.

SOLUTION

$\log_8 x^3 = 2$

$3 \log_8 x = 2$	Power Property of Logarithms
$\log_8 x = \frac{2}{3}$	Divide both sides by 3 to isolate $\log_8 x$.
$x = 8^{\frac{2}{3}}$	Definition of a logarithm
$x = (2^3)^{\frac{2}{3}}$	Write 8 as 2^3.
$x = 2^2$, or 4	Power of a Power Property

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c. Solve $\log x + \log(x - 1) = \log 6$.

SOLUTION

$$\log x + \log(x - 1) = \log 6$$

$\log[x(x - 1)] = \log 6$ Product Property of Logarithms

$x(x - 1) = 6$ If $\log_b x = \log_b y$, then $x = y$.

$x^2 - x - 6 = 0$ Multiply, and then subtract 6 from both sides.

$x = 3$ or $x = -2$ Solve the equation.

Caution

Check calculated solutions to verify that they are solutions of the original equation.

Check Check both solutions in the original equation.

$\log x + \log(x - 1) = \log 6$

$\log 3 + \log(3 - 1) \stackrel{?}{=} \log 6$

$\log 3 + \log 2 \stackrel{?}{=} \log 6$

$\log(3 \cdot 2) \stackrel{?}{=} \log 6$

$\log 6 = \log 6$ ✓

$\log x + \log(x - 1) = \log 6$

$\log(-2) + \log(-2 - 1) \stackrel{?}{=} \log 6$ ✗

$\log(-2)$ is undefined.

The only solution is $x = 3$.

To solve a logarithmic inequality, the following logarithmic property will be used.

If $\log_b x < \log_b y$, then $b^{\log_b x} < b^{\log_b y}$ and $x < y$.

Example 2 Solving a Logarithmic Inequality

Solve $\log 12x - \log 4 < \log 25$.

SOLUTION

$$\log 12x - \log 4 < \log 25$$

$\log\left(\frac{12x}{4}\right) < \log 25$ Quotient Property of Logarithms

$\log 3x < \log 25$ Simplify.

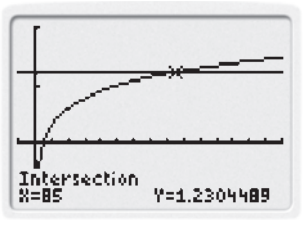
$3x < 25$ If $\log_b x < \log_b y$, then $x < y$.

$x < \frac{25}{3}$ Divide both sides by 3.

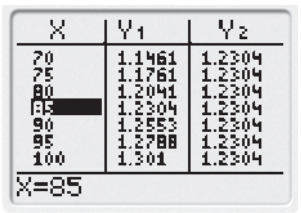
Example 3 Using a Graph and Table to Solve

Use a graph and table on a graphing calculator to solve $\log x - \log 5 = \log 17$.

SOLUTION Enter $\log x - \log 5$ as **Y1** and $\log 17$ as **Y2.**



Find the x-value of the point of intersection of the graphs.



Find the x-value for which Y1 = Y2.

The solution is $x = 85$.

Example 4 Application: Population Growth

Based on U.S. Census statistics, the population of Connecticut for the years 1950–1990 can be modeled by the equation

$$y = 147 + 812 \ln t,$$

where $10 \le t \le 50$, $t = 10$ represents 1950, and y is the population in thousands. Use the model to approximate the year in which the population of Connecticut reached 3,000,000.

SOLUTION

$147 + 812 \ln t = y$	Write the model.
$147 + 812 \ln t = 3000$	Since y is in thousands, substitute 3000 for y.
$812 \ln t = 2853$	Subtract 147 from both sides.
$\ln t = \frac{2853}{812}$	Divide both sides by 812.
$\ln t \approx 3.513$	Use a calculator to approximate.
$e^{\ln t} \approx e^{3.513}$	Use e as the base on both sides.
$t \approx e^{3.513}$	Use an inverse property of logarithms and exponents.
$t \approx 34$	Use a calculator to approximate.

Since $t = 10$ represents 1950, then $t \approx 34$ represents $1950 + 24 = 1974$.

Using the model, the population of Connecticut reached 3,000,000 in approximately 1974.

Math Reasoning

Write Note that $e^{3.513} \approx 33.5$. Explain why 1973 is another reasonable answer for Example 4.

Lesson Practice

a. (Ex 1) Solve $\log_3(x - 1) = 2$.

b. (Ex 1) Solve $\log_{125} x^3 = 4$.

c. (Ex 1) Solve $\log x + \log(x + 9) = 1$.

d. (Ex 2) Solve $\log 45x - \log 3 \ge 1$.

e. (Ex 3) Use a graph and table on a graphing calculator to solve $\log x - \log 2 = \log 75$.

f. (Ex 4) Based on U.S. Census statistics, the population of Ohio for the years 1950–1990 can be modeled by the equation $y = 3875 + 1876 \ln t$, where $10 \le t \le 50$, $t = 10$ represents 1950, and y is the population in thousands. Use the model to approximate the year in which the population of Ohio reached 10,000,000.

Practice Distributed and Integrated

1. (80) A set of scores have a mean of 74 and a standard deviation of 5. Find the percent of scores between 69 and 84.

Find the roots of each equation.

2. (85) $y = 4x^2 + 5x - 1$

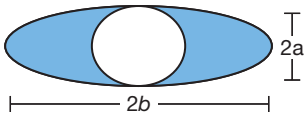
3. (85) Find the roots of $y = x^4 - 16$.

***4.** (102) **Union Membership** Beginning in the 1930s, labor union membership in the U.S. grew dramatically. During the years 1935–1955, labor union membership in the U.S. can be modeled by the equation $y = -4.4 + 12.6\ln t$, where $5 \le t \le 25$, $t = 5$ represents 1935, and y is the percent of the U.S. labor force that were union members. Use the model to approximate the year in which union membership in the U.S. reached 30%.

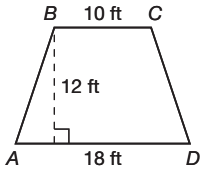
***5.** (101) **Analyze** A cubic polynomial function $f(x)$ has exactly three real zeros; they are 1, 3, and 4. It has a relative maximum at an x-value between 3 and 4. Describe the end behavior of $f(x)$.

6. (84) **National Highways** In Rhode Island, the speed limit outside a business or residential district is twice the speed limit inside a business or residential district. A driver drove 12.5 miles inside business and residential districts and 75 miles outside of business and residential districts for a total driving time of 2 hours. Suppose the driver kept a constant speed at the posted speed limits. Write and solve a rational equation to find the speed limit inside a business or residential district.

***7.** (98) **Probability** Use the equation for the area of an ellipse, πab, to find the probability of a point randomly landing within the ellipse but outside the circle.



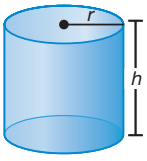
8. (96) **Construction** The diagram represents a plan for a deck in the shape of an isosceles trapezoid. Write polar coordinates for points A, B, C, and D. Use point A as the pole and $\overrightarrow{AD}$ as the polar axis.



***9.** (99) **Generalize** For two nonzero vectors $\vec{A}$ and $\vec{B}$, what must be true for $\vec{A} + \vec{B} = 0$?

10. (32) Find the inverse of the matrix $T = \begin{bmatrix} -6 & -2 \\ 11 & -3 \end{bmatrix}$, if it exists.

***11.** (100) **Geometry** Write the rational function that shows the ratio of the surface area to the volume for a right circular cylinder. Identify any asymptotes.



12. (77) **Geography** On a map of southwest Illinois, line segments connect Jerseyville, Carlinville, and Litchfield. The angle formed at Jerseyville measures 18°. The distance from Jerseyville to Carlinville is 28 miles and the distance from Jerseyville to Litchfield is 37 miles. To the nearest tenth of a mile, what is the distance from Carlinville to Litchfield?

13. (88) **Analyze** The formula for the surface area of a sphere is $S = 4\pi r^2$. The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Solve each equation for r. Explain why the first equation has a solution that cannot represent a possible value for the radius. Then tell why this is not true of the second equation.

14. (93) **Write** Explain why the equations $4^x = 100$ and $x \log 4 = 2$ are equivalent.

15. (73) **Town Government** A town council wants to know if residents support having an off-leash area for dogs in the town park. Eighty dog owners are surveyed at the park. Describe the individuals in the population and in the sample.

***16.** (102) **Multiple Choice** What is the solution to $\log 5x + \log 2 = 4$?

A $x = 1000$ **B** $x = 1428.6$ **C** $x = 1999.6$ **D** $x = 9990$

***17.** (102) Solve $\log_4(4x + 7) = \log_4 11x$.

***18.** (97) **Graphing Calculator** The first term of a geometric sequence is 9 and the common ratio is $\frac{2}{3}$.

a. Explain how to use the Table function to find the value of the 14^{th} term.

b. Explain how to use the Graph function to find the value of the 14^{th} term.

c. Rounded to the nearest ten-thousandth, what is the value of the 14^{th} term?

19. (82) Given $g(x) = 3 \sin \frac{1}{2}(x)$, identify the amplitude and period.

20. (85) **Physics** A spaceship is coming in for a landing on Mars. The equation $h = -1.885t^2 + v_o t + h_o$ can be used to find the distance h from the ground at time t based on an initial height h_o and an initial speed of v_o. To the nearest hundredth of a second, how long it will take the ship to land from an initial height of 2000 and an initial speed of 200?

***21.** (101) **Error Analysis** A student identified the degree of $P(x) = 2x^3 + 3x^2 - x^4$ as 3. What is the error?

22. (91) **Multi-Step** The map of a town is placed on a coordinate grid. The center of a small earthquake is located at $(-1, -3)$ and can be felt up to 12 miles away.

a. Write the equation of the circle that represents the border of the affected area.

b. Justify Show how to use the equation in part a to prove that people in a building located at $(11, -2)$ did not feel the earthquake.

23. (9) **Multi-Step** Use $A = \begin{bmatrix} 10 & 1 \\ 4 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 11 & 4 \\ 16 & 11 \end{bmatrix}$.

a. Find AB. **b.** Find BA. **c.** Is AB = BA?

24. (83) Write a quadratic equation whose roots are $6 \pm 2i$.

***25.** (100) Factor the terms of the rational function $y = \frac{x^2 + 3x - 4}{x - 2}$ to identify vertical asymptotes.

***26.** (98) Write an equation in standard form for the ellipse with center $(0, 0)$, $a = 4$ and $b = 6$.

Divide using synthetic division.

27. (51) $x^3 - 6$ by $x - 2$

28. (51) $x^3 + 3x^2 + 7x + 5$ by $x + 1$

29. (41) Find the distance between $(-2, 5)$ and $(3, 4)$.

30. (5) Solve $4\begin{bmatrix} 2 & 1 \\ -6 & 3.5 \end{bmatrix} - 3\begin{bmatrix} 0 & 0.4 \\ 8 & -2 \end{bmatrix}$.

LESSON
103

Graphing Reciprocal Trigonometric Functions

Warm Up

1. (46) Vocabulary The function $y = \cot(x)$ is an example of a _____ trigonometric function.

2. (90) What is the period of the function of $y = 3\tan(x)$?

3. (90) What are the asymptotes of the function $y = \tan(2x) + 2$?

4. (86) Graph the function $y = 3\cos(x)$.

New Concepts

The cosecant function is the reciprocal of the sine function. Since $\csc(x) = \frac{1}{\sin(x)}$, cosecant is undefined when $\sin(x) = 0$. The graph of cosecant has vertical asymptotes at the locations where it is undefined. Cosecant has the same period as sine. Cosecant has no absolute maxima or minima, so it has no amplitude. The graph of $y = a \cdot \csc(bx)$, where $a \neq 0$, has a period of $\frac{2\pi}{|b|}$ and asymptotes located at $x = \frac{\pi n}{|b|}$.

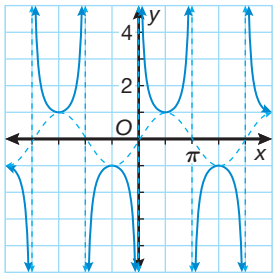
Hint

$\sin(x) = 0$ when $x = \pi n$, where n is an integer.

Example 1 Graphing the Cosecant Function

a. Graph $y = \csc(x)$. Determine its period and asymptotes.

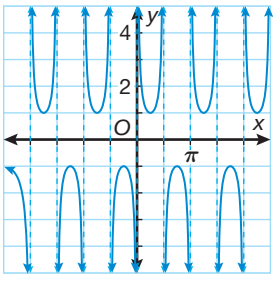
SOLUTION Graph $y = \sin x$. Vertical asymptotes of $y = \csc(x)$ occur where $\sin(x) = 0$ which are the x-intercepts. Find the reciprocals of the y-values on the sine curve to sketch $y = \csc(x)$.



x	$\sin(x)$	$\csc(x)$
0	0	undefined
$\frac{\pi}{4}$	0.707	1.414
$\frac{\pi}{2}$	1	1

$y = \csc(x)$ has a period of 2π. The asymptotes occur at $x = \pi n$, where n is an integer.

b. Graph $y = \csc(2x)$. Determine its period and asymptotes.



SOLUTION $y = \csc(2x)$ has a period of $\frac{2\pi}{|2|} = \pi$. The asymptotes occur at $x = \frac{\pi}{|2|}n = \frac{\pi}{2}n$, where n is an integer.

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The secant function is the reciprocal of the cosine function. Since $\sec(x) = \frac{1}{\cos(x)}$, secant is undefined when $\cos(x) = 0$. The graph of secant has vertical asymptotes at the locations where it is undefined. Secant has the same period as cosine. Secant has no absolute maxima or minima, so it has no amplitude. The graph of $y = a \cdot \sec(bx)$, where $a \neq 0$, has a period of $\frac{2\pi}{|b|}$ and asymptotes located at $x = \frac{\pi}{2|b|} + \frac{\pi}{|b|}n$.

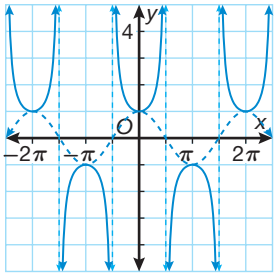
Example 2 Graphing the Secant Function

a. Graph $y = \sec(x)$. Determine its period and asymptotes.

Hint

$\cos(x) = 0$ when $x = \frac{\pi}{2} + \pi n$, where n is an integer.

SOLUTION Graph $y = \cos x$. Vertical asymptotes of $y = \sec(x)$ occur where $\cos(x) = 0$ which are the x-intercepts. Find the reciprocals of the y-values on the cosine curve to sketch $y = \sec(x)$.



x	$\cos(x)$	$\sec(x)$
0	1	1
$\frac{\pi}{4}$	0.707	1.414
$\frac{\pi}{2}$	0	undefined
$\frac{3\pi}{4}$	−0.707	−1.414
π	1	1

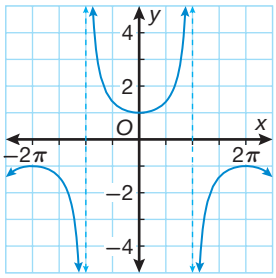
$y = \sec(x)$ has a period of 2π. The asymptotes occur at $x = \frac{\pi}{2} + \pi n$, where n is an integer.

b. Graph $y = \sec\left(\frac{x}{2}\right)$. Determine its period and asymptotes.

SOLUTION $y = \sec\left(\frac{x}{2}\right)$ has a period of $\frac{2\pi}{\left|\frac{1}{2}\right|} = 4\pi$.

The asymptotes occur at $x = \frac{\pi}{2\left|\frac{1}{2}\right|} + \frac{\pi}{\left|\frac{1}{2}\right|}n =$

$\pi + 2\pi n$, where n is an integer.



Hint

$\tan(x) = 0$ when $x = \pi n$, where n is an integer.

The cotangent function is the reciprocal of the tangent function. Since $\cot(x) = \frac{1}{\tan(x)}$, cotangent is undefined when $\tan(x) = 0$. The graph of cotangent has vertical asymptotes at the locations where it is undefined. Cotangent has the same period as tangent. Cotangent has no absolute maxima or minima, so it has no amplitude. The graph of $y = a \cdot \cot(bx)$, where $a \neq 0$, has a period of $\frac{\pi}{|b|}$ and asymptotes located at $x = \frac{\pi n}{|b|}$.

Example 3 Graphing the Cotangent Function

Graph $y = \cot(x)$. Determine its period and asymptotes.

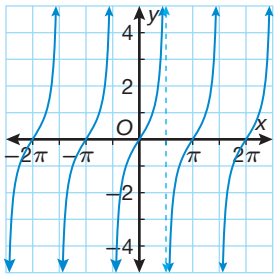
Math Reasoning

Analyze Explain why $y = \tan x$ has asymptotes where $\cos x = 0$.

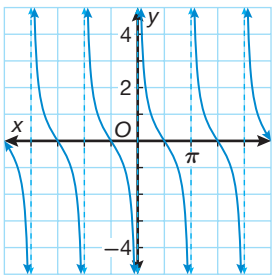
SOLUTION Graph $y = \tan x$. Vertical asymptotes for $y = \cot x$ occur where $\tan(x) = 0$ which are the x-intercepts. $y = \cot x$ has x-intercepts where $y = \tan x$ has vertical asymptotes.

$y = \cot(x)$ has a period of π. The asymptotes occur at $x = \pi n$, where n is an integer.

$y = \tan x$



$y = \cot(x)$



Example 4 Using a Graphing Calculator to Graph a Reciprocal Trigonometric Function

Use a graphing calculator to graph $y = \sec(x)$.

SOLUTION

There are no specific keys or menu items on the graphing calculator for the secant, cosecant, or cotangent functions. In order to graph them on the graphing calculator, use:

Caution

$\frac{1}{\cos(x)} \neq \cos^{-1}(x)$

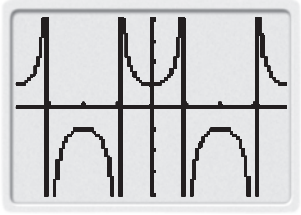
$$\csc(x) = \frac{1}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)} \qquad \cot(x) = \frac{1}{\tan(x)}$$

Step 1: On the calculator, press the **MODE** button and set the **RADIAN** mode.

Step 2: Enter the equation $Y = \frac{1}{\cos}(X)$.

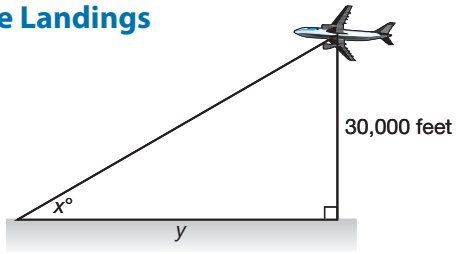
Step 3: Press the **ZOOM** button and select 7 ↓**ZTrig**

The graph of $y = \sec(x)$ is displayed.



Example 5 Application: Airplane Landings

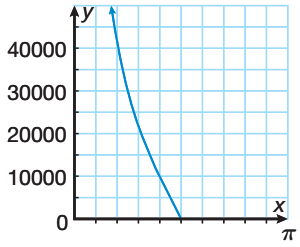
An airplane flying at an altitude of 30,000 feet is headed toward an airport. The airport's landing system sends radar signals from the runway to the airplane at an angle of elevation, x. Let y be the horizontal distance from the runway to a point directly below the airplane. Write y as a function of x and sketch the graph of the function over the interval 0 to π.



SOLUTION Use the cotangent function:

$$\cot(x) = \frac{\text{adjacent}}{\text{opposite}} = \frac{y}{30{,}000}$$

$$30{,}000 \cot(x) = y$$



Lesson Practice

(Ex 1) **a.** Graph $y = 2 \csc(x)$. Determine its period and asymptotes.

(Ex 1) **b.** Graph $y = \csc\left(\frac{1}{2}\right)x$. Determine its period and asymptotes.

(Ex 2) **c.** Graph $3 \sec(x)$. Determine its period and asymptotes.

(Ex 2) **d.** Graph $\sec(2x)$. Determine its period and asymptotes.

(Ex 3) **e.** Graph $4 \cot(x)$. Determine its period and asymptotes.

(Ex 3) **f.** Graph $\frac{1}{4} \cot(2x)$. Determine its period and asymptotes.

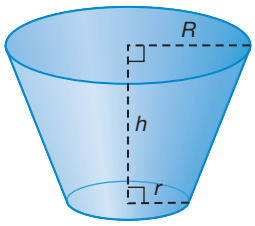
(Ex 4) **g.** Use a graphing calculator to graph $y = \frac{4}{5} \csc(x)$.

(Ex 5) **h.** An airplane is flying at an altitude of 22,000 feet. Signals are sent to the airplane from the runway at an angle of elevation, x. Let y be the distance from the runway to a point directly below the airplane. Write y as a function of x and sketch the graph of the function over the interval 0 to π.

Practice Distributed and Integrated

(103) ***1. Graphing Calculator** Use a graphing calculator to graph $y = 3 \sec(2x)$ from 0 to 2π.

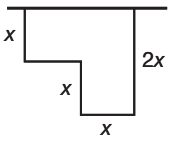
(88) **2. Multi-Step** A drinking glass is to be in the shape of a frustum of a right circular cone. The formula for the volume of a frustum of a right circular cone is $V = \frac{\pi(r^2 + rR + R^2)h}{3}$.



a. Solve the formula for h.

b. If $R = 5$ centimeters and $r = 4$ centimeters, what is the minimum height necessary, to the nearest tenth of a centimeter, so that the glass will hold at least 500 cubic centimeters?

*3. (101) **Landscape Design** A landscape architect has 96 feet of landscape timber to enclose a garden, using an existing wall for one side as shown. The area of the garden can be modeled by $A(x) = 96x - 3x^2$, where x is in feet. Find the maximum area of the garden.



4. (96) **Multiple Choice** What are the Cartesian coordinates for the point with polar coordinates $(5, -\frac{\pi}{4})$?

A $(5\sqrt{2}, -5\sqrt{2})$ **B** $(-5\sqrt{2}, 5\sqrt{2})$ **C** $\left(\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$ **D** $\left(-\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$

*5. (97) **Population** The estimated population of Illinois from 2005 to 2006 increased by about 0.52%. Assuming that the percent of increase stays the same, the populations each year form a geometric sequence. Show how to use the formula for the nth term of a geometric sequence to find the population of Illinois in 2010 given that the estimated population in 2005 was 12,765,427.

6. (33) How many 4-digit pass codes can be made using the numbers 1–5, if numbers may be repeated?

7. (89) **Business** The profit earned from selling tickets to a certain event is modeled by $y = -1.5x^2 + 80x - 600$. If the cost per ticket will be a whole number amount, what range of ticket prices will earn a profit? What range of ticket values will earn a profit of at least \$200?

8. (21) Solve the system by substitution. $\begin{aligned} 2x + 5y &= 7 \\ x + 4y &= 2 \end{aligned}$

9. (99) **Geometry** For any two vectors expressed as complex numbers, find the dot product from the components of the complex numbers.

10. (92) **Justify** When the ordered pairs (n, a_n) for the sequence $\{3, 7, 11, 15, 19\}$ are graphed, the points form a straight line. Show algebraically why the slope is equal to the common difference.

*11. (100) **Model** The graph of a rational function has vertical asymptotes at $x = 7$ and -3 and holes at $x = -2$ and 0. Write an equation of a function that can be used to model these data points.

12. (84) **Multiple Choice** To solve the equation, what should both sides of $\frac{1}{x^2 - 4} + \frac{3}{4x + 8} = \frac{6}{x}$ be multiplied by?

A $(x + 2)(x - 2)$ **B** $4(x + 2)(x - 2)$

C $4x(x + 2)(x - 2)$ **D** $4x(x + 2)^2$

13. (11) **Formulate** Give an example of two trinomials whose sum has 6 terms.

14. (25) Find the range and standard deviation of the data set 10, 12, 7, 11, 20, 7, 8, 6, 9.

*15. (98) **Formulate** The eccentricity of an ellipse centered at the origin is 0.92. The graph of the ellipse crosses the minor axis at (0, 2). Write the equation of the ellipse.

16. (83) **National Monuments** The height and span of the arch legs of the St. Louis Gateway Arch are both 630 feet. Write a quadratic function to approximate the arch.

Identify all the real roots.

17. (85) $x^9 + 1331 = 0$

18. (85) $x^3 + 10x^2 + 17x = 28$

19. (90) **Physics** Derive an equation that an air traffic controller can use to find the angle an airplane makes with the horizon, given the plane's altitude h and its horizontal distance from the airport d.

20. (95) Evaluate the polynomial $P(x)$ for $x = 4$. $P(x) = 2x^3 - 3x^2 + x - 5$

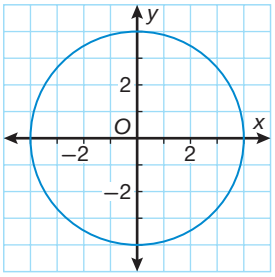
***21.** (103) Find the period and asymptotes for $y = 3 \cot \frac{2}{3}x$.

22. (93) **Multi-Step** Iron-62 has a half-life of 68 seconds.

a. Use the natural decay function $N(t) = N_0e^{-kt}$ to find the decay constant k for iron-62.

b. Write the natural decay function for iron-62 and solve for t to find how long it would take 10 grams to decay to 7 grams.

23. (91) **Error Analysis** Explain and correct the error a student made in graphing $x^2 + y^2 = 4$.



***24.** (102) **Error Analysis** A student solved $\log x + 5 = 4$ incorrectly, obtaining 9,995 as the solution. What error did the student likely make?

***25.** (102) **Write** Can a logarithmic equation have a negative number as a solution? Justify your answer. Give an example, if possible.

Simplify the expression.

26. (40) $\frac{48}{\sqrt{3}}$

27. (40) $\sqrt{108} + \sqrt{12} + \sqrt{112}$

28. (79) Write a piecewise function for a hardware store that will deliver up to 6 items for a $35 delivery charge. There is a $7 charge for each additional item.

***29.** (94) Solve $\frac{x^2 - 3x + 2}{x - 3} < x$ by finding the values of test points.

***30.** (103) A traffic helicopter is flying at an altitude of 8,000 feet. The driver spots an accident at angle of elevation, x. Let y be the distance from the accident to a point directly below the helicopter. Write y as a function of x.

LESSON
104

Finding Transformations

Warm Up

1. (17) **Vocabulary** _____ can be used to change the graph of one function into the graph of a different function.

2. (17) Write the equation that will shift $f(x) = |x|$ to the left 2 units.

3. (75) Which quadrant(s) is $f(x) = \sqrt{x} + 20$ graphed in?

New Concepts

A graph can be altered to form new equations by shifting the graph vertically or horizontally, reflecting the graph about the x- and y-axis, and stretching or compressing the graph.

The table below summarizes the transformations.

Transformation	Equation	Change in the Graph
Vertical Shifts	$y = f(x) + a$	If $a > 0$, then the graph of f shifts up by a units. If $a < 0$, then the graph of f shifts down by a units.
Horizontal Shifts	$y = f(x - a)$	If $a < 0$, then the graph of f shifts to the left a units. If $a > 0$, then the graph of f shifts to the right a units.
Reflections	$y = -f(x)$	Reflects the graph of f about the x-axis.
	$y = f(-x)$	Reflects the graph of f about the y-axis.
Vertical Stretch/ Compression	$y = af(x)$	If $a > 1$, the graph of f vertically stretches. If $0 < a < 1$, the graph of f vertically compresses.

Math Language

The function $f(x) = \sqrt{x}$ is called the **Square Root Parent Function.** A **parent function** is a basic function to which transformations are applied to create a family of functions.

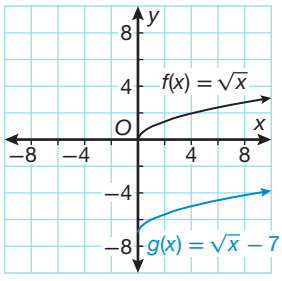
Example 1 Transforming the Square Root Function

a. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of $g(x) = \sqrt{x} - 7$.

SOLUTION Compare the equation $g(x)$ to that of $f(x)$. The equation for $g(x)$ subtracts 7 from each value of $f(x)$.

$$y = g(x) = \sqrt{x} - 7 = f(x) - 7$$

The graph of $g(x)$ will have the same shape as $f(x)$ only it will be shifted down 7 units.



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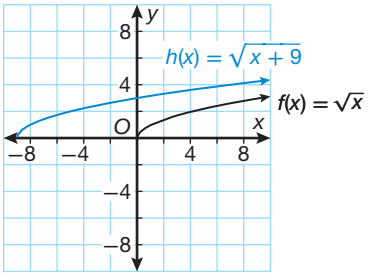
b. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of $h(x) = \sqrt{x + 9}$.

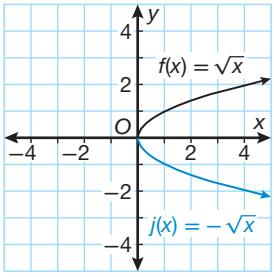


SOLUTION Compare the equation $h(x)$ to that of $f(x)$.

$y = h(x) = \sqrt{x + 9} = f(x - (-9))$

The graph of $h(x)$ will have the same shape as $f(x)$ only it will be shifted to the left 9 units.

c. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of $j(x) = -\sqrt{x}$.

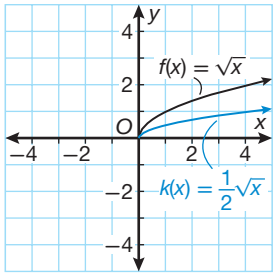


SOLUTION Compare the equation $j(x)$ to that of $f(x)$. The equation for $j(x)$ is the opposite of each value of $f(x)$.

$y = j(x) = -\sqrt{x} = -f(x)$

The graph of $j(x)$ is a reflection about the x-axis of $f(x)$.

d. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of $k(x) = \frac{1}{2}\sqrt{x}$.



SOLUTION Compare the equation $k(x)$ to that of $f(x)$. The equation for $k(x)$ multiplies each value of $f(x)$ by $\frac{1}{2}$.

$y = k(x) = \frac{1}{2}\sqrt{x} = \frac{1}{2}f(x)$

By vertically compressing $f(x)$, $k(x)$ is obtained.

Hint

To find the coordinates of a graph that is being vertically stretched or compressed, multiply each *y*-coordinate of the parent function by the constant *a*.

A function with multiple transformations can be graphed by implementing the transformations in the following order: horizontal shifts, then vertical stretching or compressing, then reflections, and then vertical shifts.

Example 2 Finding Multiple Transformations of the Square Root Function

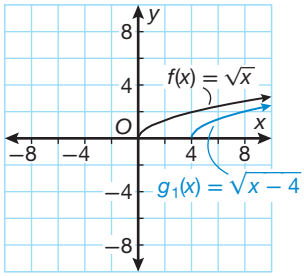
Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of $g(x) = -\sqrt{x - 4} + 8$.

SOLUTION

Step 1: Horizontal Shifts; The graph of $f(x)$ shifts to the right 4 units.

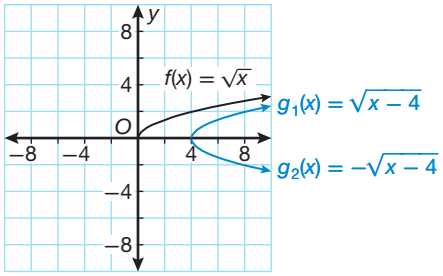
Step 2: Vertical Stretching or Compressing; Since $f(x)$ is not multiplied by a constant, there is no vertical stretching or compressing.

Step 1



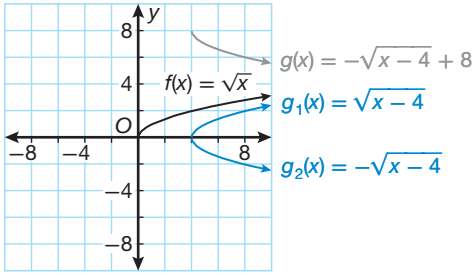
Step 3: Reflections: Since $f(x)$ is replaced by $-f(x)$, this reflects the equation about the x-axis.

Step 3



Step 4: Vertical Shifts: Since 8 is being added to $f(x)$, this will shift up the graph 8 units. See the graph at right.

Step 4



Hint

Exponential parent functions are of the form $f(x) = b^x$.

Transformations can also be performed on exponential functions to obtain new graphs. Horizontal and vertical translations can be used to shift the graph up, down, left, and right.

The table below summarizes transformations of exponential functions.

Transformation	Equation	Change in the Graph
Vertical Translations	$y = b^x + a$	If $a > 0$, raises the graph of f by a units. If $a < 0$, lowers the graph of f by a units.
Horizontal Translations	$y = b^{x-a}$	If $a < 0$, shifts the graph of f to the left a units. If $a > 0$, shifts the graph of f to the right a units.
Reflections	$y = -b^x$	Reflects the graph of f about the x-axis.
	$y = b^{-x}$	Reflects the graph of f about the y-axis.
Vertical Stretching/ Compressing	$y = ab^x$	If $a > 1$, vertically stretches the graph of f. If $0 < a < 1$, vertically compresses the graph of f.

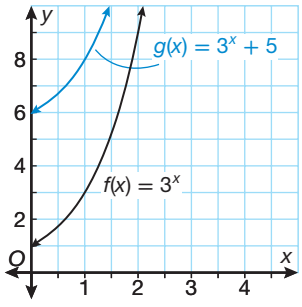
Example 3 Translating an Exponential Function

a. Use the graph of $f(x) = 3^x$ to sketch the graph of $g(x) = 3^x + 5$.

SOLUTION Compare the equation $g(x)$ to that of $f(x)$. The equation for $g(x)$ adds 5 to each value of $f(x)$.

$y = g(x) = 3^x + 5 = f(x) + 5$

The graph of $g(x)$ will have the same shape as $f(x)$ only it will be shifted up 5 units.

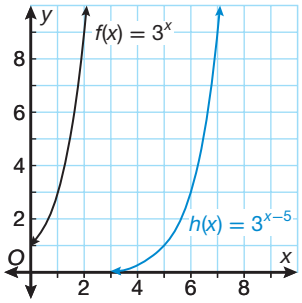


b. Use the graph of $f(x) = 3^x$ to sketch the graph of $h(x) = 3^{x-5}$.

SOLUTION Compare the equation $h(x)$ to that of $f(x)$.

$y = h(x) = 3^{x-5} = f(x - 5)$

The graph of $h(x)$ will have the same shape as $f(x)$ only it will be shifted to the right 5 units.



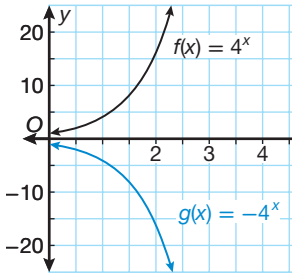
Example 4 Reflecting Exponential Functions

a. Use the graph of $f(x) = 4^x$ to sketch the graph of $g(x) = -4^x$.

SOLUTION Compare the equation $g(x)$ to that of $f(x)$. The equation for $g(x)$ is the opposite of $f(x)$.

$y = g(x) = -4^x = -f(x)$

The graph of $g(x)$ is a reflection about the x-axis of $f(x)$.



Math Reasoning

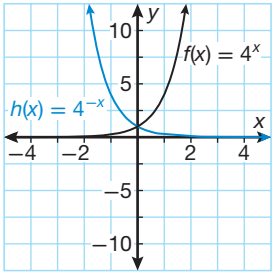
Write Explain the difference between the change in a function that is reflected about the *x*-axis and about the *y*-axis.

b. Use the graph of $f(x) = 4^x$ to sketch the graph of $h(x) = 4^{-x}$.

SOLUTION Compare the equation $h(x)$ to that of $f(x)$. The equation for $h(x)$ is the opposite at each value of x.

$y = h(x) = 4^{-x} = f(-x)$

The graph of $h(x)$ is a reflection about the y-axis of $f(x)$.



Example 5 Application: IQ Formula

The function $n(x) = 2\sqrt{x} - 9$ is used by psychologists to determine the number of nonsense syllables, $n(x)$, a person with an IQ of x can repeat. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of $n(x) = 2\sqrt{x} - 9$. Then use the graph of $n(x) = 2\sqrt{x} - 9$ to estimate the number of nonsense syllables a person with an IQ of 121 can repeat.

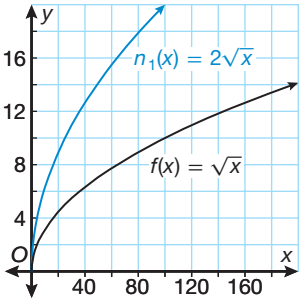
SOLUTION

To find the graph of $n(x)$, implement transformations to the graph of $f(x)$. The function $n(x)$ involves multiple transformations.

Implement the transformations in the correct order.

Step 1: Horizontal Shifts: none

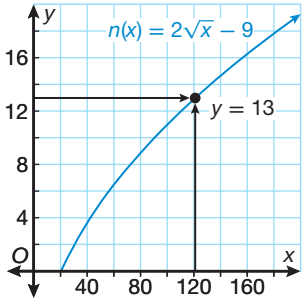
Step 2: Vertical Stretching or Compressing: $f(x)$ is being multiplied by 2, so the graph is vertically stretched by a factor 2.



Step 3: Reflections: none

Step 4: Vertical Shifts: 9 is being subtracted from $f(x)$. This will lower the graph 9 units.

Now use the graph to approximate the value of $n(x)$ when $x = 121$.



Substitute for the value of x in the function $n(x)$. See if this value is close to the value found.

$n(121) = 2\sqrt{121} - 9 = 2(11) - 9 =$
$22 - 9 = 13$

A person with an IQ of 121 can repeat 13 nonsense syllables.

Lesson Practice

Use the graph of $f(x) = \sqrt{x}$ to sketch each of the following.

a. *(Ex 1)* $h(x) = \sqrt{x} + 10$

b. *(Ex 1)* $h(x) = \sqrt{x - 1}$

c. *(Ex 1)* $h(x) = -\sqrt{x}$

d. *(Ex 1)* $h(x) = 4\sqrt{x}$

e. *(Ex 2)* $h(x) = \frac{1}{5}\sqrt{2 - x}$

f. *(Ex 3)* Use the graph of $f(x) = 5^x$ to sketch the graph of $g(x) = 5^x - 4$.

g. *(Ex 3)* Use the graph of $f(x) = 4^x$ to sketch the graph of $g(x) = 4^{x+6}$.

h. *(Ex 4)* Use the graph of $f(x) = 10^x$ to sketch the graph of $g(x) = -10^x$.

i. *(Ex 4)* Use the graph of $f(x) = 7^x$ to sketch the graph of $g(x) = 7^{-x}$.

j. *(Ex 5)* The time T, in seconds, for the period of a simple pendulum can be estimated by the equation $T(x) = 2.009\sqrt{x}$, where x is the length, in meters, of the pendulum. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of $T(x) = 2.009\sqrt{x}$. Use the graph of $T(x)$ to estimate the length of a pendulum whose period is 4 seconds.

Practice Distributed and Integrated

1. *(94)* **Multiple Choice** When solving $\frac{x^2 + 5x - 6}{x^2 - 12x + 32}$ by using a sign table, for how many intervals will a test point need to be chosen?

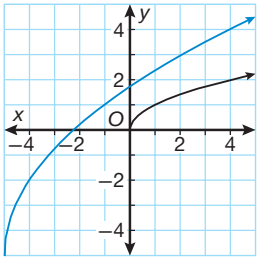
A 3 **B** 4 **C** 5 **D** 6

2. *(77)* A triangle has side lengths of 7.5 meters, 11 meters, and 13 meters. Use Heron's formula to find the area of the triangle.

***3.** *(103)* Graph $y = 4\cot\left(\frac{1}{2}x\right)$. Determine its period and asymptotes.

***4.** (90) **Graphing Calculator** Use a graphing calculator to graph $y = \tan(x + 6)$. Identify the period and undefined values of the function.

***5.** (104) **Error Analysis** In the coordinate plane shown, functions $f(x) = \sqrt{x}$ and $g(x) = 3\sqrt{x - 5} - 5$ are graphed. Is the graph of g correct? If not, find and explain the error and sketch the correct graph of g.



***6.** (101) **Finance** The NASDAQ Composite is an indicator of stock prices. The monthly low NASDAQ Composite for the year 2002 can be modeled by $f(x) = 2.229x^3 - 34.667x^2 + 60.339x + 1787$, where $x = 1$ represents January, $x = 2$ represents February, and so on, with $x = 12$ representing December. Use the model to approximate the lowest monthly NASDAQ Composite during 2002. Based on the model, in which month did it occur?

7. (95) **Analyze** Suppose that for polynomial $P(x)$, $P(a) = 0$ for constant a. What can you conclude about the graph of rational function $f(x) = \frac{1}{P(x)}$?

8. (96) **Error Analysis** A student incorrectly converted the polar coordinates $(-1, \pi)$ to the Cartesian coordinates $(-1, 0)$. What is the error? Find the correct Cartesian coordinates.

9. (87) **Radioactive Decay** A radioactive substance decays based on $d = d_0e^{-0.0063t}$. The time, in days, is represented by t, and d_0 represents the initial amount of the radioactive substance. Solve for t.

10. (97) **Chemistry** The half-life of Barium-131 is 12 days. The amounts of Barium-131 after every half-life (every 12^{th} day) form a geometric sequence. What is the common ratio of this sequence?

11. (80) **Geometry** The students in a geometry class each chose a polygon that has between 3 and 12 sides and wrote a paragraph about its side and angle measures. The table shows how many students chose each type of polygon. Tell if the data is normally distributed. Explain your answer.

Number of Sides in Polygon	3	4	5	6	7	8	9	10	11	12
Number of Students Who Chose Polygon	2	5	7	6	2	5	6	8	4	1

12. (98) **Formulate** The eccentricity of an ellipse centered at $(-3, 5)$ is 0.9682. The graph of the ellipse crosses the major axis at $(-3, 21)$. Write the equation of the ellipse.

13. (28) Identify any excluded values for the expression $\frac{12m^7}{3m^9}$, then simplify.

***14.** (99) **Multiple Choice** In what quadrant is the vector $22 + (-33)i$?

A I **B** II **C** III **D** IV

15. (34) **Formulate** Rewrite the following equation in slope-intercept form: $6x - 8y = 24$.

***16.** (100) **Model** The graph of a rational function has vertical asymptotes at $x = 0, -1$, and 3 and a hole at $x = 5$. Write an equation of a function that can be used to model these data points.

17. (79) **Data Analysis** The rates for media mail are shown in the table. Write a piecewise function that represents the cost of mailing a first-class letter.

Fee	Weight
\$1.30	up to 1 ounce
\$1.64	greater than 1 ounce and up to 2 ounces
\$1.98	greater than 2 ounces and up to 3 ounces

***18.** (103) **Error Analysis** Describe and correct the error in finding $\csc\theta$, given that θ is an acute angle of a right triangle and $\cos\theta = \frac{7}{11}$.

$$\csc\theta = \frac{1}{\cos\theta} = \frac{11}{7}$$

19. (92) **Salaries** An employee's salary is structured so that after the first year, the employee receives an annual raise of a set amount. What was the employee's starting salary if the salary in the eighth year was \$42,710 and the salary in the 17^{th} year was \$50,135?

***20.** (102) **Multi-Step** The Richter magnitude M of an earthquake is expressed by $M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$, where E is the energy released in ergs.

a. Find the energy released by an earthquake of magnitude 6.

b. Find the energy released by an earthquake of magnitude 7.

c. By what factor is the energy multiplied when the magnitude is increased by 1?

21. (88) **Simple Interest** The simple interest formula is $A = P + Prt$, where P is the principal, or original amount invested, r is the annual simple interest rate, and A is the value of the investment at the end of t years. Solve the formula for P. Then determine the amount P that must be invested at 5% annual simple interest to have a value of \$10,000 at the end of 2 years.

22. (10) Darryl lives 2 miles from the mall and Taylor lives 8 miles from the same mall. Write an inequality to determine how far Darryl and Taylor live from each other.

***23.** (104) **Write** Explain in complete sentences what must be done to an equation to vertically shift a graph down.

Use matrices $B = \begin{bmatrix} 3\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \end{bmatrix}$ and $C = \begin{bmatrix} 2\frac{1}{2} & \frac{1}{2} & -3 \end{bmatrix}$.

24. (5) Find $-C$.

25. (5) Find $\frac{1}{2}C + 4B$.

26. (31) Divide, and identify all values of x that make the expression $\frac{x^2 + 6x + 9}{x^2 - 9} \div (x - 7)$ undefined.

Solve and classify each of the following systems.

27. (24)
$$\begin{aligned} 4x - 3y &= 10 \\ 8x - 6y &= 20 \end{aligned}$$

28. (24)
$$\begin{aligned} 7x + 2y &= 11 \\ -2x + 3y &= 29 \end{aligned}$$

29. (82) Using $\sin(x)$ as a guide, find the phase shift in $g(x) = 2\sin 3(x - 8) + 5$.

***30.** (103) Find the period and asymptotes of $y = -\sec 4x$.

LESSON
105

Finding Arithmetic Series

Warm Up

1. (92) **Vocabulary** In an arithmetic sequence, the _____ is the difference between successive terms.

2. (92) Find the next three terms of the arithmetic sequence $-2, -5, -8, -11, \ldots$

3. (92) Find the 14^{th} term of the arithmetic sequence $11, 18, 25, 32, \ldots$

New Concepts

Recall that a sequence is an ordered list of numbers, or terms. Some sequences are arithmetic, with common differences, and some are geometric, with common ratios.

A **series** is the indicated sum of the terms of a sequence. An **arithmetic series** is the sum of the terms of an arithmetic sequence. It can be finite or infinite.

Math Language

The *indicated sum* is the expression that shows each addend. The *sum* is the actual result of adding the terms.

Arithmetic Sequence	Arithmetic Series
$4, 9, 14, 19$	$4 + 9 + 14 + 19$
$8, 5, 2, -1, \ldots$	$8 + 5 + 2 + (-1) + \ldots$

The sum, S, of a finite sequence is the sum of its terms. An infinite series does not have a sum, but it does have partial sums. A partial sum is the sum of the first n terms, indicated by S_n. For $8 + 5 + 2 + (-1) + \ldots$, S_4 is the sum of the first four terms, so $S_4 = 14$.

A shorthand way of writing a series is to use **summation notation**, sometimes called **sigma notation**, because it uses the Greek letter sigma, Σ.

Summation Notation

For the series $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$, the integer k is called the index, 1 is the lower limit of summation, and n is the upper limit of summation.

Example 1 Finding the Sum of an Arithmetic Series Given in Summation Notation

a. Find the sum of $\sum_{k=1}^{6} 2k$.

SOLUTION Write the indicated sum by replacing k with the given limits of summation. Then add the terms of the series.

$$\begin{aligned}\sum_{k=1}^{6} 2k &= 2(1) + 2(2) + 2(3) + 2(4) + 2(5) + 2(6)\\ &= 2 + 4 + 6 + 8 + 10 + 12\\ &= 42\end{aligned}$$

Online Connection
www.SaxonMathResources.com

b. Find the sum of $\sum_{k=3}^{7} 10 - k$.

SOLUTION

$$\sum_{k=3}^{7} 10 - k = (10 - 3) + (10 - 4) + (10 - 5) + (10 - 6) + (10 - 7)$$
$$= 7 + 6 + 5 + 4 + 3$$
$$= 25$$

Caution

The lower limit of summation is often 1, but it does not have to be.

Example 2 Writing an Arithmetic Series in Summation Notation

Write each arithmetic series in summation notation.

a. $5 + 10 + 15 + 20$

SOLUTION *Think:* The terms increase by 5. Each is a multiple of 5, so the rule is $5k$.

$\sum 5k$ Write the rule.

$\sum_{k=1}^{4} 5k$ Determine the limits of summation.

b. $14 + 11 + 8 + 5 + 2 + (-1)$

SOLUTION *Think:* The terms decrease by 3, so $3k$ is part of the rule. The first term is 14, so $3k$ must be 0 for the first term.

$\sum 14 - 3k$ Write the rule.

$\sum_{k=0}^{5} 14 - 3k$ Determine the limits of summation.

Hint

Notice that the vertical addends are the first and last term, then the second and next to the last term, and so on.

A formula for the sum of an arithmetic series can be found by writing an arithmetic series forwards and backwards and then adding the series.

Consider the series $3 + 7 + 11 + 15 + 19 + 23 + 27$.

$$S_7 = 3 + 7 + 11 + 15 + 19 + 23 + 27$$
$$S_7 = 27 + 23 + 19 + 15 + 11 + 7 + 3$$
$$2S_7 = 30 + 30 + 30 + 30 + 30 + 30 + 30$$
$$2S_7 = 7(30)$$
$$S_7 = \frac{7(30)}{2}$$

Notice that $7 = n$ and $30 = a_1 + a_n$. Then, $S_n = \frac{n(a_1 + a_n)}{2} = \frac{n}{2}(a_1 + a_2)$.

Sum of a Finite Arithmetic Series

The sum of the first n terms of an arithmetic series, S_n, is $\frac{n}{2}(a_1 + a_2)$.

A **partial sum** is the sum S_n of a specified number of terms, n, of an infinite series.

Example 3 Finding the Sum of the First *n* Terms of a Series

a. Find the partial sum S_{12} for the infinite arithmetic series $6 + 14 + 22 + 30 + \ldots$.

SOLUTION The formula for the sum requires a_1, which is given, and a_{12}, which can be found once the common difference, d, is known.

$d = 30 - 22 = 8$ Subtract to find d.

$a_{12} = 6 + (12 - 1)(8)$ $a_n = a_1 + (n - 1)d$

$a_{12} = 94$ Simplify.

Now use $S_n = \frac{n}{2}(a_1 + a_2)$.

$S_{12} = \frac{12}{2}(6 + 94)$ Substitute 12 for n, 6 for a_1, and 94 for a_{12}.

$S_{12} = 600$ Simplify.

b. Evaluate the series $\sum_{k=1}^{20}(3 + 2k)$.

SOLUTION Substitute 1 and 20 for k to find a_1 and a_{20}.

$a_1 = 3 + 2(1) = 5$

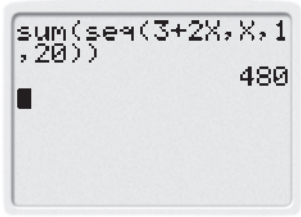
$a_{20} = 3 + 2(20) = 43$

Now use $S_n = \frac{n}{2}(a_1 + a_2)$.

$S_{20} = \frac{20}{2}(5 + 43)$ Substitute 20 for n, 5 for a_1, and 43 for a_{20}.

$S_{20} = 480$ Simplify.

Check by using a graphing calculator.



Graphing Calculator Tip

The **sum(** and **seq(** commands can be accessed by pressing 2nd STAT. Choose MATH 5, then 2nd STAT **OPS** 5.

Example 4 Application: Exercise

An athlete is on a 10-day running plan. On the plan, the athlete is to run 3 miles on the first day, and on each day thereafter, run 0.5 mile longer than the previous day. How many total miles will the athlete have run while on the plan?

SOLUTION

1. **Understand** The miles ran each day form a finite arithmetic sequence. The sum of the miles form an arithmetic series.

2. **Plan** Find a_{10} so that S_{10} can be calculated.

3. Solve Use $a_n = a_1 + (n-1)d$ to find a_{10}.

$$a_{10} = 3 + (10-1)(0.5)$$

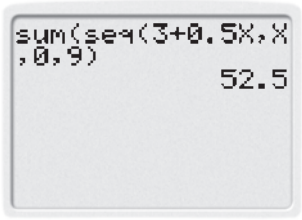
$$a_{10} = 7.5$$

Use $S_n = n\left(\frac{a_1 + a_n}{2}\right)$ to find S_{10}.

$$S_{10} = 10\left(\frac{3 + 7.5}{2}\right)$$

$$S_{10} = 52.5$$

The athlete will have run a total of 52.5 miles on the plan.



4. Check Use a graphing calculator.

Math Reasoning

Justify In the Check, why do the limits of summation of $3 + 0.5x$ go from 0 to 9?

Lesson Practice

(Ex 1) **a.** Find the sum of $\sum_{k=1}^{10} k$.

(Ex 1) **b.** Find the sum of $\sum_{k=1}^{4} 5 + 2k$.

Write each arithmetic series in summation notation.

(Ex 2) **c.** $8 + 16 + 24 + 32 + 40 + 48$

(Ex 2) **d.** $25 + 28 + 31 + 34$

(Ex 3) **e.** Find S_{18} for the arithmetic series $5 + (-5) + (-15) + (-25) + \ldots$

(Ex 3) **f.** Evaluate the series $\sum_{k=1}^{33} 6k$.

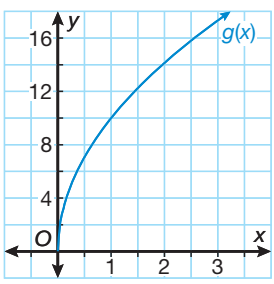
(Ex 4) **g.** An athlete is on a 7-day bicycling plan. On the plan, the athlete is to bike 6 miles on the first day, and on each day thereafter, bike 1.5 miles longer than the previous day. How many total miles will the athlete have biked while on the plan?

Practice Distributed and Integrated

(95) ***1.** Let $f(x) = x^4$ and $g(x) = x^2 - 5x + 6$. Find $P(3)$ for $P(x) = f(g(x))$.

(105) ***2. Verify** The first and second terms of an arithmetic sequence are 48 and 30 respectively. Show that the sum of the first 16 terms is -1392.

(104) ***3. Multiple Choice** The graph is a translation of the graph of $f(x) = \sqrt{x}$. Choose the letter of the equation that best represents the graph.



A $g(x) = \sqrt{x + 10}$

B $g(x) = \frac{1}{10}\sqrt{x}$

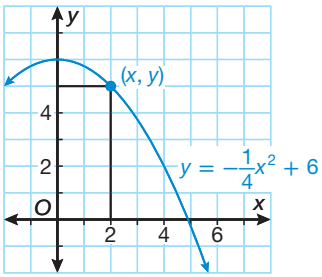
C $g(x) = \sqrt{x} + 10$

D $g(x) = 10\sqrt{x}$

***4.** (104) **Retirement Funds** A Teachers Retirement System paid out an interest rate of 4% compounded annually from 1920 to 1955. The function $A(x) = P \cdot 1.04^x$ can be used to calculate the amount in the account, $A(x)$, after x years with the original amount placed in the account, P. Use the function $f(x) = 1.04^x$ to sketch a graph of the equation $A(x) = P \cdot 1.04^x$ if the original amount in the account is \$1,000.

5. (83) Write a quadratic equation whose root is 7.

***6.** (101) **Coordinate Geometry** A rectangle has a side on the x-axis, a side on the y-axis, and a vertex in Quadrant I on the graph of $y = -\frac{1}{4}x^2 + 6$. Find the maximum possible area of the rectangle and the corresponding vertex (x, y).



7. (10) **Geometry** A triangle has a base of 10 cm and a height of x cm. Write and solve an inequality for all the values of x that will result in a triangle with an area of at least 95 cm.

8. (69) Graph each of the complex numbers on the same coordinate plane.

a. $2i$ **b.** $-4i$ **c.** $3 + i$

9. (92) Find the 4th term of an arithmetic sequence if the first term is 6.2 and the 19th term is 33.2.

10. (9) **Cafeteria** C.J., Lyle, and Avery had lunch together. C.J. bought 1 drink, 3 sandwiches, and 3 snacks. Lyle bought 2 drinks, 1 sandwich, and 4 snacks. Avery bought 2 drinks, 2 sandwiches, and 5 snacks. The cost of a drink is \$1.00, a sandwich is \$3.50, and a snack is \$1.50. Use matrices to determine how much each person spent and who spent the most money.

11. (32) Use an inverse matrix to solve the linear system given by $\begin{aligned} -3x + y - 8z &= 18 \\ x - 2y + z &= 11 \\ 2x - 2y + 5z &= -17 \end{aligned}$.

12. (88) **Jet Stream** The jet stream is a wind pattern. Between London and New York the jet stream generally blows west to east, sometimes at hundreds of miles per hour. The total time t (in hours) of a round-trip flight is modeled by the equation $t = \frac{d}{r - w} + \frac{d}{r + w}$, where r is the average speed of the plane in miles per hour, w is the average speed of the jet stream in miles per hour, and d is the one-way distance in miles.

a. Solve the equation for d.

b. If the average speed of a plane is 500 miles per hour and the average speed of the jet stream is 100 miles per hour, then it would take 14.4 hours for a round-trip flight from London to New York. Find the distance from London to New York.

***13.** (105) **Graphing Calculator** Use the **sum(seq(** commands to find the sum of $\sum_{k=1}^{100} 4.25k$.

***14.** (105) Evaluate the series $\sum_{k=1}^{10}(8 - 3k)$.

15. (84) **Error Analysis** Explain and correct the error made in the steps shown below.

$$\frac{2}{3x+6} + 4 = \frac{1}{2} \rightarrow {}^{2}\cancel{6}\cancel{(x+2)}\left(\frac{2}{\cancel{3}\cancel{(x+2)}} + 4\right) = {}^{3}\cancel{6}(x+2)\left(\frac{1}{\cancel{2}}\right) \rightarrow 4 + 8 = 3x + 6$$

***16.** (94) **Government** Suppose counties in the state of California had the option to donate an equal amount of money towards a $500,000 donation to be given to a charity. Write and solve an inequality to show the number of counties whose participation would bring the cost per county to less than $14,000. There are 58 counties in California.

17. (85) **Multiple Choice** Which of the following is not a root of $f(x) = 8x^3 - 18x^2 + 3x + 2$?

A -3 **B** $-\frac{1}{4}$ **C** $\frac{1}{2}$ **D** 2

18. (87) **Multi-Step** Use the properties of logarithms to rewrite the expression $\log_3(9x)^5$. Evaluate the expression when $x = 81$.

19. (90) **Surveying** Derive an equation that a surveyor can use to find the height (h) of a mountain given the angle of elevation (θ) and the distance from the mountain (d).

20. (21) Solve the system $\begin{matrix} 3x - y = 2 \\ 6x + 3y = 14 \end{matrix}$ by substitution.

21. (91) **Analyze** Is a circle a function? Why or why not? Sample:

***22.** (93) **Write** Explain why the inequalities $2(3)^x < 310$ and $\log 2 + x \log 3 < \log 310$ are equivalent.

23. (10) Solve the compound inequality $4 < 3x + 1 \leq 10$ by writing and solving two inequalities. Show the steps you followed to reach your answers.

24. (26) A line has slope 5 and crosses the y-axis at (0, 3). What is the equation of the line written in slope-intercept form?

25. (18) **Write** Explain how to simplify this problem: 12.65 in. + 30.1 in. + 15.843 in.

26. (39) Determine if (7, 19) is a solution of the inequality $y > 4x + 8$.

Evaluate each piecewise function for $x = -2$ and $x = 0$.

27. (79) $f(x) = \begin{cases} 7x, & \text{if } x < -2 \\ 0, & \text{if } x \geq -2 \end{cases}$

28. (79) $f(x) = \begin{cases} 15, & \text{if } x \leq -2 \\ -x^2, & \text{if } -2 < x \leq 0 \\ 2 + x, & \text{if } x > 0 \end{cases}$

***29.** (102) **Estimate** Estimate the solution to $1 + \log x = 3.998$. Explain your method.

30. (87) Convert $\log_8 (9x)^2$ to base 10. Then evaluate when $x = 4$.

LESSON
106

Using the Fundamental Theorem of Algebra

Warm Up

1. (66) **Vocabulary** The _____ of a root r is the number of times $x - r$ appears as a factor.

2. (61) Use synthetic division to determine if $x + 1$ is a factor of $P(x) = 15x^3 + 22x^2 + 5x - 2$. If so, give the remaining quadratic factor.

3. (66) Use the Rational Root Theorem to identify the possible rational roots of $x^3 - 2x^2 - 9x + 18 = 0$

4. (66) Identify all the real roots of $4x^5 - 8x^4 - 32x^3 = 0$.

New Concepts

Exploration **Making Conjectures About Roots of Polynomial Functions**

Complete the table. Include multiplicities in the last column.

$P(x)$	Factored Form of $P(x)$	Roots of $P(x) = 0$	Number of Roots
$x^2 - 16$			
$x^3 - 5x^2 + 8x - 4$	$(x - 1)(x - 2)^2$		
$x^5 - x^4 - 2x^3$			
$x^4 - 9x^2$			
	$x^3(x - 5)(x + 2)$		

Look for a pattern in the table. Make a conjecture about the number of roots and $P(x)$. Based on your conjecture, what would you say about the roots of $x^{10} - 5x^7 + 12x = 0$?

Hint

Remember that the set of complex numbers includes the real and the imaginary numbers.

The degree of every function in the exploration is the same as the number of roots of the related equation. This is true for all polynomials and is described in the following theorem.

Fundamental Theorem of Algebra

The **Fundamental Theorem of Algebra** states that every polynomial function of degree $n \geq 1$ has at least one zero in the set of complex numbers.

A corollary of the theorem is that, including multiplicities, a polynomial function of degree $n \geq 1$ has exactly n zeros.

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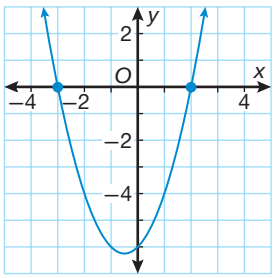
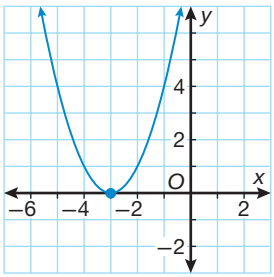
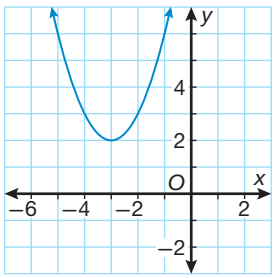
In other words, a function of degree n has n zeros, so a polynomial equation of degree n has n roots. These roots may be real or complex.

A quadratic function always has two zeros, so a quadratic equation always has two roots: either two distinct real roots, one real root with multiplicity of 2, or two complex roots.

Two Distinct Roots	Two Identical Roots	Two Complex Roots
		

A cubic function also has three possible cases. These are: three distinct roots, three real roots, one of which has a multiplicity of 2, one root of multiplicity 3, or one real root and two complex roots.

Example 1 Finding Roots of Cubic Equations

a. Solve $x^3 + 12x^2 + 44x + 48 = 0$ by finding all the roots.

Hint

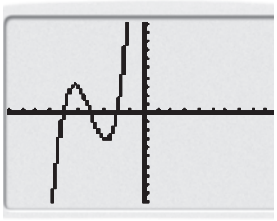
Start with the smaller possible roots, so that the calculations are easier.

SOLUTION Identify possible roots by using the Rational Root Theorem: $\pm 48, \pm 24, \pm 16, \pm 12, \pm 8, \pm 6, \pm 4, \pm 3, \pm 2, \pm 1$. Test roots by using synthetic division.

$$\begin{array}{r|cccc} -6 & 1 & 12 & 44 & 48 \\ & & -6 & -36 & -48 \\ \hline & 1 & 6 & 8 & 0 \end{array} \quad (x + 6) \text{ is a factor.}$$

The equation can be written as $(x + 6)(x^2 + 6x + 8) = 0$.

Factor the quadratic, and the equation is $(x + 6)(x + 2)(x + 4) = 0$, making the solutions -6, -2, and -4.



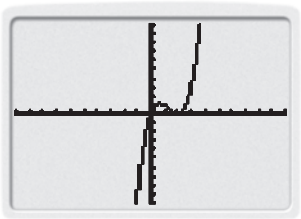
The graph of $y = x^3 + 12x^2 + 44x + 48$ has three real roots.

b. Solve $x^3 - 4x^2 + 4x = 0$ by finding all the roots.

SOLUTION Possible roots are $\pm 4, \pm 2, \pm 1$.

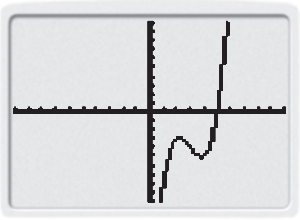
$$\begin{array}{r|ccc} 2 & 1 & -4 & 4 \\ & & 2 & -4 \\ \hline & 1 & -2 & 0 \end{array} \quad (x - 2) \text{ is a factor.}$$

The equation can be written as $(x - 2)(x^2 - 2x) = 0$. The quadratic has a GCF of x, so the equation can be written as $(x - 2)(x)(x - 2) = 0$, or $x(x - 2)^2 = 0$. The solutions are 0 and 2.



The graph of $y = x^3 - 4x^2 + 4x$ has three real roots, one of which has a multiplicity of 2.

c. Solve $x^3 - 9x^2 + 25x - 25 = 0$ by finding all the roots.



SOLUTION Possible roots are $\pm 25, \pm 5, \pm 1$.

$$\begin{array}{r|rrrr} 5 & 1 & -9 & 25 & -25 \\ & & 5 & -20 & 25 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

$(x - 5)$ is a factor.

The equation can be written as
$(x - 5)(x^2 - 4x + 5) = 0$.
The quadratic cannot be factored; use the Quadratic Formula.

$$x = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

The solutions are 5, $2 + i$, and $2 - i$. The graph of $y = x^3 - 9x^2 + 25x - 25$ illustrates one real root and two complex roots.

Math Reasoning

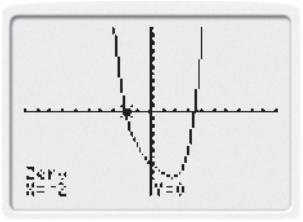
Justify Why must a cubic equation have at least one real root?

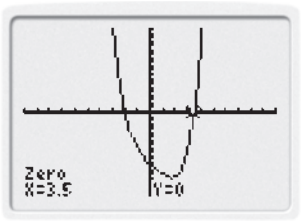
To find roots of equations whose polynomial is of a degree greater than three, it helps to use a graphing calculator to first identify the rational roots. Then use algebra to find any complex roots and/or the exact values of the irrational roots, since a graphing calculator only approximates irrational roots.

Example 2 Finding Roots of Polynomials with Degree > 3

a. Solve $2x^4 - 3x^3 + 4x^2 - 27x - 126 = 0$ by finding all the roots.

SOLUTION A graphing calculator shows that -2 and 3.5 are roots.





Use synthetic division to find the remaining unfactored polynomial.

$$\begin{array}{r|rrrrr} -2 & 2 & -3 & 4 & -27 & -126 \\ & & -4 & 14 & -36 & 126 \\ \hline & 2 & -7 & 18 & -63 & 0 \end{array} \qquad \begin{array}{r|rrrr} 3.5 & 2 & -7 & 18 & -63 \\ & & 7 & 0 & 63 \\ \hline & 2 & 0 & 18 & 0 \end{array}$$

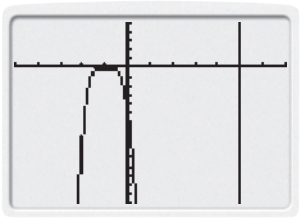
The equation can be written as $(x + 2)(x - 3.5)(2x^2 + 18) = 0$ or $2(x + 2)(x - 3.5)(x^2 + 9) = 0$. Solve $x^2 + 9 = 0$ to find the complex factors.

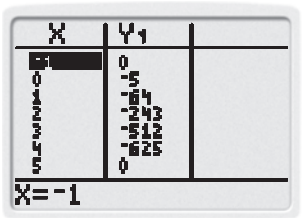
$$x^2 + 9 = 0$$
$$x^2 = -9$$
$$x = \pm\sqrt{-9} = \pm 3i$$

The four solutions are -2, 3.5, $3i$ and $-3i$.

b. Solve $x^5 - x^4 - 14x^3 - 26x^2 - 19x - 5 = 0$ by finding all the roots.

SOLUTION A graphing calculator shows that -1 and 5 are roots.





Use synthetic division to find the remaining unfactored polynomial.

5	1	−1	−14	−26	−19	−5
		5	20	30	20	5
	1	4	6	4	1	0

−1	1	4	6	4	1
		−1	−3	−3	−1
	1	3	3	1	0

Math Reasoning

Verify Show by factoring that $x^2 + 2x + 1 = 0$ has a double root of -1.

By the Rational Root Theorem, the equation $x^3 + 3x^2 + 3x + 1 = 0$ has possible roots of -1 or 1. Continue to use synthetic division to check if -1 is a multiple root.

−1	1	3	3	1
		−1	−2	−1
	1	2	1	0

−1	1	2	1
		−1	−1
	1	1	0

$x^5 - x^4 - 14x^3 - 26x^2 - 19x - 5 = 0$

$(x + 1)(x + 1)(x + 1)(x + 1)(x + 5) = 0$

The root of -1 has a multiplicity of 4, and the root of 5 has a multiplicity of 1.

The solutions are -1 and 5.

Notice in the graph of $y = x^5 - x^4 - 14x^3 - 26x^2 - 19x - 5$ above, the graph touches but does not cross the root of -1, while it crosses at the root of 5. A graph will always touch but not cross when the root has an even-numbered multiplicity and cross at an odd-numbered multiplicity.

Remember, just as irrational roots come in conjugate pairs, $a \pm b\sqrt{c}$, complex roots always come in conjugate pairs, $a \pm bi$.

Example 3 Using Conjugates to Write a Polynomial Function

Write the lowest degreed polynomial function with zeros $5 + i$ and $1 + \sqrt{3}$.

SOLUTION Because both irrational and complex roots come in pairs, there are four roots: $5 + i$, $5 - i$, $1 + \sqrt{3}$,$1 - \sqrt{3}$. Begin by writing the function as the product of the factors. Then multiply carefully.

$$P(x) = (x - (5 + i))(x - (5 - i))\left(x - \left(1 + \sqrt{3}\right)\right)\left(x - \left(1 - \sqrt{3}\right)\right)$$

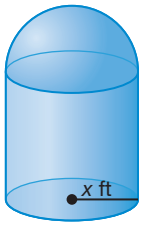
$$P(x) = \left((x - 5 - i)(x - 5 + i)\right)\left(\left(x - 1 - \sqrt{3}\right)\left(x - 1 + \sqrt{3}\right)\right)$$

$$P(x) = (x^2 - 5x + ix - 5x + 25 - 5i - ix + 5i - i^2) \cdot \left(x^2 - x + x\sqrt{3} - x + 1 - \sqrt{3} - x\sqrt{3} + \sqrt{3} - 3\right)$$

$$P(x) = (x^2 - 10x + 26)(x^2 - 2x - 2) = x^4 - 12x^3 + 44x^2 - 32x - 52$$

Example 4 Application: Farming

A grain silo is shaped like a cylinder with a hemisphere on top. The height of the cylindrical part is 25 feet. The radius of the hemisphere is x feet. The volume of the entire silo is 4752π cubic feet. Find the radius of the silo.



SOLUTION

1. Understand The volume of the hemisphere plus the volume of the cylinder equals the volume of the entire silo.

2. Plan $V_{\text{hemisphere}} = \frac{2}{3}\pi r^3$ and $V_{\text{cylinder}} = \pi r^2 h$.

3. Solve Substitute x for r and 25 for h.

$$\frac{2}{3}\pi x^3 + \pi x^2(25) = 4752\pi$$ Add the volumes.

$$\frac{2}{3}\pi x^3 + 25\pi x^2 - 4752\pi = 0$$ Write in standard from.

$$\frac{2}{3}x^3 + 25x^2 - 4752 = 0$$ Divide both sides by π.

Use a graphing calculator.

It appears that 12 is a root.

Use synthetic division to check.

12	$\frac{2}{3}$	25	0	−4752
		8	396	4752
	$\frac{2}{3}$	33	396	0

Because the polynomial is cubic, there must be two other roots. Use the quadratic formula: $x = \frac{-33 \pm \sqrt{33^2 - 4\left(\frac{2}{3}\right)(396)}}{\frac{4}{3}} = \frac{-33 \pm \sqrt{33}}{\frac{4}{3}} \approx -20.4$ and -29.1.

The radius must be positive, so the radius is 12 feet.

4. Check Substitute 12 for r: $V = \frac{2}{3}\pi(12)^3 + \pi(12)^2(25) = 4752\pi$.

Math Reasoning

Analyze How is the volume of a hemisphere derived from the volume of a sphere?

Lesson Practice

Solve the following equations by finding all the roots.

a. (Ex 1) $x^3 + x^2 - 17x + 15 = 0$

b. (Ex 1) $x^3 - 2x^2 - 4x + 8 = 0$

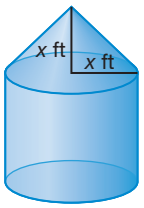
c. (Ex 1) $x^3 - 7x^2 + 9x + 17 = 0$

d. (Ex 2) $x^4 - 8x^3 + 13x^2 + 16x - 30 = 0$

e. (Ex 2) $x^5 - 5x^4 - 5x^3 + 45x^2 - 108 = 0$

f. (Ex 3) Write the simplest polynomial function with zeros $2 - 3i$ and $\sqrt{7}$.

g. (Ex 4) A grain silo is shaped like a cylinder with a cone on top. The height of the cylindrical part is 20 feet. The radius and height of the cone is x feet. The volume of the entire silo is 1863π cubic feet. Find the radius of the silo.



Practice Distributed and Integrated

*1. (106) **Medicine** A vitamin is shaped like a cylinder with congruent hemispheres on either end. If the volume of the vitamin is 22.5π cubic millimeters, and the height of the cylindrical part is 8 millimeters, what is the radius of the vitamin?

*2. (102) **Population Growth** Based on U.S. Census statistics, the population of Maryland for the years 1950–1990 can be modeled by the equation $y = -1202 + 1495\ln t$, where $10 \le t \le 50$, $t = 10$ represents 1950, and y is the population in thousands. Use the model to approximate the year in which the population of Maryland reached 3,000,000.

3. (94) **Generalize** How is solving a quadratic inequality similar to and different from solving a rational inequality?

*4. (105) **Analyze** Explain the difference between S_9 and a_9 for an arithmetic series.

*5. (105) **Error Analysis** Explain and correct the error a student made in writing the arithmetic series $\{9 + 13 + 17 + 21 + 25\}$ in summation notation.

$$\sum_{k=1}^{5} 9 + 4k$$

6. (87) **Multiple Choice** Using the change of base formula, $\log_{12} 5$ is equivalent to:

A $\frac{\log 12}{\log 5}$ **B** $\frac{\log 5}{\log 12}$ **C** $\frac{\log_5 10}{\log_{12} 10}$ **D** $\frac{\ln 12}{\ln 5}$

7. (103) Graph $y = \frac{2}{3}\csc(x)$. Determine its period and asymptotes.

8. (49) Use the Binomial Theorem to expand $(x + 3y)^3$

*9. (104) **Retirement Funds** A teachers' retirement program paid out an interest rate of 3% compounded annually from 1955 to 1963. The function $A(x) = P \cdot 1.03^x$ can be used to calculate the amount in the account, $A(x)$, after x years with the original amount placed in the account, P. Use the function $f(x) = 1.03^x$ to sketch a graph of the equation $A(x) = P \cdot 1.03^x$ if the original amount in the account is \$10,000.

10. (91) The endpoints of a diameter of a circle are located at $(-8, 4)$ and $(2, 4)$. Write an equation for this circle.

*11. (101) **Analyze** A polynomial function $f(x)$ of degree 4 has exactly four real zeros; they are -1, 2, 3, and 6. It also has a negative y-intercept. Describe the end behavior of $f(x)$.

12. (98) **Geometry** For the equation to calculate eccentricity, $e = \sqrt{1 - \frac{b^2}{a^2}}$, what is the value of e as a approaches infinity? For this limiting value, is the figure still an ellipse?

13. (97) **Multiple Choice** What is the 11^{th} term of a geometric sequence that starts $-4, 4, \ldots$?

A $-4{,}194{,}304$ **B** -4 **C** 4 **D** 4,194,304

14. (27) Convert $f(x) = (x - 3)^2 + 6$ into standard form.

Simplify the expression.

15. (87) $\log_6 36^x$

16. (87) $\log_5 125^x$

***17.** (100) **Multi-Step** Let $f(x) = x^2 - 9$ and $g(x) = x + 4$. Find $\frac{f(x)}{f(g(x))}$ and then determine the vertical asymptotes and holes for it.

***18.** (30) **Graphing Calculator** Graph $f(x) = -x^2 + 2$ using a graphing calculator. Compare with the graph of x^2, and explain how the graph is shifted.

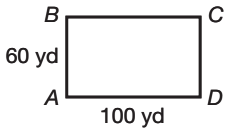
19. (85) **Physics** A spaceship is coming in for a landing on Jupiter. The equation $h = -12.95t^2 + v_0t + h_0$ can be used to find the distance h from the ground at time t based on an initial height h_0 and an initial speed of v_0. To the nearest hundredth of a second, how long it will take the ship to land from a height of 2000 and an initial speed of 200?

Simplify.

20. (48) $\frac{6}{x} + \frac{2x}{x+2}$

21. (48) $\frac{\frac{1}{x+5}}{\frac{x}{2}}$

22. (96) **Athletic Field** The diagram represents a soccer field. Write polar coordinates for points A, B, C, and D. Use point A as the pole and $\overrightarrow{AD}$ as the polar axis.



23. (42) Find the number of permutations of 9 objects taken 6 at a time.

24. (Inv 8) Estimate the area under the curve $y = -x^2 + 5x + 5$ from $0 \le x \le 5$. Use 5 partitions.

25. (33) A teacher needs 3 volunteers to go to the cafeteria to get ice for the next chemistry lab. With a class of 26 students, how many different ways can the teacher choose the volunteers for the task?

26. (80) **Statistics** A set of results are normally distributed with a mean of 50 and a standard deviation of 4. Find the percent of results between 46 and 54.

***27.** (106) **Multi-Step** Consider the polynomial equation $x^3 + x = 0$.

a. How many roots must it have and why?

b. Factor the polynomial. Name a real root.

c. Find the remaining two roots.

d. Graph the polynomial on a graphing calculator and tell how the graph supports your answer.

28. (91) **Write** Explain how to write the equation of a circle given the endpoints of a diameter of the circle.

***29.** (106) Write the simplest polynomial function given the following roots: $-\frac{3}{4}$, 6, -1

30. (99) **Formulate** For two nonzero vectors $\vec{A}$ and $\vec{B}$ what must be true for $\vec{A} \cdot \vec{B} = 0$?

LESSON
107 Graphing Rational Functions II

Warm Up

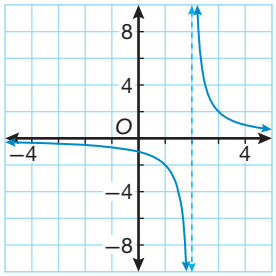
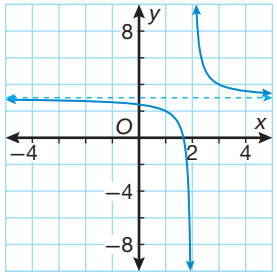
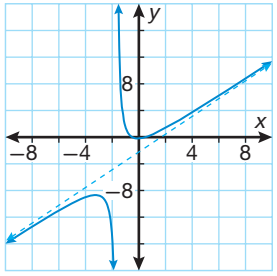
1. (38) **Vocabulary** To determine that $\frac{2x^2 + 12x - 15}{x - 3} = 2x + 18 + \frac{39}{x - 3}$ use _____.
2. (38) Simplify $\frac{x^2 - 6x + 9}{x^2 - 9}$.
3. (100) What are the vertical asymptotes of $f(x) = \frac{2x - 1}{3x^2 - 28x + 32}$?

New Concepts

A graph of a rational function $f(x) = \frac{g(x)}{h(x)}$, for polynomials $g(x)$ and $h(x) \neq 0$, can exhibit three types of asymptotes. Each type of asymptote has a characteristic equation.

Math Language

A **slant asymptote** is a linear asymptote that is neither horizontal nor vertical: can be called an oblique asymptote.

Vertical Asymptotes	Horizontal Asymptotes	Slant Asymptotes
		
$x = constant$	$y = constant$	$y = mx + b$

The equations for slant asymptotes are linear equations with a defined, nonzero slope that are derived from the rational function itself.

Math Reasoning

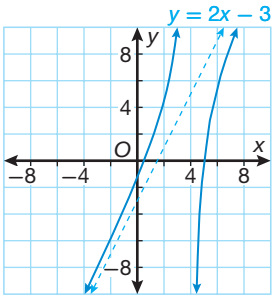
Analyze Suppose a rational function $g(x)$ has a vertical asymptote at $x = a$, such that $f(a)$ is not defined. If this same function has a slant asymptote $y = mx + b$, why is $y(a)$ defined?

Example 1 Finding the Equation of a Slant Asymptote

Find the equation of the slant asymptote for the following rational function.

$$f(x) = \frac{2x^2 - 11x + 5}{x - 4}$$

SOLUTION



Step 1: Graph the equation to verify that there is a slant asymptote.

Step 2: Divide the numerator by the denominator using polynomial division.

$$\begin{array}{r} 2x - 3 \\ (x - 4)\overline{)\,2x^2 - 11x + 5} \\ \underline{-(2x^2 - 8x)} \\ 0 - 3x + 5 \\ \underline{-(-3x + 12)} \\ 0 - 7 \end{array}$$

Step 3: The quotient, without the remainder, is the equation of the slant asymptote.

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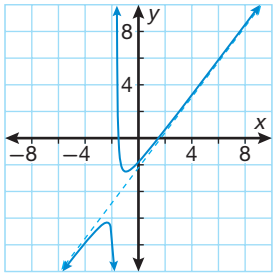
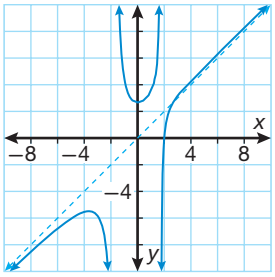
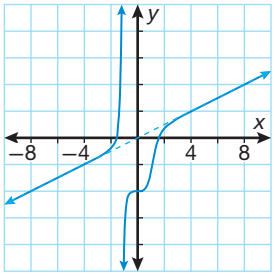
Math Reasoning

Verify Explain why functions of the form $y = \frac{ax^3 + bx^2 + cx + d}{ax + b}$ for nonzero values of a, b, c, and d, do not have slant asymptotes.

A slant asymptote will occur when the degree of the numerator is one greater than the degree of the denominator.

Numerator: Quadratic Denominator: Linear	Numerator: Cubic Denominator: Quadratic
 $y = \frac{4x^2 - 9}{3x + 5}$	 $y = \frac{x^3 - 8}{x^2 - 3}$

Numerator: Degree n, Denominator: Degree $n - 1$



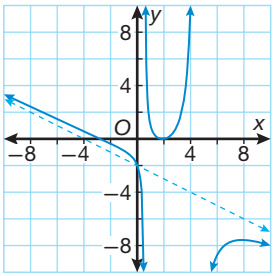
$$f(x) = \frac{g(x)}{h(x)}$$

Example 2 Slant Asymptotes with Quadratic and Cubic Terms

Find the equation of the slant asymptote for the following rational function.

$$f(x) = \frac{x^3 - x^2 - 7x + 10}{-2x^2 + 10x - 5}$$

SOLUTION



Step 1: Graph the equation to verify that there is a slant asymptote. Or, verify that the degree of the numerator is one greater than the degree of the denominator. Since the numerator is a cubic and the denominator is a quadratic, the numerator is one greater than the denominator.

Step 2: Divide the numerator by the denominator using polynomial division.

$$\begin{array}{r} -0.5x - 2 \\ (-2x^2 + 10x - 5)\overline{)\,x^3 - x^2 - 7x + 10} \\ \underline{-(x^3 - 5x^2 + 2.5x)} \\ 0 + 4x^2 - 9.5x + 10 \\ \underline{-(4x^2 - 20x + 10)} \\ 0 \quad + 10.5x \end{array}$$

$$f(x) = \frac{x^3 - x^2 - 7x + 10}{-2x^2 + 10x - 5} = (-0.5x - 2) + \frac{10.5x}{-2x^2 + 10x - 5}$$

Step 3: The quotient, without the remainder, is the equation of the slant asymptote.

$$f(x) = -0.5x - 2$$

To graph a rational function, first find all asymptotes, holes, and intercepts. Calculate the values of y for values of x near the asymptotes. Then plot the points and sketch the graph.

Example 3 Graphing a Rational Function

Graph $f(x) = \dfrac{x^4 - 256}{x^3 + x^2 - 14x - 24}$.

SOLUTION **Step 1:** Determine if there is a horizontal or slant asymptote. Since the degree of the numerator is one more than the degree of the denominator, there is no horizontal asymptote, but there is a slant asymptote.

Divide to find the slant asymptote.

$$\frac{x^4 - 256}{x^3 + x^2 - 14x - 24} = (x + 1) + \frac{(15x + 70)}{(x^3 + x^2 - 14x - 24)}$$

The slant asymptote is $y = x - 1$.

Step 2: Factor the numerator and denominator to find vertical asymptotes and holes.

Numerator

$$x^4 - 256 = (x^2 - 16)(x^2 + 16) = (x - 4)(x + 4)(x^2 + 16)$$

Denominator

Use the Rational Root Theorem to factor $x^3 + x^2 - 14x - 24$.

Possible Rational Roots:

$$\pm 1, \pm 2, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 24$$

Find the first root that yields a factor through synthetic division:

1	1	1	−14	−24	
		1	2	−12	
	1	2	−12	−36 ✗	

−1	1	1	−14	−24	
		−1	0	14	
	1	0	−14	10 ✗	

−2	1	1	−14	−24	
		−2	2	24	
	1	−1	−12	0 ✓	

$$f(x) = \frac{(x - 4)(x + 4)(x^2 + 16)}{(x + 2)(x^2 - x - 12)} = \frac{(x - 4)(x + 4)(x^2 + 16)}{(x + 2)(x - 4)(x + 3)}$$

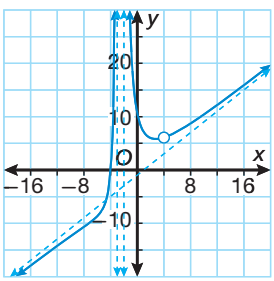
There are vertical asymptotes at $x = -2$ and $x = -3$.

There is a hole at $x = 4$.

Step 3: Find the x- and y-intercepts.

The x-intercept occurs at $x = -4$, since $f(-4) = 0$. The y-intercept occurs at $x = 10\frac{2}{3}$, since $f(0) = \dfrac{(0)^4 - 256}{(0)^3 + (0)^2 - (0)x - 24} = 10\frac{2}{3}$.

Step 4: Graph the asymptotes, holes, and intercepts. Calculate the values of y for values of x near the asymptotes. Then plot the points and sketch the graph.



Math Reasoning

Analyze Consider rational function

$$f(x) = \frac{g(x)}{h(x)}$$

If $f(a)$ is undefined for $x = a$, what needs to be true for there to be a vertical asymptote at $x = a$?

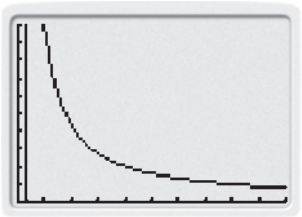
Example 4 Application: Consumer Finance

As the weight of a car increases, its miles per gallon (mpg) decreases. A reasonable model for this behavior is shown in the rational function below, where x is the weight of the car, in pounds. Use the function to identify the mpg for cars weighing 1750 pounds and cars weighing 2140 pounds. Determine the cost savings if the car is used 20,000 miles per year and gas costs on average \$3 per gallon.

$$f(x) = \frac{7500}{0.1x - 25}$$

SOLUTION

Step 1: Graph the function. Set the WINDOW as shown below.



$x_{min} = 0$

$x_{max} = 10{,}000$

$y_{min} = 0$

$y_{max} = 100$

Step 2: Use the TRACE feature to find the mpg. For a car weighing 1750 pounds, the mpg is 50. For a car weighing 2140 pounds, the mpg is approximately 40.

Step 3: Determine the number of gallons of fuel needed per year for each weight of car using the formula:

$$\text{Gallons of fuel} = \frac{\text{Miles traveled}}{\text{Miles per gallon}}$$

40 mpg vehicle	50 mpg vehicle
$\frac{20{,}000}{40} = 500$	$\frac{20{,}000}{50} = 400$

Step 4: Determine the cost savings.

$$(500 - 400)\$3 = \$300$$

Math Reasoning

Verify Use the equation $f(x) = \frac{7500}{0.1x - 25}$ to verify the values you found for Step 2.

Lesson Practice

(Ex 1) **a.** Find the equation of the slant asymptote for the following rational function: $f(x) = \frac{3x^2 + 12x - 5}{x - 7}$.

(Ex 2) **b.** Find the equation of the slant asymptote for the following rational function: $f(x) = \frac{-15x^3 - 2x^2 - 9x + 17}{x^2 - 5}$.

(Ex 3) **c.** Graph $f(x) = \frac{x^8 - 1}{x^7 - x^4 - 16x^3 + 16}$.

(Ex 4) **d.** Use the rational function $f(x) = \frac{7500}{0.1x - 25}$ to determine the mpg for cars weighing 1313 pounds and 1484 pounds.

Practice Distributed and Integrated

1. (31) Multiply and then evaluate for $t = 2$, $k = 5$: $\frac{4t^3k^7}{8t} \cdot \frac{3k^3}{6t^4k^9}$.

***2.** (105) Find S_{30} for $101 + 95 + 89 + 83 + \ldots$

3. (96) **Estimate** Write approximate Cartesian coordinates for the point with polar coordinates $\left(2, \frac{1}{2}\right)$. Explain your method.

***4.** (107) **Multiple Choice** Which of the following rational functions has a slant asymptote of $y = 2x - 4$?

A $y = \frac{2x^2 + 1}{2x + 1}$ **B** $y = \frac{x^3 - 8}{0.5x^2 + x + 5}$

C $y = \frac{(2x - 4)^2}{x + 8}$ **D** $y = \frac{2x^2 - 1}{x + 2}$

5. (49) 1 in 6 cars sold on a lot are silver. Today, 4 cars were sold. What is the probability that 2 of the cars sold were silver?

***6.** (97) If the 1st term of a geometric sequence is 5 and the common ratio is -4, what is the 6th term?

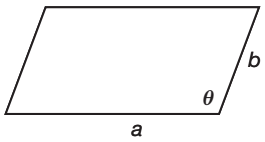
***7.** (47) **Graphing Calculator** Graph $y = 6^x$ and $y = \left(\frac{1}{6}\right)^x$ on a graphing calculator and describe the graphs.

8. (92) **Error Analysis** A student said that the finite sequence {5, 10, 20, 35, 55, 80} is arithmetic because it is formed by adding numbers in a pattern. Explain and correct the student's reasoning.

9. (2) **Multiple Choice** Evaluate $-5x^2 + 2y$ if $x = 1$ and $y = -1$.

A -7 **B** -3 **C** 3 **D** 7

10. (71) **Geometry** Derive the function that can be used to find the area of the parallelogram. Express in simplified form.



Divide.

11. (38) $5x^4 - 2x^3 - 7x^2 - 39$ by $2x$

12. (38) $3x^2 - 11x - 26$ by $(x - 5)$

13. (85) **Error Analysis** A student solves for the roots of $y = x^3 - 3x + 2$; what was his mistake?

Factors of 1: ± 1 ← Values of q Factors of 2: $\pm 1, 2$ ← Values of p $\frac{p}{q} = \{\pm 1, 2\}$

$$\begin{array}{r|rrr} 1 & 1 & -3 & 2 \\ & & 1 & -2 \\ \hline & 1 & -2 & 0 \end{array} \checkmark$$

$y = (x - 1)(x^2 - 2)$ The roots of $y = x^3 - 3x + 2$ are $x = 1$ and $\pm\sqrt{2}$.

14. (106) **Error Analysis** A student said that a polynomial equation with roots of 4 and $6 + i$ will have exactly four roots because each has a conjugate pair, -4 and $6 - i$. Explain the error in the student's thinking.

15. (Inv 9) **Multi-Step** To join the local gym, there is a \$60 initial sign-up fee and an additional monthly fee. If you join the gym for 6 months, it is \$50 per month. If you join the gym for 1 year, the monthly fee is \$45 per month. There is a \$5 monthly discount for each year added to your membership, up to 5 years.

a. If you join the gym for 6 months, and then sign up for an additional year, what will be the total cost?

b. What will the total cost be if you later add an additional 6 months?

c. If you join the gym for 2 years, what will be the total cost?

d. How much money will you save by joining the gym for 2 years initially?

16. (91) **Analyze** How is the graph of $x^2 + (y - 7)^2 = 20$ similar to, and different from, the graph of $x^2 + (y - 7)^2 = 24$?

***17.** (100) **Generalize** Show that rational functions of the form $f(x) = \frac{1}{x^2n}$, for positive integers n, are even functions.

18. (94) **Write** Describe two ways to use a sign table to solve rational inequalities. When should each be used?

19. (95) **Space Science** If you throw a ball in the air, the height (h) it travels is based on the acceleration due to gravity (g), the initial height (h_o) the object is thrown from, its initial speed (v_o), and the time (t) that the object is in the air. The table shows different values of g for different planets. Use the Remainder Theorem to compare $h(10)$ for Earth and Saturn, assuming that $h_o = 100$ and $v_o = 100$. Account for any differences.

$$h(t) = -\frac{1}{2}gt^2 + v_o t + h_o$$

Planet	Gravity, g(m/s^2)
Mercury	3.59
Venus	8.87
Earth	9.81
Mars	3.77
Jupiter	25.95
Saturn	11.08
Uranus	10.67
Neptune	14.07

20. (87) **Commodities** P represents the selling price, in dollars, of a commodity and x represents the demand, in number sold per day. This relationship is indicated by the equation $p = p_0e^{-ax}$. Solve the equation for x.

21. (93) **Population Growth** The population of Japan grew at an average annual rate of approximately 0.26% from 1990 to 2000, and the population in 2000 was 127,000,000 (rounded to the nearest million). If the population continues to increase at the same rate, in what year will it reach 150,000,000?

***22.** (107) **Consumer Finance** You want to compare the cost of buying a more expensive car that has a higher miles per gallon (mpg) rating to a less expensive car with low mpg. Use the data in the table to generate two rational functions. Assume that the number of miles traveled is 30,000 and the cost of fuel is \$4.50 per gallon. Use the graph of both functions to determine the number of years it takes for the car with 40 mpg to be a better deal than the car with 20 mpg.

MPG	Car Price
20	\$20,000
40	\$40,000

$$\text{Yearly cost} = \frac{\text{Price} + \frac{\text{Miles traveled}}{\text{MPG}} \times \text{Cost of Fuel per Gallon} \times \text{Number of Years}}{\text{Number of Years}}$$

23. (99) **Multi-Step** Find the magnitude of vectors $\begin{bmatrix}3\\6\end{bmatrix}$ and $\begin{bmatrix}2\\-5\end{bmatrix}$. Calculate the dot product and then calculate the angle between the two vectors.

***24.** (104) **Probability** The probability that a single coin tossed in the air will not land tails up until the n^{th} toss can be found using the equation $P(n) = 0.5(0.5)^{n-1}$, where $P(x)$ is the probability and n is the number of times the coin is tossed. Use the graph of $f(n) = 0.5^n$ to sketch a graph of $P(n)$. Use the graph of $P(n)$ to estimate the probability that the coin will not land tails up until the third toss.

***25.** (107) **Formulate** Write a possible rational function that fits these parameters: Vertical asymptotes $x = \pm 1$; slant asymptote $y = 3x + 1$.

Find the zeros of the quadratic function.

26. (35) $f(x) = x^2 + 21x - 46$

27. (35) $f(x) = x^2 + \frac{5}{12}x - \frac{1}{6}$

Simplify.

28. (44) $\dfrac{1}{5 + \sqrt{6}} =$

29. (44) $\dfrac{3 + \sqrt{7}}{2 - \sqrt{10}} =$

30. (89) **Tourist Attractions** The world's tallest thermometer is located in Baker, California, with a height of 134 feet. Suppose a maintenance worker at the top drops a nail. The height of the nail, in feet, after x seconds, is modeled by $y = -16x^2 + 134$. Write and solve a quadratic inequality that can be solved to find the time span for which the nail remains at least 50 feet above the ground. Round to the nearest hundredth of a second.

LESSON
108

Using Fundamental Trigonometric Identities

Warm Up

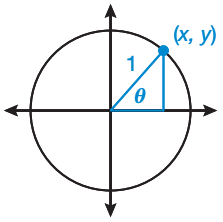
(46) **1.** **Vocabulary** The ratio $\frac{\sin(x)}{\cos(x)}$ is the _____ ratio.

(46) **2.** True or false. For a right triangle the ratio $\frac{\text{adjacent side}}{\text{hypotenuse}}$ is the sine ratio.

(46) **3.** True or false. For a right triangle the ratio $\frac{\text{opposite side}}{\text{hypotenuse}}$ is the cosine ratio.

New Concepts

A **trigonometric identity** is a trigonometric equation that is true for all values of the variable for which the statement is defined. The trigonometric ratios derived from the unit circle give rise to trigonometric identities based on those ratios.



By the Pythagorean Theorem, for any point (x, y) on the unit circle $y^2 + x^2 = 1$.

Since $\cos(\theta) = \frac{x}{1} = x$ and $\sin(\theta) = \frac{y}{1} = y$, substituting these terms into the equation of the unit circle yields the first trigonometric identity,

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Math Reasoning

Verify Show that this trigonometric identity is true even for a circle whose radius is not equal to 1.

Example 1 Using the Trigonometric Identity $\cos^2(\theta) + \sin^2(\theta) = 1$

a. Show that $\sin^2(\theta) + \cos^2(\theta) = 1$ for $\theta = 30°$.

SOLUTION

$$\sin^2(30°) + \cos^2(30°) = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

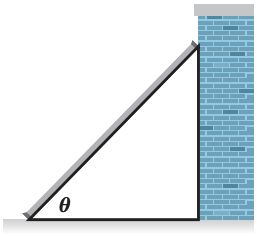
$$= \frac{1}{4} + \frac{3}{4}$$

$$= 1$$

Tip

For help evaluating trigonometric functions use the unit circle in Lesson 63.

b. A 12-foot ladder is tipped against a building at angle $\theta = \frac{\pi}{4}$, as shown. Show that for this angle $\sin^2(\theta) + \cos^2(\theta) = 1$.



SOLUTION

$$\sin^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2$$

$$= \frac{2}{4} + \frac{2}{4}$$

$$= 1$$

Online Connection
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Take the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and divide both sides by $\cos^2(\theta)$ to get another trigonometric identity.

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$
$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

Hint

Recall that $\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$ and $\frac{1}{\cos(\theta)} = \sec(\theta)$.

Example 2 Using the Trigonometric Identity $\tan^2(\theta) + 1 = \sec^2(\theta)$

a. Using the information in the diagram, show that $\tan^2(\theta) + 1 = \sec^2(\theta)$ for $\theta = 45°$.

SOLUTION

$$\tan^2(45°) + 1 = (1)^2 + 1 = 2$$

$$\sec^2(45°) = \frac{1}{\cos^2(45°)} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2$$

Therefore $\tan^2\theta + 1 = \sec^2\theta$ for $\theta = 45°$.

b. Show that $\tan^2\left(\frac{\pi}{3}\right) + 1 = \sec^2\left(\frac{\pi}{3}\right)$.

SOLUTION

$$\tan^2\left(\frac{\pi}{3}\right) + 1 = \left(\sqrt{(3)}\right)^2 + 1 = 3 + 1 = 4$$

$$\sec^2\left(\frac{\pi}{3}\right) = \frac{1}{\cos^2\frac{\pi}{3}} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$$

Therefore $\tan^2(\theta) + 1 = \sec^2(\theta)$ for $\theta = \frac{\pi}{3}$.

Take the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ and divide both sides by $\sin^2(\theta)$.

$$1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$
$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Example 3 Using the Trigonometric Identity $1 + \cot^2(\theta) = \csc^2(\theta)$

a. Show that $1 + \cot^2(\theta) = \csc^2(\theta)$ for 135°.

SOLUTION

$$1 + \cot^2(135°) = 1 + \frac{1}{\tan^2(135°)} = 1 + \frac{1}{(-1)^2} = 2$$

$$\csc^2(135°) = 1 + \frac{1}{\sin^2(135°)} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2$$

b. Show that $1 + \cot^2\left(\frac{2\pi}{3}\right) = \csc^2\left(\frac{2\pi}{3}\right)$.

SOLUTION

$$1 + \cot^2\left(\frac{2\pi}{3}\right) = 1 + \left(\frac{1}{-\sqrt{3}}\right)^2 \qquad \csc^2\left(\frac{2\pi}{3}\right) = \left(\frac{2}{\sqrt{3}}\right)^2$$

$$= 1 + \frac{1}{3} \qquad = \frac{4}{3}$$

$$= 1\frac{1}{3} \qquad = 1\frac{1}{3}$$

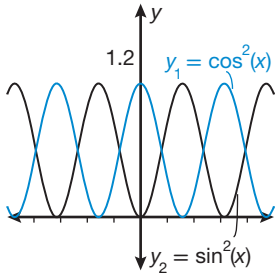
Example 4 Application: Graphical Analysis

Show graphically that the following trigonometric identity is true.

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

SOLUTION

Graph $y_1 = \cos^2(x)$ and $y_2 = \sin^2(x)$ on the same coordinate system.



Each graph has an amplitude of 0.5 and is symmetric about the line $y = 0.5$. Therefore, at any given point on the graph, y_1 and y_2 are the same distance from $y = 0.5$.

For $y_1 > y_2$,

$$y_1 - 0.5 = 0.5 - y_2$$
$$\cos^2(x) - 0.5 = 0.5 - \sin^2(x)$$
$$\cos^2(x) + \sin^2(x) = 0.5 + 0.5$$
$$\cos^2(x) + \sin^2(x) = 1$$

For $y_2 < y_1$,

$$0.5 - y_1 = y_2 - 0.5$$
$$0.5 - \cos^2(x) = \sin^2(x) - 0.5$$
$$\cos^2(x) + \sin^2(x) = 0.5 + 0.5$$
$$\cos^2(x) + \sin^2(x) = 1$$

Lesson Practice

a. (Ex 1) Show that $\sin^2(\theta) + \cos^2(\theta) = 1$ for $\theta = 60°$.

b. (Ex 1) Show that $\sin^2(\theta) + \cos^2(\theta) = 1$ for $\theta = \pi$.

c. (Ex 2) Show that $\tan^2(\theta) + 1 = \sec^2(\theta)$, for $\theta = 98.5°$.

d. (Ex 2) Show that $\tan^2(\theta) + 1 = \sec^2(\theta)$, for $\theta = \frac{5\pi}{4}$.

e. (Ex 3) Show that $1 + \cot^2(\theta) = \csc^2(\theta)$, for $\theta = 212°$.

f. (Ex 3) Show that $1 + \cot^2(\theta) = \csc^2(\theta)$, for $\theta = \frac{\pi}{6}$.

g. (Ex 4) Explain why $1 + \cot^2(\theta) = \csc^2(\theta)$. (Hint: graph $y_1 = \cot^2(\theta)$ and $y_2 = \csc^2(\theta)$.

Practice Distributed and Integrated

Write the exponential equations in logarithmic form.

1. (64) $6^5 = 7776$

2. (64) $x^3 = 6859$

***3.** (108) Let $f(x) = \sin(x)$ and $g(x) = (x - 1)(x + 1)$. Find $g(f(x)) + 1$.

4. (89) **Construction** A construction team has 230 feet of fencing material for which to fence in a rectangular play area. They were told that the play area should be at least 3000 square feet. Find the range of possible widths they could use.

***5.** (104) **Write** Explain in complete sentences what must be done to an equation to vertically stretch the graph.

6. (33) **Lockbox** A homeowner is choosing a combination to the lockbox for the house. Four digits (1–9) can be chosen. How many combinations can be made if a digit cannot be repeated?

***7.** (108) **Generalize** Use the trig ratios $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$ to show that $\frac{1}{1 - \frac{1}{1 + \tan^2(\theta)}} = \csc^2(\theta)$.

8. (11) **Analyze** A quadratic trinomial in standard form is added to another quadratic trinomial in standard form and the result is written in standard form. The leading coefficients of the two trinomials are opposites. What are all the possible types of the resulting polynomial (classified by degree)?

9. (88) **Multiple Choice** Solve $S = B + \frac{1}{2}pl$ for l. Which equation shows the correct solution?

A $l = \frac{1}{2}(S - B)p$ **B** $l = S - B - \frac{1}{2}p$ **C** $l = \frac{2S - B}{p}$ **D** $l = \frac{2S - 2B}{p}$

10. (35) Write a function that has zeros −6, −2, and 1.

***11.** (107) **Formulate** Write a possible rational function that fits these parameters: Vertical asymptotes $x = \pm 2$; slant asymptote $y = -2x + 3$.

12. (49) Use the Binomial Theorem to expand $(2x + y)^2$.

13. (101) **Multiple Choice** Which ordered pair contains a local maximum of $g(x) = 2x^3 - 3x^2 - 12x$?

A (−1, 7) **B** (0, 0) **C** (4, 32) **D** (5, 115)

14. (92) **Analyze** Is the sequence {5, 5, 5, 5} an arithmetic sequence? Why or why not?

15. (51) Use synthetic substitution to find $f(4)$ for $f(x) = 3x^4 + 6x^2 - x - 22$.

16. (91) **Soccer** A drafter is creating the plans for a sport and fitness center, which includes an official size soccer field. On her grid, the units represent yards and the endpoints of a diameter of the center circle are located at (23, −4) and (29, −4). Write an equation for this circle.

***17.** (107) Find the real zeros of $y = x^5 - 4x^3 - x^2 + 4$.

18. (94) Solve $\frac{x + 3}{x - 6} > 0$ by finding the sign of the numerator and denominator.

*19. (106) **Multi-Step** A polynomial equation has roots of 6 and $7i$.

a. What is the minimum degree of the polynomial? Name the other roots if there are any.

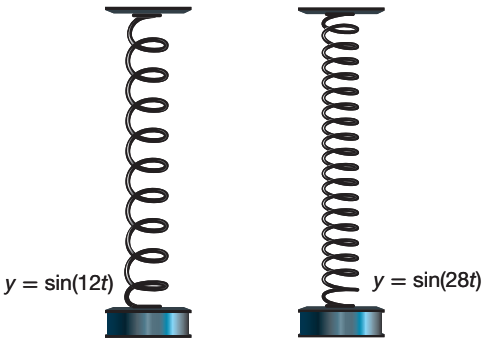
b. What is the simplest polynomial equation with these roots?

c. Give another polynomial equation with these roots.

20. (17) **Coordinate Geometry** Use a table of values to graph $f(x) = |x - 4| + 2$.

21. (88) Given $a^2 + b^2 = c^2$. Solve for b.

22. (82) **Physics** Two springs have two different weights attached at the ends. The movement of each weight is modeled by the periodic functions shown, where t is the time in seconds. How often will both weights be at their lowest point at the same time.



23. (99) **Error Analysis** Two students are subtracting two vectors written as complex numbers but they get different results. Which student made the mistake?

Student A	Student B
$(3 + 4i) - (-5 + 3i)$ $= (3 - 5) + (4 - 3)i$ $= -2 + i$	$(3 + 4i) - (-5 + 3i)$ $= (3 + 5) + (4 - 3)i$ $= 8 + i$

*24. (102) **Sound** Two measures of sound are loudness and intensity. They are related by the function $L(I) = 10 \log \frac{I}{I_0}$, where $L(I)$ is loudness in decibels, I is intensity in watts per square meter, and $I_0 \approx 10^{-12}$ watt per square meter, the intensity of a barely audible sound. The loudness of a quiet library is about 40 decibels. Find the intensity that corresponds to a loudness of 40 decibels.

25. (84) **Geometry** The length of $\overline{BC}$ is four times the length of $\overline{AB}$, and $\frac{1}{AB} + \frac{1}{BC} = \frac{5}{32}$. Find the length of $\overline{AB}$.

*26. (108) **Error Analysis** A student is simplifying the expression below using trigonometric identities. What mistake did the student make?

$$\frac{1 - \cos^2(x)}{1 - \dfrac{1}{1 + \cot^2(x)}} = \frac{\sin^2(x)}{1 - \dfrac{1}{\csc^2(x)}} = \frac{\sin^2(x)}{1 - \cos^2(x)} = \frac{\sin^2(x)}{\sin^2(x)} = 1$$

Solve.

27. (78) $63x^3 - 7x = 0$

28. (78) $4x^3 + 6x^2 - 40x = 0$

29. (26) A line passes through points $(-2, 4)$ and $(0, 5)$. Write the equation of the line in slope-intercept form.

*30. (103) **Graphing Calculator** Graph $y = 2 \sec(x)$ on your graphing calculator. Determine its period and asymptotes.

LESSON
109 Making Graphs and Using Equations of Hyperbolas

Warm Up

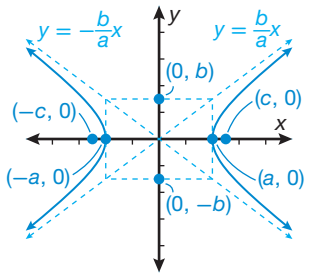
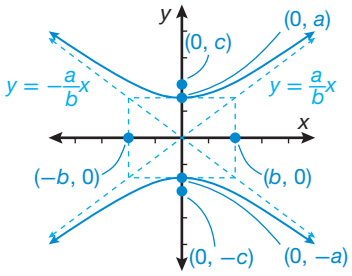
1. (58) **Vocabulary** To convert the equation $x^2 + 10x + 3 = 0$ to $(x + 5)^2 = 22$ involves using the technique of _____.

2. (26) What is the equation of the line with slope 3 that passes through $(2, -25)$?

3. (58) Solve by completing the square: $2x^2 - 4x + 10 = 0$.

New Concepts

A **hyperbola** is the set of points P in a plane such that the difference of the distances from P to two fixed points F_1 and F_2, called the **foci,** is a constant. The standard form of the equation of an hyperbola depends on the orientation of the hyperbola. The **transverse axis** is the segment that lies on the line containing the foci and has endpoints on a hyperbola. The endpoints are the **vertices** of a hyperbola.

Horizontal Orientation	Vertical Orientation
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
	
$c = \sqrt{a^2 + b^2}$, Eccentricity: $e = \sqrt{1 + \frac{b^2}{a^2}} = \frac{c}{a}$	
Foci: $(-c, 0)$ and $(c, 0)$ Vertices: $(-a, 0)$ and $(a, 0)$ Asymptotes: $y = \pm\frac{b}{a}x$	Foci: $(0, -c)$ and $(0, c)$ Vertices: $(0, a)$ and $(0, -a)$ Asymptotes: $y = \pm\frac{a}{b}x$

Math Reasoning

Verify Show that $\sqrt{1 + \frac{b^2}{a^2}} = \frac{c}{a}$.

Math Reasoning

Analyze Why are the values of c and e not dependent on the orientation of the hyperbola?

Example 1 Graphs of Hyperbolas Centered at the Origin

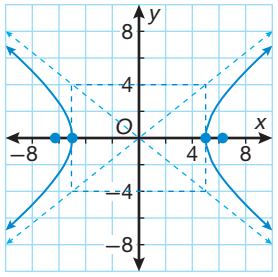
a. Graph $\frac{x^2}{5^2} - \frac{y^2}{4^2} = 1$. Identify the values of a, b, and c, as well as the orientation of the graph. Determine the eccentricity e and the asymptotes.

SOLUTION Horizontal orientation

$a = 5; b = 4$

$c = \sqrt{5^2 + 4^2} \approx 6.4031, e = \sqrt{1 + \frac{4^2}{5^2}} \approx 1.2806$

Asymptotes: $y = \pm\frac{4}{5}x$



b. Graph $\frac{y^2}{8^2} - \frac{x^2}{6^2} = 1$. Identify the values of a, b, and c, as well as the orientation of the graph. Determine the eccentricity e and the asymptotes.

SOLUTION Vertical orientation

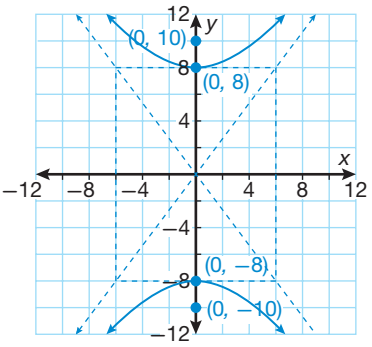
$a = 8$

$b = 6$

$c = \sqrt{8^2 + 6^2} = 10$

$e = \sqrt{1 + \frac{6^2}{8^2}} = 1.25$

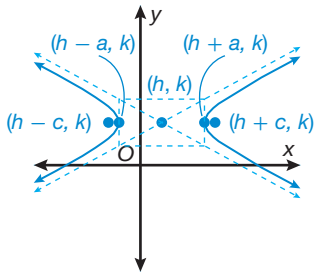
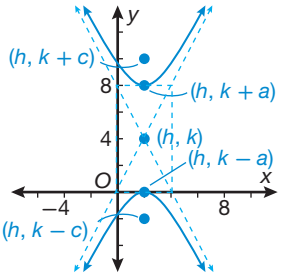
Asymptotes: $y = \pm\frac{4}{3}x$



Math Language

One of the two symmetrical parts of the hyperbola is called a **branch.** A hyperbola has two branches.

A hyperbola does not need to be centered at the origin.

Horizontal Orientation	Vertical Orientation
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
	

Example 2 Graphs of Hyperbolas Not Centered at the Origin

Graph each equation. Identify the values of a, b, and c, as well as the orientation of the graph. Determine the eccentricity and the asymptotes.

a. $\frac{(x-3)^2}{5^2} - \frac{(y-5)^2}{4^2} = 1$

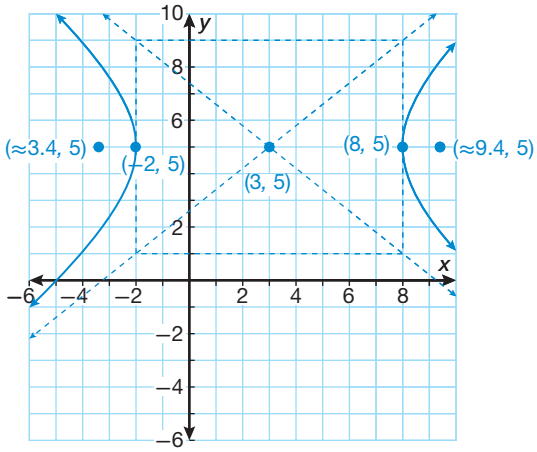
SOLUTION Horizontal orientation

$a = 5 \qquad b = 4$

$c = \sqrt{5^2 + 4^2} \approx 6.4031$

$e = \sqrt{1 + \frac{4^2}{5^2}} \approx 1.2806$

Finding the equations of the asymptotes requires using the slope formula, since the hyperbola is not centered at the origin.



Step 1: Calculate the slope.

$m = \pm\frac{b}{a} = \pm\frac{4}{5}$

Step 2: Use the slope formula for each asymptote. Use (h, k).

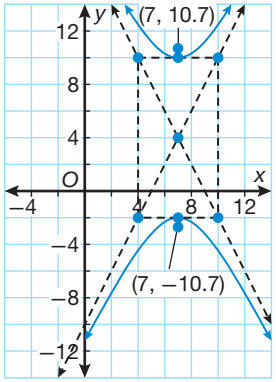
$$m = \frac{y - y_1}{x - x_1} \qquad m = \frac{y - y_1}{x - x_1}$$

$$\frac{4}{5} = \frac{y - 5}{x - 3} \qquad -\frac{4}{5} = \frac{y - 5}{x - 3}$$

$$4x - 12 = 5y - 25 \qquad -4x + 12 = 5y - 25$$

$$y = \frac{4}{5}x + \frac{13}{5} \qquad y = -\frac{4}{5}x + \frac{37}{5}$$

b. $\dfrac{(y-4)^2}{6^2} - \dfrac{(x-7)^2}{3^2} = 1$



SOLUTION Vertical orientation

$a = 6 \qquad b = 3$

$c = \sqrt{6^2 + 3^2} \approx 6.7082$

$e = \sqrt{1 + \dfrac{3^2}{6^2}} \approx 1.1180$

Asymptotes: $y = 2x - 10 \quad y = -2x + 18$

When the equation of a hyperbola is in standard form, it is easier to graph.

Example 3 Deriving the Equation of a Hyperbola in Standard Form

Change $54y - 9y^2 + 4x^2 - 32x - 53 = 0$ to the equation of a hyperbola in standard form.

SOLUTION **Step 1:** Organize the terms into x- and y-terms.

$$(4x^2 - 32x) - (9y^2 - 54y) - 53 = 0$$

$$4(x^2 - 8x) - 9(y^2 - 6y) - 53 = 0$$

Step 2: Complete the square for the x- and y-terms in the parentheses. Be sure to subtract the values that are added to complete the square. See the highlighted numbers.

$$4(x^2 - 8x + \mathbf{16}) - 9(y^2 - 6y + \mathbf{9}) - 53 \mathbf{- 64 + 81} = 0$$

$$4(x - 4)^2 - 9(y - 3)^2 - 36 = 0$$

Step 3:

$$4(x - 4)^2 - 9(y - 3)^2 = 36$$ Put the constant term on the right side of the equation.

$$\frac{4(x-4)^2}{36} - \frac{9(y-3)^2}{36} = 1$$ Divide both sides by that number to get 1 on the right.

$$\frac{(x-4)^2}{9} - \frac{(y-3)^2}{4} = 1$$

$$\frac{(x-4)^2}{3^2} - \frac{(y-3)^2}{2^2} = 1$$ Express the denominators as squares.

The properties of hyperbolas give rise to what is known as hyperbolic geometry. **Hyperbolic geometry** has its own set of trigonometric identities, as shown below.

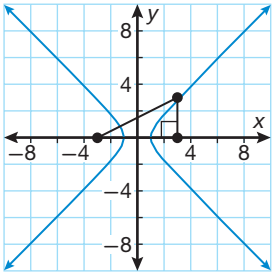
Hyperbolic Geometry

For any point (x, y) on the unit hyperbola,

$$x^2 - y^2 = 1$$

$$\cos h(x) = \frac{e^x + e^{-x}}{2}$$

$$\sin h(x) = \frac{e^x - e^{-x}}{2}$$



The function $\cos h(x)$ is known as the hyperbolic cosine, and $\sin h(x)$ is known as the hyperbolic sine.

Example 4 Application: Hyperbolic Geometry

Show that in hyperbolic geometry,

$\cos h^2(x) - \sin h^2(x) = 1$

SOLUTION Using the definitions of the hyperbolic trigonometric functions,

$$\cos h^2(x) - \sin h^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}(e^{2x} + 2e^x e^{-x} + e^{-2x} - e^{2x} + 2e^x e^{-x} - e^{-2x})$$

$$= \frac{1}{4}(2 + 2) = 1$$

Lesson Practice

Graph each equation. Identify the values of *a*, *b*, and *c*, as well as the orientation of the graph. Determine the eccentricity *e* and the asymptotes.

a. *(Ex 1)* $\frac{x^2}{6^2} - \frac{y^2}{4^2} = 1$

b. *(Ex 1)* $\frac{y^2}{7^2} - \frac{x^2}{5^2} = 1$

c. *(Ex 2)* $\frac{(x-5)^2}{7^2} - \frac{(y-4)^2}{5^2} = 1$

d. *(Ex 2)* $\frac{(y-8)^2}{9^2} - \frac{(x-5)^2}{6^2} = 1$

e. *(Ex 3)* Take the following equation and change it to the equation of a hyperbola in standard form. $54y - 9y^2 + 4x^2 - 32x - 161 = 0$

f. *(Ex 4)* What is $\sin h^2(x) - \cos h^2(x)$?

Practice Distributed and Integrated

Simplify.

1. (48) $\dfrac{x+4}{x^2-x-12}+\dfrac{2x}{x+3}$

2. (48) $\dfrac{\dfrac{x-1}{x+5}}{\dfrac{x+6}{x-3}}$

Convert from Cartesian to polar coordinates.

3. (96) $(-5, -5)$

4. (96) $(0, -4)$

***5.** (109) **Formulate** A hyperbola has asymptotes with equations $y = \pm x$. The graph of the hyperbola crosses the x-axis at $(5, 0)$. Write the equation of the hyperbola.

6. (32) Determine whether the matrices are inverses.

$$M=\begin{bmatrix}10 & -3\\ 3 & -1\end{bmatrix} \qquad N=\begin{bmatrix}1 & -3\\ 3 & -10\end{bmatrix}$$

7. (Inv 9) **Data Analysis** The general sizing chart for infant/toddler shoe sizes is shown to the right. Lana is buying shoes for her 9-month-old brother. What shoe size(s) can she buy? Does this table form a function? Explain.

Age (in months)	Size
$3 \le x \le 6$	2
$6 \le x \le 9$	3
$9 \le x \le 12$	4

8. (101) **Analyze** Explain why the leading coefficient and degree of the highest-degree term of a polynomial function are the only characteristics that determine end behavior.

9. (93) **Error Analysis** To evaluate $\dfrac{\log 2}{12\log 1.05}$ with a calculator, a student entered $\log(2) \div 12\log(1.05)$. What is the error? What is a correct expression to enter on a calculator?

***10.** (109) **Space Science** A satellite is following a hyperbolic path, and there are two monitoring stations located at the foci of the hyperbola. What data can be used to ensure that the satellite is maintaining its hyperbolic trajectory?

11. (96) **Analyze** Point P has polar coordinates (r, θ). Describe all the polar coordinates of point P that have r as the first coordinate.

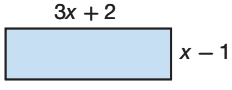
Solve.

12. (5) $\dfrac{1}{4}\begin{bmatrix}2 & 4\\ 6 & 8\end{bmatrix}+\begin{bmatrix}-2 & 1\\ -\frac{1}{2} & 5\end{bmatrix}$

13. (84) $\dfrac{x^2-x-12}{x-2}=0$

***14.** (108) **Graphing Calculator** Graph $y_1 = \cot^2(x)$ and $y_2 = \csc^2(x)$. Use the information in the graphs to prove the trigonometric identity $1 + \cot^2(x) = \csc^2(x)$.

15. (85) **Construction** The sheet of metal shown is rolled into a cylinder along the longer side. The area of the sheet is 22 square units. Find the radius of the cylinder.



16. (36) Find the equation of the line perpendicular to the graph of $y = 19$ that passes through the point $(6, -5)$.

17. (79) Evaluate the piecewise function $f(x) = \begin{cases} 2 - x \text{ if } x \leq 5 \\ -x^2 \text{ if } 5 < x < 8 \\ 6 \text{ if } 8 \leq x \end{cases}$ for $x = -8$ and $x = 5$.

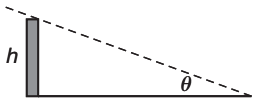
18. (88) **Error Analysis** A student solved the formula $A = P + Prt$ for P as shown below. What is the error? Solve the formula for P correctly.

$$A = P + Prt$$
$$A - P = Prt$$
$$A - \frac{P}{rt} = P$$

***19.** (108) **Multiple Choice** Which expression is equivalent to $\frac{1}{1 + \tan^2(\theta)}$?

A $\tan^2(\theta)$ **B** $\csc^2(\theta)$ **C** $\sin^2(\theta)$ **D** $\cos^2(\theta)$

20. (46) **Geometry** A wall casts a shadow depending on the angle of the sun. Derive the equation that can be used to find the length of the shadow, L.



21. (91) **Multiple Choice** Concentric circles have the same center but different radii. Which pair of equations represents a pair of concentric circles?

A $(x - 1)^2 + (y + 2)^2 = 6$ and $(x - 1)^2 + (y + 2)^2 = 6$

B $(x - 1)^2 + (y + 2)^2 = 6$ and $(x - 1)^2 + (y + 2)^2 = 8$

C $(x - 1)^2 + (y + 2)^2 = 6$ and $(x - 2)^2 + (y + 1)^2 = 6$

D $(x - 1)^2 + (y + 2)^2 = 6$ and $(x - 2)^2 + (y + 8)^2 = 12$

***22.** (102) **Population Growth** Based on U.S. Census statistics, the population of New Jersey for the years 1950–1990 can be modeled by the equation $y = 628 + 1842 \ln t$, where $10 \leq t \leq 50$, $t = 10$ represents 1950, and y is the population in thousands. Use the model to approximate the year in which the population of New Jersey reached 7,000,000.

23. (93) **Multi-Step** The table below is based on census statistics from India and China.

	China	India
Population in 2000 (to nearest million)	1,267,000,000	1,005,000,000
Average annual increase (1990–2000)	1.04%	1.83%

a. Write a function of the form $y_1 = ab^t$, where y_1 represents China's population t years after 2000.

b. Write a function of the form $y_2 = ab^t$, where y_2 represents India's population t years after 2000.

c. Write and solve an equation to predict the year in which India's population will overtake China's population.

***24.** *(107)* **Physics** If you throw a ball in the air, the vertical distance d that it travels over time t is found using the equation $d(t) = -4.9t^2 + v_0t + d_0$. The term v_o is the initial speed that the object is thrown, and d_o is the initial height the object is thrown from. The rational function $y = \frac{d(t)}{t}$ is the average speed the object has traveled. Find the slant asymptotes of this equation for the function that has $v_o = 65$ and $d_o = 25$.

***25.** *(97)* **Write** How is a geometric sequence similar to an arithmetic sequence?

26. *(100)* **Generalize** An odd function is a function with a graph that is symmetric with respect to the origin. A function is odd if and only if $f(-x) = -f(x)$. Show that rational functions of the form $f(x) = \frac{1}{x(2n+1)}$, for positive integers n, are odd functions.

***27.** *(105)* **Physics** The number of feet an object falls during each second after it is dropped is given by the sequence 16, 48, 80, 112, … It takes between 8 and 9 seconds for an object that falls off the top of the John Hancock Center in Chicago to hit the ground. Show how to use an arithmetic series to estimate the height of the John Hancock Center.

28. *(28)* Identify any excluded values and then simplify the expression $\frac{4x + 28}{x^2 - 49}$.

***29.** *(106)* Determine where the graph of $y = (x + 4)(x + 4)(x + 4)(x - 2)(x - 6)(x - 6)$ passes through the x-axis and where it just touches the x-axis.

***30.** *(109)* The distance between the two foci of a hyperbola is 10. The slopes of the asymptotes are $\pm\frac{3}{4}$. Write one possible equation for such a hyperbola.

LESSON
110

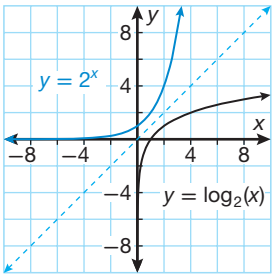
Graphing Logarithmic Functions

Warm Up

1. (72) **Vocabulary** A log base e is known as a _____.
2. (72) Write $\log \frac{(x^2 - 1)}{x - 3}$ as the sum and difference of logs.
3. (72) Rewrite $\log_6(78)$ as a natural logarithm.

New Concepts

A **logarithmic function** is the inverse of an exponential function. The graph of the logarithmic function has a similar shape to the graph of the corresponding exponential function. The graphs are reflected across the graph of $y = x$.



Axis of reflection: $y = x$

Recall that when a graph is reflected across the line $y = x$, the domain and range of each function are switched. So if the point (0, 1) is on the graph of $f(x)$, then (1, 0) is on the graph of $f^{-1}(x)$.

Example 1 Graphing Functions of the Form $f(x) = \log_b x$

Make a table using the given values to graph the function and its inverse. Describe the domain and range of the function.

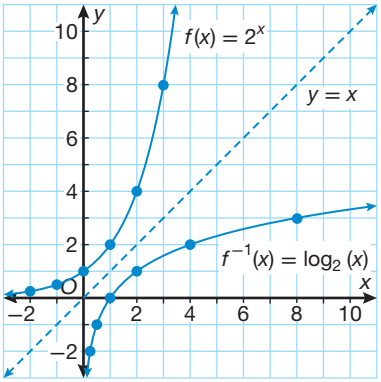
$f(x) = 2^x$; $x = -2, -1, 0, 1$, and 2

SOLUTION

x	-2	-1	0	1	2
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

Graph the line $y = x$. Reverse the ordered pairs to graph the inverse function $f^{-1}(x) = \log_2 x$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f^{-1}(x) = \log_2 x$	-2	-1	0	1	2



The domain of $f(x) = 2^x$ is all real numbers. The range is $f(x) > 0$. So, the domain of the inverse function $f^{-1}(x) = \log_2 x$ is $x > 0$, and the range $f^{-1}(x)$ is all real numbers.

Math Reasoning

Generalize Explain how to graph a logarithmic function with a base other than 10 or e.

When graphing a logarithmic function, make a note of three key constants, the coefficient a of the function, the base b of the logarithm, and the coefficient c of the x-term.

$$f(x) = a\log_b(cx)$$

Recall that e is the natural base, which is equal to approximately 2.7182818…. The **natural logarithmic function** is the function $f(x) = \ln x$, which is the inverse of the natural exponential function $f(x) = e^x$. The domain of $f(x) = \ln x$ is $x > 0$ and the range is all real numbers.

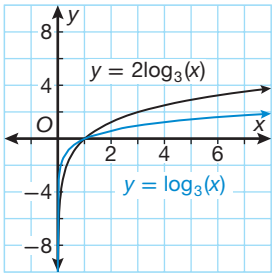
Example 2 Graphing Functions of the Form $f(x) = a\log_b(cx)$

a. Compare the graphs of the two logarithmic functions to see the effect of the variable a on the graph of the function.

$$y = \log_3(x) \text{ and } y = 2\log_3(x)$$

SOLUTION

From the graphs, note that the larger the value of the coefficient a, the taller the height of the graph.

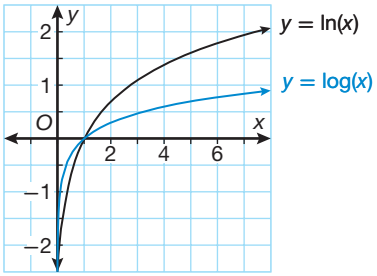


b. Compare the graphs of the logarithmic functions to see the effect of the base b on the graph of the function.

$$y = \ln(x) \qquad y = \log(x)$$

SOLUTION

From the graphs, note that the larger the value of the base, the shorter the height of the graph.

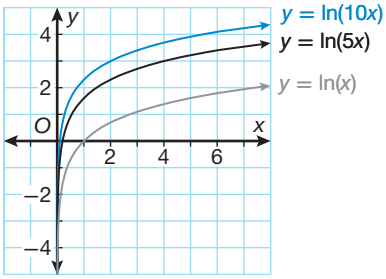


c. Compare the graphs of the logarithmic functions to see the effect of the coefficient c on the graph of the function.

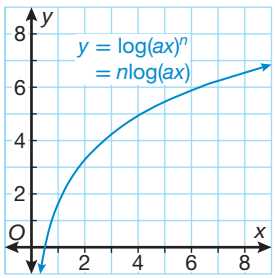
$$y = \ln(x) \qquad y = \ln(5x) \qquad y = \ln(10x)$$

SOLUTION

From the graphs, note that the larger the value of the coefficient, the taller the height of the graph.



When a logarithmic function is a composite function, look for ways of simplifying the function using the laws of logarithms.



Math Reasoning

Verify Why do the graphs intersect at $x = 1$? What can you conclude about all logarithmic functions?

Math Reasoning

Analyze Suppose that for two logarithmic functions $f(x)$ and $g(x)$,

$|f(x_1)| > |g(x_1)|$

for $x_1 > 1$. What can you conclude about the bases of $f(x)$ and $g(x)$?

Math Reasoning

Analyze For logarithmic functions of the form $f(x) = \log(ax)^n$, which values of n allow for input values of $x < 0$?

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Example 3 Application: Sound

Sound is measured in the decibel system, a logarithmic system using the equation $\text{db} = 10\log\left(\frac{I}{I_0}\right)$, where $I_0 = 1 \times 10^{-12}$. The sound of a jet is 140 dB and the sound of a vacuum cleaner is 80 dB. Find how much more intense the sound of a jet taking off is than the sound of a vacuum cleaner.

SOLUTION The difference between two sound levels, db_1 and db_2, results in

$$\text{db}_2 - \text{db}_1 = 10\log\left(\frac{I_2}{I_0}\right) - 10\log\left(\frac{I_1}{I_0}\right) = 10\log\left(\frac{I_2}{I_1}\right).$$

Solve this equation for the db levels of 140 and 80.

$$\text{db}_2 - \text{db}_1 = 10\log\left(\frac{I_2}{I_1}\right)$$

$$140 - 80 = 10\log\left(\frac{I_2}{I_1}\right)$$

$$60 = 10\log\left(\frac{I_2}{I_1}\right)$$

$$6 = \log\left(\frac{I_2}{I_1}\right)$$

$$10^6 = \frac{I_2}{I_1}$$

$$I_2 = 10^6 I_1$$

The intensity of sound of a jet is a million times louder than a vacuum cleaner.

Lesson Practice

a. *(Ex 1)* Make a table using the given values to graph the function and its inverse. Describe the domain and range of the function.

$$f(x) = 3^x;\ x = -2, -1, 0, 1, \text{ and } 2$$

b. *(Ex 2)* Compare the graphs of the two logarithmic functions to see the effect of the variable a on the graph of the function.

$$y = 12\log(x) \text{ and } y = -12\log(x)$$

c. *(Ex 2)* Compare the graphs of the three logarithmic functions to see the effect of the base b on the graph of the function.

$$y = 5\ln(x) \qquad y = 5\log(x)$$

d. *(Ex 2)* Compare the graphs of the three logarithmic functions to see the effect of the coefficient c on the graph of the function.

$$y = \ln(-x) \qquad y = \ln(-5x) \qquad y = \ln(-10x)$$

e. *(Ex 3)* The sound of a whisper is 20 dB and the threshold of hearing is 0 dB. How much louder is a whisper than the threshold of hearing?

Practice Distributed and Integrated

*1. (85) Use the Rational Root Theorem to find the roots of

$$y = 12x^{11} - 23x^{10} + 10x^9 - 12x^8 + 23x^7 - 10x^6 - 12x^5 + 23x^4 - 10x^3 + 12x^2 - 23x + 10.$$

2. (92) **Multiple Choice** The first term of an arithmetic sequence is -12 and the fifteenth term is -54. What is the common difference?

A -4.4 **B** -3 **C** 3 **D** 4.4

Write the polynomials in standard form.

3. (27) $16x = 2 - x^2 + y$

4. (27) $-9 + 13x = 4x^2 - y$

5. (37) For what values of x is the denominator of the expression $\frac{3}{x^2 + x - 30}$ zero?

6. (106) **Multi-Step** A shipping crate has dimensions of x, $x - 1$, and $x - 2$.

a. What is the simplest polynomial function that represents the volume of the crate?

b. Describe the possible zeros of the function.

c. What are the dimensions of the crate if the volume is 120 cubic units?

*7. (110) **Analyze** A chemical process goes at a logarithmic pace to create a chemical by-product. How long will it take to go from 1 unit of by-product to 2 units? Express your answer as a function of the base of the logarithm.

8. (14) **Analyze** Find the determinant of the multiplicative identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

*9. (84) **Optics** The focal length of a lens, f, the distance from the lens to the object, o, and the distance from the lens to the image, i, are related by the formula $\frac{1}{f} = \frac{1}{o} + \frac{1}{i}$. Find i if f is 4 cm and o is 48 cm.

*10. (32) **Graphing Calculator** Solve the system using a graphing calculator.

$$\begin{aligned} x + y + z &= -2 \\ 3x + 3y + z &= -18 \\ 4x + 2y + z &= -20 \end{aligned}$$

11. (42) Find the number of outfits possible using 2 of 4 shirts and 2 of 3 pants.

*12. (102) **Motor Vehicle Ownership** The number of motor vehicles owned in U.S. households during the years 1970–2000 can be modeled by the equation $y = -82{,}110 + 70{,}059 \ln t$, $10 \le t \le 40$, where $t = 10$ represents 1970, and y is the number of motor vehicles owned in households, in thousands. Use the model to approximate the year in which the number of motor vehicles owned in U.S. households reached 100 million.

*13. (109) **Generalize** A hyperbola has the form $ax^2 + by^2 + cx + dy + e = 0$. Complete the square for the x-term.

14. (53) Let $f(x) = 12\log(4x)$ and $g(x) = \frac{1}{\sqrt{x}}$. Find the x-intercept for $f(g(x))$.

15. (91) **Analyze** How can the equation of a circle located at the origin be derived from the equation of a circle located at any point?

16. (93) A club's membership increases 4% each year. There are 800 members. How long until it reaches 1000?

Divide using synthetic division.

17. (51) $x^4 - 1$ by $x + 1$

18. (51) $x^3 - 7$ by $x - 5$

***19.** (109) **Error Analysis** A student is finding the equations of the asymptotes for the hyperbola with following equation: $\frac{(x-15)^2}{9^2} - \frac{(y+5)^2}{7^2} = 1$. What is her mistake?

Calculate the slopes: $m = \pm\frac{7}{9}$

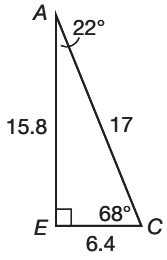
Use the slope formula:

$$\frac{7}{9} = \frac{y-5}{x-15} \quad y = \frac{7}{9}x - \frac{20}{3} \qquad -\frac{7}{9} = \frac{y-5}{x-15} \quad y = -\frac{7}{9}x - \frac{50}{3}$$

Use the triangle to solve problems 20 and 21.

20. (46) Find the cosecant of $\angle A$.

21. (46) Find the secant of $\angle A$.



22. (89) **Multiple Choice** What are the solutions of $x^2 + 4x < 5$?

A $-5 < x < 1$ **B** $-1 < x < 5$ **C** $x < -5$ or $x > 1$ **D** $x < -1$ or $x > 5$

***23.** (108) **Model** Do the trigonometric identities hold true even if each function is phase shifted by angle θ? Provide examples to support your claim.

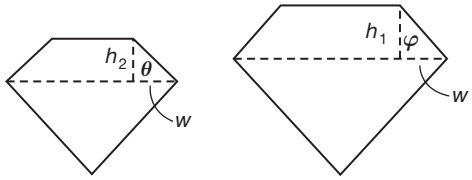
24. (85) **Geometry** A pyramid with a square base has a height of $3x$ and a volume of $x^3 + 4x^2 + 4x$. Find the dimensions of the square base.

***25.** (107) **Algebra** The sum of the first n consecutive integers can be found by using the function $f(n) = \frac{1}{2}n(n+1)$. The sum of the first n consecutive odd integers can be found by using the function $f(n) = n^2$. Write the rational function that is the ratio of the sum of the first n consecutive integers to the sum of the first n consecutive odd integers. Graph the function to find the horizontal asymptote. Explain the significance of the asymptote.

26. (86) **Error Analysis** Two students are finding the period of $y = \cos\left(\frac{2}{3}x\right)$ but get different results. Which student made the mistake?

Student A	Student B
Period $= \frac{2\pi}{b} = \frac{2\pi}{\frac{2}{3}} = \frac{6\pi}{2} = 3\pi$	Period $= \frac{2\pi}{b} = \frac{2\pi}{\frac{2}{3}} = \frac{4\pi}{3}$

27. *(90)* **Jewelry Making** A jeweler is changing the angle of a facet for a diamond ring from θ to φ. The vertical distances change as shown in the diagram, while the width remains constant. Derive the function that the jeweler can use to find values of θ when the other values are known.



28. *(101)* **Finance** The Dow Jones Industrial Average (DJIA) is an indicator of stock prices. The quarterly high DJIA for 1999–2003 can be modeled by $f(x) = 0.195x^4 - 5.035x^3 + 6.205x^2 + 338.331x + 10219$, where $x = 1$ represents the first quarter (January–March 1999), $x = 2$ represents the second quarter (April–June 1999), and so on, with $x = 20$ representing the last quarter (October–December 2003). Use the model to approximate the highest quarterly Dow Jones Industrial Average during 1999–2003. Based on the model, in which quarter did it occur?

Solve by factoring.

29. *(35)* $0 = x^2 + 4x - 21$

30. *(35)* $x^2 - 16 = 0$

INVESTIGATION

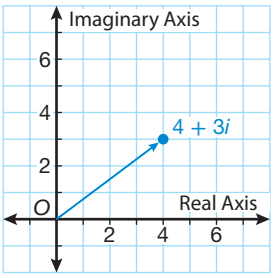
11

Using De Moivre's Theorem

Hint

When graphing complex numbers, the horizontal axis is the real axis and the vertical axis is the imaginary axis.

All complex numbers can be graphed on a complex plane. A complex number of the form $a + bi$ can be graphed with a as the x-coordinate and b as the y-coordinate. This is called the rectangular form of a complex number.



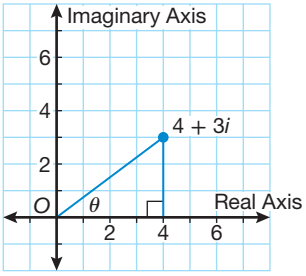
1. On a sheet of graph paper, create an x-y grid to graph the following complex numbers.

   a. $3 + 2i$
   b. $-10 + 5i$
   c. $6i$
   d. $-10i$
   e. 7
   f. -9

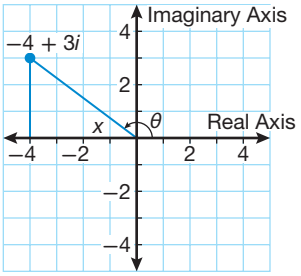
Hint

When graphing points, $6i = 0 + 6i$ and $7 = 7 + 0i$

Refer again to the point on the imaginary plane represented by $4 + 3i$. If a segment is drawn from the origin to this point, and a segment is also drawn from the point straight down to the real axis, a right triangle is formed.



The measure of the angle between the real axis and the hypotenuse of this right triangle is θ. That is, θ is the angle with the positive x-axis as its initial side and the hypotenuse of the right triangle as its terminal side. Using the graph of the complex number to find the corresponding coordinate on the standard coordinate grid, the trigonometric ratios, and a graphing calculator, the measure of θ can be determined.



For example, the graph of $-4 + 3i$ corresponds to the graph of $(-4, 3)$. The length of the side adjacent to angle x is 4 units and the length of the side opposite angle x is 3 units. So, $\tan x = \frac{3}{4}$. Using a graphing calculator and $\tan^{-1}\left(\frac{3}{4}\right) = x$, $x \approx 36.87°$. Subtract 36.87° from 90° to find the complement of angle x (53.13°). Last, to find the approximate value of θ, add: $53.13° + 90° = 143.13°$. Angle $\theta \approx 143.13°$. (Notice that the same result can be found by subtracting the measure of angle x from 180°. That is, $180° - 53.13° = 143.13°$.)

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2. Find the measure of θ when each complex numbers is graphed on the complex plane.

a. $4 + 4i$
b. $-7 + 14i$
c. $3 + (-9i)$
d. $-2 + 8i$
e. $2 - 5i$
f. $-4 + i$

Caution

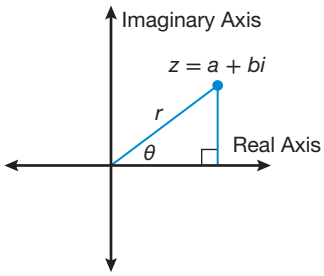
To calculate θ based on the values of a and b, use $\tan^{-1}\left(\frac{b}{a}\right)$. Also, keep track of the quadrant in which the complex number is found.

3. For each of the complex numbers question 2, complete the table calculate the hypotenuse r of corresponding right triangle.

a	b	$r = \sqrt{a^2 + b^2}$

Another way of writing and graphing complex numbers is to use polar coordinates. Recall that polar coordinates take two arguments: the magnitude of a segment and the counterclockwise angle from the x-axis to that segment.

Polar form for complex numbers works in the same way. In the graph shown here, the right triangle and θ can be used to express the real and imaginary parts of the complex number z, where $z = a + bi$, with trigonometric ratios.



Let r be the length of the hypotenuse. Then the real part of this coordinate, a, can be found using the equation:

$$\cos\theta = \frac{a}{r}$$

$$a = r\cos\theta$$

And the imaginary part is:

$$\sin\theta = \frac{b}{r}$$

$$b = r\sin\theta$$

So $z = a + bi$ in polar form is $z = r\cos\theta + i(r\sin\theta)$. To convert from rectangular coordinates to polar coordinates, find the value of r using the Pythagorean Theorem and the value of θ as shown above, then substitute these values into the polar form of the equation $z = r\cos\theta + i(r\sin\theta)$.

4. Convert the following complex numbers in rectangular form to polar form.

a. $z = 4 - 2i$

b. $z = -1 - i$

c. $z = 3 + 4i$

To convert from polar form to rectangular form, simplify the polar form by evaluating the trigonometric ratios. For example:

$$\sqrt{2}\cos(45°) + i\sqrt{2}\sin(45°)$$
$$= \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) + i\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)$$
$$= 1 + i$$

5. Convert the complex numbers in polar form to rectangular form.

a. $7\cos(45°) + i\,7\sin(45°)$

b. $10\cos(90°) + i\,10\sin(90°)$

c. $5\cos(120°) + i\,5\sin(120°)$

d. $17\cos(200°) + i\,17\sin(200°)$

Graphing Calculator Tip

Make sure the calculator is set to degree mode. Incorrect values will result if the radian mode is used.

6. For each of the complex numbers in question 5, complete the table for values a, b, r, and θ.

r	θ	a	b

7. For each of the complex numbers written in $a + bi$ form, convert to polar form. (Hint: The magnitude of z, $|z|$ is equal to r.)

$a + bi$	r	θ	Polar Form
$4 + 6i$			
$7 + 7i$			
$-3 + 4i$			
15			
$28i$			

While the rectangular and polar forms of a complex number are interchangeable, complex numbers of the form $(a + bi)^n$, for any integer n, are much easier to calculate in polar form. The reason for this is De Moivre's Theorem, which states,

$$(r\cos(\theta) + ir\sin(\theta))^n = r^n\cos(n\theta) + ir^n\sin(n\theta)$$

According to De Moivre's Theorem, a complex number in trigonometric form, when raised to any integer power, remains a binomial and is therefore much easier to calculate. Compare the De Moivre form to the binomial expansion of the $a + bi$ form:

$$(a + bi)^2 = (a^2 - b^2) + i2ab$$
$$(a + bi)^3 = (a^3 - 3ab^2) + i(-b^3 + 3a^2b)$$
$$(a + bi)^4 = (a^4 + b^4 - 6a^2b^2) + i4(a^3b - ab^3)$$
$$(a + bi)^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} a^{n-k}b^k$$

8. Use De Moivre's Theorem to write each of the expressions below in the standard form $a + bi$.

Hint

Make sure that $n\theta < 360°$. If necessary, subtract multiples of 360°.

a. $[7\cos(20°) + i7\sin(20°)]^3$.
b. $[3\cos(45°) + i3\sin(45°)]^2$.
c. $[2\cos(120°) + i2\sin(120°)]^4$.
d. $[\cos(210°) + i\ \sin(210°)]^3$.
e. $[4\cos(180°) + i4\sin(180°)]^2$.
f. $[2\cos(30°) + i2\sin(30°)]^6$.

9. Use De Moivre's Theorem to evaluate the following complex numbers. Express the results in polar form.

a. $(3 + 4i)^3$
b. $(10 - 10i)^4$
c. $(5 + 12i)^6$
d. $(-4 + 0i)^8$
e. $(0 + 7i)^{12}$

Investigation Practice

a. For each of the complex numbers written in rectangular form, convert to polar form.

$a + bi$	r	θ	Polar Form
$5 + (-5i)$			
$-7 + 10i$			
$0 + 18i$			
$25 + 0i$			
$-9 + (-3i)$			

Use De Moivre's Theorem to evaluate the following complex numbers. Express the results in polar form.

b. $(2 + 7i)^4$
c. $(1 - 13i)^7$
d. $(6 + 8i)^8$
e. $(-9 + 0i)^{12}$
f. $(0 + 19i)^{15}$

LESSON 111

Transforming Polynomial Functions

Warm Up

1. (20) **Vocabulary** A shift transformation is a _____.
2. (20) If $f(x) = 5x^2 - x + 1$, then $f(x) + 3 =$ _____.
3. (20) If $f(x) = x - 6$, then $2 \cdot f(x) =$ _____.

New Concepts

You can perform the same transformations on all polynomial functions that you performed on quadratic and linear functions.

Transformations of $f(x)$		
Transformation	**$f(x)$ notation**	**Examples for $f(x) = x^3$**
Vertical translation	$f(x) + k$	$g(x) = x^3 + 2$ 2 units up $g(x) = x^3 - 5$ 5 units down
Horizontal translation	$f(x - h)$	$g(x) = (x - 1)^3$ 1 unit right $g(x) = (x + 3)^3$ 3 units left
Vertical stretch/ compression	$af(x)$	$g(x) = 4x^3$ stretch by a factor of 4 $g(x) = \frac{1}{3}x^3$ compress by a factor of $\frac{1}{3}$
Horizontal stretch/ compression	$f\left(\frac{1}{b}x\right)$	$g(x) = \left(\frac{1}{5}x\right)^3$ stretch by a factor of 5 $g(x) = (2x)^3$ compress by a factor of $\frac{1}{2}$
Reflection	$-f(x)$ $f(-x)$	$g(x) = -x^3$ across x-axis $g(x) = (-x)^3$ across y-axis

Example 1 Translating a Polynomial Function

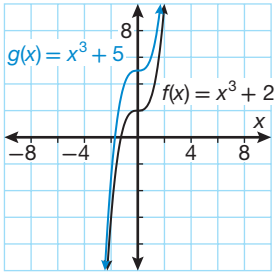
For $f(x) = x^3 + 2$, write the rule for each function and sketch its graph.

a. $g(x) = f(x) + 3$

SOLUTION

$g(x) = (x^3 + 2) + 3 = x^3 + 5$

Translate the graph of $f(x)$ 3 units up.

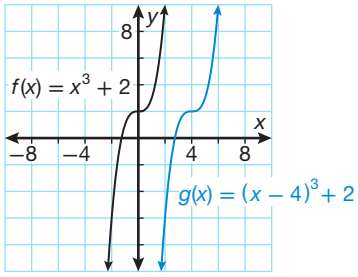


b. $g(x) = f(x - 4)$

SOLUTION

$g(x) = (x - 4)^3 + 2$

Translate the graph of $f(x)$ 4 units to the right.



Hint

You can use a calculator to check your graph.

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Example 2 Reflecting a Polynomial Function

Let $f(x) = x^3 - 8x^2 + 4x - 10$. Write a function g that performs each transformation.

Hint

To write the rule for $-f(x)$, change every sign of $f(x)$.

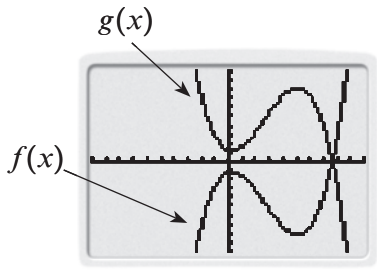
a. Reflect $f(x)$ across the x-axis.

SOLUTION

$g(x) = -f(x)$

$g(x) = -(x^3 - 8x^2 + 4x - 10)$

$g(x) = -x^3 + 8x^2 - 4x + 10$



Check Graph both functions. The graph of $g(x)$ is a reflection of $f(x)$ across the x-axis.

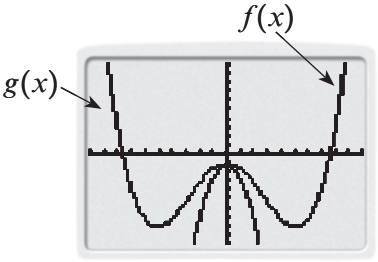
b. Reflect $f(x)$ across the y-axis.

SOLUTION

$g(x) = f(-x)$

$g(x) = (-x)^3 - 8(-x)^2 + 4(-x) - 10$

$g(x) = -x^3 - 8x^2 - 4x - 10$



Check Graph both functions. The graph of $g(x)$ is a reflection of $f(x)$ across the y-axis.

Example 3 Compressing and Stretching a Polynomial Function

Let $f(x) = x^4 - 5x^2 + 4$. Graph f and g on the same coordinate plane. Describe g as a transformation of f.

a. $g(x) = 2f(x)$

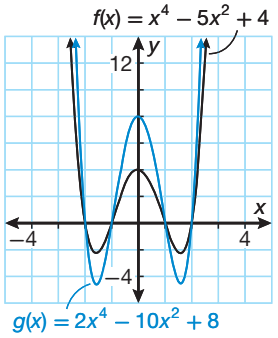
SOLUTION

$g(x) = 2f(x)$

$g(x) = 2(x^4 - 5x^2 + 4)$

$g(x) = 2x^4 - 10x^2 + 8$

$g(x)$ is a vertical stretch of $f(x)$.



b. $g(x) = f(3x)$

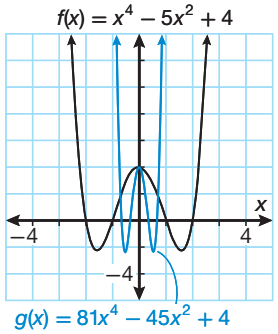
SOLUTION

$g(x) = f(3x)$

$g(x) = (3x)^4 - 5(3x)^2 + 4$

$g(x) = 81x^4 - 45x^2 + 4$

$g(x)$ is a horizontal compression of $f(x)$.



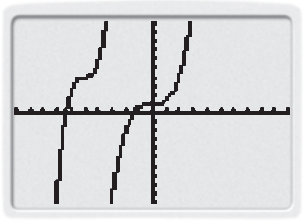
Example 4 Combining Transformations

Write a function that transforms $f(x) = 2x^3 + 5$ in each of the following ways. Support your solution by using a graphing calculator.

a. Stretch vertically by a factor of 4, and shift 5 units left.

SOLUTION

A vertical stretch is represented by $af(x)$, and a horizontal shift is represented by $f(x - h)$. Combining the two transformations gives $g(x) = af(x - h)$. Substitute 4 for a and -5 for h.



$g(x) = 4f(x - (-5))$

$g(x) = 4f(x + 5)$

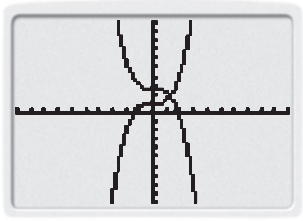
$g(x) = 4(2(x + 5)^3 + 5)$

$g(x) = 8(x + 5)^3 + 20$

b. Reflect across the x-axis and shift 18 units up.

SOLUTION

A reflection across the x-axis is represented by $-f(x)$, and a vertical shift is represented by $f(x) + k$. Combining the two transformations gives $h(x) = -f(x) + k$.



Substitute 18 for k.

$h(x) = -f(x) + 18$

$h(x) = -(2x^3 + 5) + 18$

$h(x) = -2x^3 + 13$

Hint

To get the graph of $f(x - h)$, shift the graph of $f(x)$:
- left if h is negative.
- right if h is positive.

Example 5 Application: A Height Function

After a certain object is thrown with an upward velocity of 40 feet per second, its height is given by the function $h(t) = -16t^2 + 40t + 10$, where t is the elapsed time in seconds and air resistance is neglected. Let $g(t) = h(t) + 5$. Write the rule for g, and explain what the transformation represents.

SOLUTION

$g(t) = h(t) + 5$

$g(t) = -16t^2 + 40t + 10 + 5$

$g(t) = -16t^2 + 40t + 15$

The rule is $g(t) = -16t^2 + 40t + 15$. The graph of $g(t)$ is a vertical shift 5 units up of the graph of $h(t)$. If the height of object H is given by $h(t)$ and the height of object G is given by $g(t)$, then object G is 5 feet higher than object H at all times that both objects are in the air.

Caution

The graph of $h(t) = -16t^2 + 40t + 10$ does not represent the path of the object; it only represents the height.

Lesson Practice

a. *(Ex 1)* Given $f(x) = x^3 - 1$, write the rule for $g(x) = f(x) + 5$. Sketch the graphs of f and g.

b. *(Ex 1)* Given $f(x) = x^3 - 1$, write the rule for $g(x) = f(x + 3)$. Sketch the graphs of f and g.

c. *(Ex 2)* Let $f(x) = x^3 + x^2 - 6x - 1$. Write a function g that is the reflection of $f(x)$ across the x-axis.

d. *(Ex 2)* Let $f(x) = x^3 + x^2 - 6x - 1$. Write a function g that is the reflection of $f(x)$ across the y-axis.

e. *(Ex 3)* Let $f(x) = 16x^4 - 24x^2 + 4$ and $g(x) = \frac{1}{4}f(x)$. Graph f and g on the same coordinate plane. Describe g as a transformation of f.

f. *(Ex 3)* Let $f(x) = 16x^4 - 24x^2 + 4$ and $g(x) = f\left(\frac{1}{2}x\right)$. Graph f and g on the same coordinate plane. Describe g as a transformation of f.

g. *(Ex 4)* Write a function g that transforms $f(x) = 8x^3 - 2$ as follows: Compress vertically by a factor of $\frac{1}{2}$, and move the x-intercept 3 units right. Support your solution by using a graphing calculator.

h. *(Ex 4)* Write a function h that transforms $f(x) = 8x^3 - 2$ as follows: Reflect across the x-axis, and move the x-intercept 4 units left. Support your solution by using a graphing calculator.

i. *(Ex 5)* After a certain object is thrown with a downward velocity of 15 feet per second, its height is given by the function $h(t) = -16t^2 - 15t + 200$, where t is the elapsed time in seconds and air resistance is neglected. Let $g(t) = h(t) - 40$. Write the rule for g, and explain what the transformation represents.

Practice Distributed and Integrated

1. *(26)* A line has slope -4 and passes through $(6, 8)$. What is the equation of the line written in slope intercept form?

***2.** *(111)* **Baseball Attendance** The *cumulative* attendance at home baseball games for the years 1996–2005 can be modeled by $f(x) = 1.922x^4 - 36.769x^3 + 132.212x^2 + 3558.615x - 11$, where x is the number of years since 1995 and $f(x)$ is in thousands. (Cumulative means the statistic for each year is the sum of that year's attendance and all the previous years' attendances.) Describe the transformation $g(x) = f(x - 1)$ by writing the rule for $g(x)$ and explaining the change in the context of the problem.

***3.** *(99)* Use matrix multiplication to show that the dot product of two vectors that are perpendicular to each other is zero.

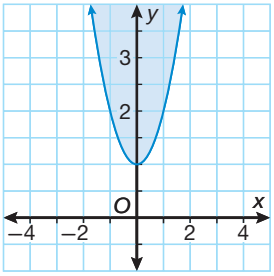
***4.** (110) **Multiple Choice** Which of the following functions has a graph with greater y-values?

A $y = \ln(x)$ **B** $y = \log_2(x)$ **C** $y = \log_{20}(x)$ **D** $y = \log_{15}(x)$

***5.** (Inv 11) Given the complex number $8 + 24i$ written in $a + bi$ form, convert it to the trigonometric and polar forms.

6. (10) Solve and graph the compound inequality $-8 < 4(x + 1)$ and $12 > 4x$.

7. (89) **Error Analysis** Explain the error or errors a student made in graphing $x^2 + 1 > y$.



8. (90) **Multiple Choice** Which of the following has a period of $\frac{1}{5}\pi$?

A $y = 5\tan(x)$ **B** $y = \tan(x - 5)$ **C** $y = \tan(5x)$ **D** $y = \tan\left(\frac{x}{5}\right)$

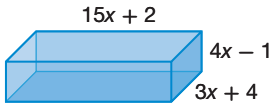
***9.** (109) **Geometry** In hyperbolic geometry, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$. Show that $1 - \tanh^2(x) = \frac{1}{\cosh^2(x)}$.

10. (92) **Theater** The number of seats in the first 14 rows of the center orchestra aisle of the Marquis Theater in New York City form an arithmetic sequence. The third row has 13 seats and the last row has 24 seats. Find the number of seats in the 7th row.

11. (50) **Temperature** The formula $C = \frac{5}{9}(F - 32)$ relates Celsius temperature C and Fahrenheit temperature F. Solve the formula for F. Then write and solve a new equation to find the number that represents the same temperature on both scales. (Hint: Substitute x for both C and F.)

12. (93) **Analyze** Cobalt-58ml has a half-life of 9.04 hours. Calculate mentally how long it would take for 4 grams of cobalt-58ml to decay to 1 gram. Explain your method.

13. (95) **Construction** Use the Remainder Theorem to determine the volume of the box if $x = 3$.



14. (29) How many solutions does the system of equations have, and what type of solution is it?

$$x + 5y - 2z = 1$$
$$-x - 2y + z = 6$$
$$-2x - 7y + 3z = 7$$

15. (97) **Write** Describe how to find the nth term of a geometric sequence, given the first and second terms.

***16.** (110) **Predict** Which function has the greater x-intercept, $f(x) = \log_b(x)$ or $g(x) = \log_b(\log_b(x))$? What can you conclude about the function with the greater x-intercept.

17. (96) **Multi-Step** Complete parts a–c to graph a polar equation and analyze the graph.

a. Convert the Cartesian equation $y = x^2$ to a polar equation by using $x = r\cos\theta$ and $y = r\sin\theta$. (Hint: Factor and then eliminate the case $r = 0$.)

b. Graph the polar equation on a calculator, using these window settings: $\theta\text{min} = 0$, $\theta\text{max} = \frac{\pi}{2}$, $\theta\text{step} = \frac{\pi}{24}$, $X\text{min} = -6$, $X\text{max} = 6$, $Y\text{min} = -1$, $Y\text{max} = 7$. Do you get the same graph as the graph of $y = x^2$? Explain why or why not.

c. If $\theta\text{min} = 0$ is used as a window setting, what is the least value of θmax needed to get the same graph as the graph of $y = x^2$? Explain.

***18.** (98) Write the equation $5x^2 + 2y^2 = 10$ as an ellipse in standard form.

***19.** (100) **Graphing Calculator** Graph the function $y = \frac{x^2 - 7x - 60}{x + 3}$ on a graphing calculator. Identify any vertical asymptote(s).

20. (102) **Analyze** Without graphing, determine the point in which the graphs of $y = \log x$ and $y = 3\log x$ intersect. Explain your reasoning.

21. (61) Determine whether the linear binomial $x + 3$ is a factor of the polynomial $P(x) = x^3 + 19x^2 + 79x - 35$.

***22.** (111) **Justify** Give an example of a polynomial function of degree 3 or higher whose reflection across the x-axis is the same as its reflection across the y-axis. Justify your answer.

23. (35) Write a quadratic function that has zeros 18 and 3.

24. (85) Let $f(x) = x^3 - 125$ and $g(x) = 2x - 1$. Find the roots of $f(g(x))$.

Find the roots of the equations.

25. (90) $0 = 3x^3 + 2x^2 + 1$

26. (90) $0 = 5x^4 + 2x^3 + x^2$

Solve the equations.

***27.** (102) $\log(12x - 11) = \log(3x + 13)$

***28.** (102) $\log_2(x - 4) = 6$

29. (80) Scores from a test have a mean of 87 and a standard deviation of 4.3. A randomly selected test has a score of 82.5. Find the z-score for this test.

30. (33) **Probability** Five folded pieces of paper are in a hat. Three are to be chosen at random. Is it a dependent or independent event if the papers are not returned to the hat once they are chosen?

LESSON 112

Using Sum and Difference Identities

Warm Up

1. (46) **Vocabulary** The ratio $\frac{x}{r}$ is the _____ ratio.

2. (52) $\sin 30° =$ _____

3. (52) $\tan 45° =$ _____

New Concepts

A trigonometric identity is a trigonometric equation that is true for all values of the variables for which every expression in the equation is defined.

Sum and Difference Identities are used to simplify and evaluate expressions.

Sum and Difference Identities	
Sum Identities	**Difference Identities**
$\sin(A + B) = \sin A \cos B + \cos A \sin B$	$\sin(A - B) = \sin A \cos B - \cos A \sin B$
$\cos(A + B) = \cos A \cos B - \sin A \sin B$	$\cos(A - B) = \cos A \cos B + \sin A \sin B$
$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$	$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Example 1 Evaluating Expressions with Sum and Difference Identities

a. Find the exact value of $\cos 75°$.

SOLUTION Write $75°$ as $30° + 45°$ so that known trigonometric function values of $30°$ and $45°$ can be used.

$\cos 75° = \cos(30° + 45°)$ Write $75°$ as $30° + 45°$.

$= \cos 30° \cos 45° - \sin 30° \sin 45°$ Apply the $\cos(A + B)$ identity.

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$ Evaluate.

$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$ Simplify.

b. Find the exact value of $\sin\left(-\frac{\pi}{12}\right)$.

SOLUTION

$\sin\left(-\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$ Write $\sin\left(-\frac{\pi}{12}\right)$ as $\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$.

$= \sin\frac{\pi}{4} \cos\frac{\pi}{3} - \cos\frac{\pi}{4} \sin\frac{\pi}{3}$ Apply the $\sin(A - B)$ identity.

$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$ Evaluate.

$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$ Simplify.

Hint

In Example 1b, there is more than one expression that can be substituted for $-\frac{\pi}{12}$. For example, $-\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{3}$ and $-\frac{\pi}{12} = \frac{\pi}{6} - \frac{\pi}{4}$.

Online Connection
www.SaxonMathResources.com

Example 2 Using the Pythagorean Theorem with Sum and Difference Identities

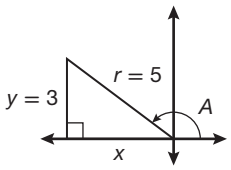
Evaluate $\tan(A - B)$ if $\sin A = \frac{3}{5}$, $90° < A < 270°$, $\cos B = -\frac{15}{17}$, and $180° < B < 360°$.

SOLUTION

Step 1: Find $\tan A$ and $\tan B$.

$90° < A < 270°$ and $\sin A$ is positive, so A is in Quadrant II.

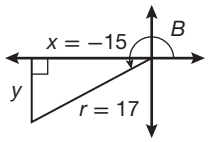
$$\sin A = \frac{3}{5} = \frac{y}{r}$$



$$x^2 + y^2 = r^2$$
$$x^2 + 3^2 = 5^2$$
$$x = -\sqrt{25 - 9} = -4$$
$$\tan A = \frac{y}{x} = \frac{3}{-4} = -\frac{3}{4}$$

$180° < B < 360°$ and $\cos B$ is negative, so B is in Quadrant III.

$$\cos B = \frac{-15}{17} = \frac{x}{r}$$



$$x^2 + y^2 = r^2$$
$$(-15)^2 + y^2 = 17^2$$
$$y = -\sqrt{289 - 225} = -8$$
$$\tan B = \frac{y}{x} = \frac{-8}{-15} = \frac{8}{15}$$

Step 2: Use the difference identity to find $\tan(A - B)$.

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$
$$= \frac{-\frac{3}{4} - \frac{8}{15}}{1 + \left(-\frac{3}{4}\right)\left(\frac{8}{15}\right)} = -\frac{77}{36}$$

Substitute $-\frac{3}{4}$ for $\tan A$ and $\frac{8}{15}$ for $\tan B$. Then simplify.

Math Reasoning

Write Explain why x is negative in the diagram for angle A.

Hint

Use an inverse trigonometric function and a calculator to verify that $A \approx 143.13°$. Then find the measures of B and $(A - B)$ to check the value of $\tan(A - B)$.

The sum identities for sine and cosine can be used to derive a matrix for rotating a point in a plane. To rotate a point $P(x, y)$ through an angle θ, you can use the **rotation matrix** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

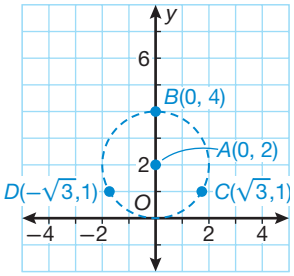
Using a Rotation Matrix

Let $P(x, y)$ be any point in a plane. Let $P'(x', y')$ be the image of P after a rotation of θ degrees counterclockwise about the origin. Then

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Example 4 Using a Rotation Matrix

Find the coordinates of the images of the points in the figure after a 60° rotation counterclockwise about the origin.



SOLUTION

Step 1: Write a rotation matrix and a matrix of the coordinates of the points in the figure.

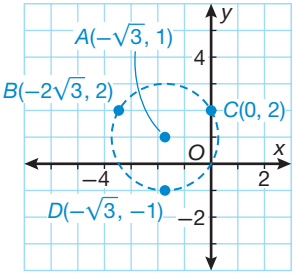
$$R_{60°} = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix}$$ Rotation matrix

$$S = \begin{bmatrix} 0 & 0 & \sqrt{3} & -\sqrt{3} \\ 2 & 4 & 1 & 1 \end{bmatrix}$$ Matrix of point coordinates

Step 2: Find the matrix product.

$$R_{60°} \times S = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \begin{bmatrix} 0 & 0 & \sqrt{3} & -\sqrt{3} \\ 2 & 4 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{3} & -2\sqrt{3} & 0 & -\sqrt{3} \\ 1 & 2 & 2 & -1 \end{bmatrix}$$



The image points after a 60° rotation counterclockwise about the origin are $A'(-\sqrt{3}, 1)$, $B'(-2\sqrt{3}, 2)$, $C'(0, 2)$, and $D'(-\sqrt{3}, -1)$.

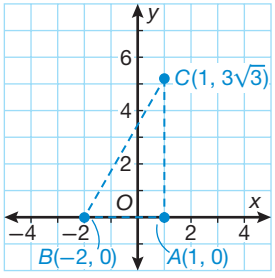
Lesson Practice

a. (Ex 1) Find the exact value of $\tan 105°$.

b. (Ex 1) Find the exact value of $\cos\left(\frac{11\pi}{12}\right)$.

c. (Ex 2) Evaluate $\cos(A + B)$ if $\sin A = \frac{24}{25}$, $-90° < A < 90°$, $\tan B = -1$, and $-90° < B < 90°$.

d. (Ex 3) Find the coordinates of the images of the points in the figure after a 45° rotation counterclockwise about the origin.



Practice Distributed and Integrated

Write the expression in simplest form.

1. (40) $\dfrac{3}{\sqrt[4]{144}}$

2. (40) $\dfrac{\sqrt[3]{9}}{\sqrt[5]{27}}$

3. (82) **Data Analysis** The number of days it takes to complete a year for different planets is shown in the table. Let Earth's yearly motion be modeled by $y = \sin(0.017x)$. How would you model Venus' motion?

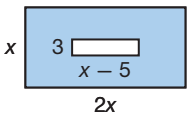
Planet	Days in a Year
Mercury	88.03
Venus	224.63
Earth	365.25
Mars	686.67
Jupiter	4331.87
Saturn	10760.27
Uranus	30681.00
Neptune	60193.20

4. (18) Convert 30 hours into seconds.

***5.** (112) Find the exact value of sin 75°.

***6.** (106) **Graphing Calculator** Graph $y = x^7 - 3x^4 - 2$ and describe the solutions of $x^7 - 3x^4 - 2 = 0$. Round real solutions to the nearest thousandth if needed.

7. (11) **Geometry** Write a polynomial in standard form that represents the shaded area formed by the two rectangles.



8. (95) **Multiple Choice** Which of the following polynomials has $P(3) = 533{,}386$?

A $P(x) = x^{12} + 3x^6 + 5x^2 - 100x + 13$

B $P(x) = x^{14} + 3x^7 + 5x^2 - 100x + 13$

C $P(x) = x^{11} + 3x^5 + 5x^2 - 100x + 13$

D $P(x) = x^9 + 3x^5 + 5x^4 - 100x + 13$

***9.** (109) **Formulate** A hyperbola with vertical orientation has asymptotes with equations $y = \frac{4}{3}x - \frac{11}{3}$ and $y = -\frac{4}{3}x + \frac{29}{3}$. Write the equation of the hyperbola.

10. (94) **Error Analysis** Find the error(s) in the sign chart for $\dfrac{x+4}{(x-5)(x-1)} < 0$. What is the solution of the inequality?

	$x < -4$	$-4 < x < 1$	$1 < x < 5$	$x > 5$
	−5	0	3	8
$x + 4$	−	+	+	+
$x - 5$	−	−	−	+
$x - 1$	+	−	+	+
Value of expression	+	−	−	+

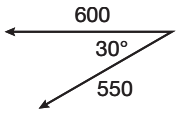
11. (98) **Analyze** Describe a method for converting a polynomial of the form $ax^2 + by^2 + cx + dy + e = 0$ to an ellipse in standard form, assuming the graph is an ellipse. Use this method with the polynomial $x^2 + 9y^2 - 10x - 18y + 7 = 0$.

***12.** (111) **Multi-Step** $f(x) = x^3 - 6x^2 + 4$ has turning points at (0, 4) and (4, −28). Identify the turning points of each function and justify your answers.

a. $\frac{1}{4}f(x)$ **b.** $f\left(\frac{1}{4}x\right)$ **c.** $f\left(x - \frac{1}{4}\right)$

13. (31) Multiply, then evaluate for $a = 2$, $b = 3$. $\frac{6a^2b}{4ab} \cdot \frac{3ab^3}{5a^2b^2}$

14. (99) **Navigation** An airplane travels 225 mph 30° south of due west for two hours. The pilot makes an adjustment so that the plane then flies for another three hours at 268 mph 20° north of due west. What is the distance and direction of the plane from its original position?



***15.** (111) **Error Analysis** To graph $g(x) = (x + 4)^3 + 1$, a student shifted the graph of $f(x) = x^3 + 1$ right 4 units. What is the error?

16. (100) **Generalize** Show that for rational functions $y = \frac{1}{x} + n$, with constant n, the horizontal asymptote is $y = n$.

17. (102) **Unions** From the 1930s on, labor union membership in the U.S. grew dramatically. During the years 1935–1955, labor union membership in the U.S. can be modeled by the equation $y = -4.4 + 12.6 \ln t$, where $5 \le t \le 25$, $t = 5$ represents 1935, and y is the percent of the U.S. labor force that were union members. Use the model to approximate the year in which union membership in the U.S. reached 30%.

***18.** (104) **Savings Accounts** A share certificate pays 5% interest, compounded yearly. The function $A(x) = P \cdot 1.05^x$ can be used to calculate the amount in the account, $A(x)$, after x years with the original amount, placed in the account, P. Use the function $f(x) = 1.05^x$ to sketch a graph of the equation $A(x) = P \cdot 1.05^x$ if the original amount in the account is $50,000.

19. (96) **Analyze** P has polar coordinates (r, θ). Write polar coordinates for P that have a different 1st coordinate.

***20.** (112) **Multiple Choice** Given that $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{4}$, what is the value of $\tan(A + B)$?

A $\frac{2}{9}$ **B** $\frac{2}{7}$ **C** $\frac{2}{3}$ **D** $\frac{6}{7}$

*21. (106) **Farming** A grain silo is shaped like a cylinder with a hemisphere on top. The height of the cylindrical part is 28 feet. The radius of the hemisphere is x feet. The volume of the entire silo is 8550π cubic feet. Find the radius of the silo.

22. (37) Find any values of x for which the following expression is undefined: $\frac{1}{x^3+4x^2-21x}$.

*23. (112) **Write** What is the value of $\cos\frac{2\pi}{5}\cos\frac{3\pi}{5}-\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}$? Explain how to answer without a calculator.

*24. (105) **Organizing** A homeowner organized wooden planks in his backyard into 12 rows where each row has 2 more planks than the row above it. How many planks are there if the bottom row has 32 planks?

25. (88) Given $d=\sqrt{l^2+w^2+h^2}$, find l.

26. (77) In $\triangle LMN$, $LM=24$, $MN=29$, and $m\angle M=65°$. Find LN.

27. (35) Write a cubic function that has zeros 0, 16, and -9.

28. (97) **Salaries** An employee's salary is structured so that he earns \$25,925 in the first year and a 2.75% raise each year thereafter. How much can the employee expect to earn in his 10th year?

29. (32) Solve the system.
$$\begin{aligned} y+z+w&=6\\ -y+3z-w&=2\\ 2y-z+w&=5 \end{aligned}$$

30. (87) Evaluate $\log_{10}(15x)^2$ when $x=1$.

LESSON
113

Using Geometric Series

Warm Up

1. (97) **Vocabulary** In a geometric sequence, the _____ is found by dividing a term by the previous term.

2. (97) Find the next three terms of the geometric sequence $-2, 10, -50, 250, \ldots$

3. (97) Find the 9th term of the geometric sequence $\frac{2}{3}, \frac{1}{6}, \frac{1}{24}, \ldots$

New Concepts

Recall that a series is the indicated sum of the terms of a sequence. In an arithmetic series, the terms are those of an arithmetic sequence. In a **geometric series,** the terms are those of a geometric sequence.

Arithmetic Series	Geometric Series
$4 + 9 + 14 + 19$	$4 + 8 + 16 + 32$
$8 + 5 + 2 + (-1) + \ldots$	$1 + (-3) + 9 + (-27) + \ldots$

A formula for the sum of a finite geometric series can be found by multiplying the partial sum of a geometric series by the common ratio, and then subtracting.

Consider the series $S_n = ar^0 + ar^1 + ar^2 + ar^3 + ar^4 + \ldots + ar^n$ where a is the first term of the series and r is the common ratio.

Multiply the entire series by r, using rules of exponents to simplify, and then subtract the result from the original series. Notice that all but two terms will cancel out.

$$\begin{aligned} S_n &= ar^0 + \cancel{ar^1} + \cancel{ar^2} + \cancel{ar^3} + \cancel{ar^4} + \ldots + \cancel{ar^{n-1}} \\ -\ rS_n &= \cancel{ar^1} + \cancel{ar^2} + \cancel{ar^3} + \cancel{ar^4} + \ldots + \cancel{ar^{n-1}} + ar^n \\ S_n - rS_n &= ar^0 - ar^n \\ (1 - r)S_n &= ar^0 - ar^n \end{aligned}$$

Solving for S_n yields a formula to find the sum of n terms of any finite geometric series.

$$S_n = \frac{ar^0 - ar^n}{1 - r} = a\left(\frac{1 - r^n}{1 - r}\right).$$

Hint

Remember, the nth term of a geometric series, a_n, equals $a_1(r)^{n-1}$, for example $a_7 = a_1r^6$.

Sum of a Finite Geometric Series
The sum of the first n terms of a geometric series, S_n, is $a_1\left(\frac{1 - r^n}{1 - r}\right)$.

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Note that the formula gives a partial sum if the series is infinite. It gives the actual sum if the series is finite with n terms.

Example 1 Finding the Sum of a Geometric Series

a. Find S_9 for the geometric series $1 + 4 + 16 + 64 + \ldots$

SOLUTION The formula for the sum requires a_1, which is 1, n, which is 9, and r which can be found by dividing: $64 \div 16 = 4$.

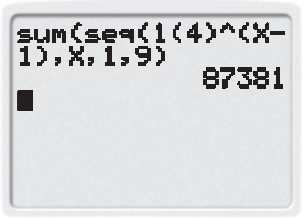
Now use $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$.

$$S_9 = 1\left(\frac{1 - 4^9}{1 - 4}\right) \quad \text{Substitute.}$$

$$S_9 = 1\left(\frac{1 - 262{,}144}{-3}\right) \quad \text{Simplify.}$$

$$S_9 = 1(87{,}381) = 87{,}381$$

Check by using a graphing calculator.



Hint

Notice that the equation is in the form $a_n = a_1(r)^{n-1}$, so $a_1 = 3$

b. Find S_{11} for the sequence $a_k = 3(-2)^{k-1}$.

SOLUTION Substitute 1 for k to find a_1.

$a_1 = 3(-2)^{1-1} = 3$

$n = 11$ and r is -2.

Now use $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$.

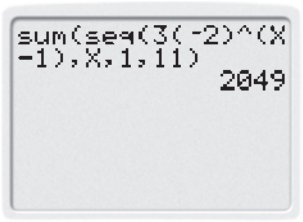
$$S_{11} = 3\left(\frac{1 - (-2)^{11}}{1 - (-2)}\right) \quad \text{Substitute.}$$

$$S_{11} = 3\left(\frac{1 - (-2048)}{3}\right) \quad \text{Simplify.}$$

$$S_{11} = 3\left(\frac{2049}{3}\right) = 2049$$

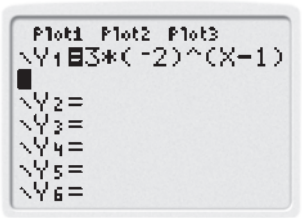
Check by using a graphing calculator.

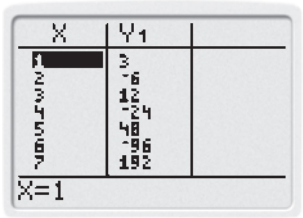
View the individual terms in the TABLE.



Graphing Calculator Tip

Be sure to enclose -2 in parentheses when raising it to the 11th power.





Hint

In the first row of the chart, the cumulative sum of the perimeters is the same as the perimeter.

Exploration Exploring an Infinite Geometric Series

Materials needed: graph paper, graphing calculator

Square	Perimeter	Cumulative Sum of Perimeters
16 by 16		
8 by 8		

Step 1: Copy the chart. Then draw a 16 by 16 unit square on graph paper. Enter the perimeter in the chart.

Step 2: Starting at one corner of this square, draw a new square with sides that are half as long as the first. Find the perimeter of this square. Continue making new squares with sides as half as long as the previous square and finding their perimeters. Complete the chart.

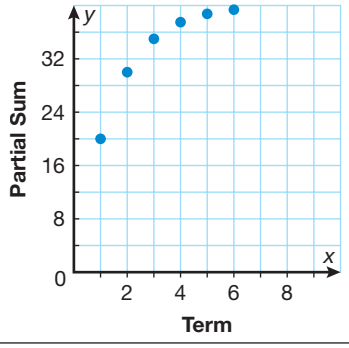
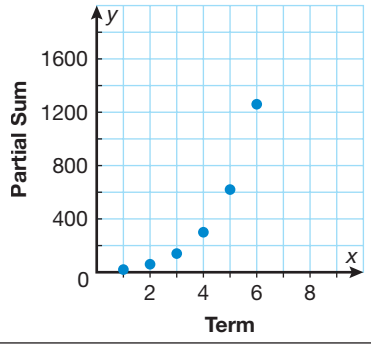
Step 3: Write a geometric series for the first n terms of the series in summation notation. Then use a graphing calculator to find the first 10, 15, and 25 terms of the series.

Step 4: Evaluate $\frac{64}{1-\frac{1}{2}}$. How is this value related to the answers in Step 3?

An **infinite geometric series** has infinitely many terms, and as the exploration above indicates, the partial sums of some infinite geometric series can get closer and closer to a fixed number. The fixed number is called the **limit,** and is considered the sum of the infinite series.

An infinite series only has a sum when the absolute value of the common ratio is between 0 and 1. These series are said to **converge.**

An infinite series does not have a sum when the absolute value of the common ratio is greater than 1. These series are said to **diverge.**

Convergent Series	Divergent Series
$20 + 10 + 5 + 2.5 + 1.25 + 0.625 + \ldots$	$20 + 40 + 80 + 160 + 320 + 640 + \ldots$
$r = \frac{1}{2}$	$r = 2$
	

Example 2 Determining if a Series is Convergent or Divergent

Determine if the geometric series converges or diverges.

a. $\frac{6}{5} + \frac{4}{5} + \frac{8}{15} + \frac{16}{45} + \ldots$

SOLUTION Find r and compare it to 1: $r = \frac{16}{45} \div \frac{8}{15} = \frac{\overset{2}{\cancel{16}}}{\underset{3}{\cancel{45}}} \cdot \frac{\overset{1}{\cancel{15}}}{\underset{1}{\cancel{8}}} = \frac{2}{3}$.

Because $\left|\frac{2}{3}\right| < 1$, the series converges and has a sum.

b. $3 + (-9) + 27 + (-81) + \ldots$

SOLUTION Find r and compare it to 1: $r = \frac{-9}{3} = -3$.

Because $|-3| > 1$, the series diverges and does not have a sum.

Math Reasoning

Analyze Use the formula for the sum of a finite geometric series to verify the sum formula for the infinite geometric with $< |r| < 1$.

As the ratio in Step 4 of the Exploration on the previous page implies, the sum of an infinite geometric series, if there is one, is the ratio of the first term and the difference between 1 and the common ratio.

Sum of an Infinite Geometric Series

The sum of an infinite geometric series, S, is $\frac{a_1}{1-r}$, where r is the common ratio and $0 < |r| < 1$.

Example 3 Finding the Sum of an Infinite Geometric Series

Find the sum of the geometric series $16 + (-4) + 1 + \left(-\frac{1}{4}\right) + \ldots$

SOLUTION Find r: $r = \frac{-4}{16} = -\frac{1}{4}$.

Now use $S = \frac{a_1}{1-r}$.

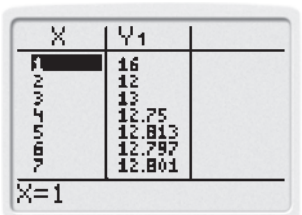
$$S = \frac{16}{1 - \left(-\frac{1}{4}\right)} \qquad \text{Substitute 16 for } a_1 \text{ and } -\frac{1}{4} \text{ for } r.$$

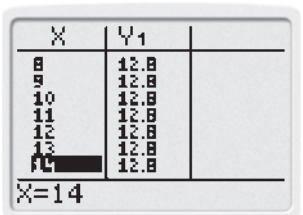
$$S = 12\frac{4}{5} \qquad \text{Simplify.}$$

Graphing Calculator Tip

The calculator rounds; the sums are numbers close to 12.8, but not actually 12.8.

Check by using a graphing calculator. Graph the equation given by substituting into the formula for the sum of a finite geometric series, $y = 16\left(\frac{1-(-0.25)^x}{1-(-0.25)}\right)$. Then look at the table for this graph. As x increases, y approaches 12.8.





Example 4 Application: Salaries

An employee's salary is structured so that he earns $43,400 in the first year and a 3.5% raise each year thereafter. How much can the employee expect to earn in total after 15 years with this company?

SOLUTION

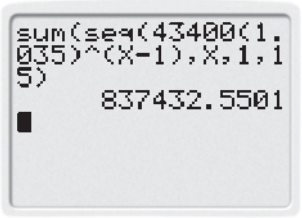
1. Understand The indicated sum of the yearly salaries form a finite geometric series with $r = 1.035$.

2. Plan Use $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$.

3. Solve $S_{15} = 43{,}400\left(\frac{1 - (1.035)^{15}}{1 - 1.035}\right) \approx 837{,}432.55$

The employee can expect to earn $837,432.55 over the course of 15 years.

4. Check Use a graphing calculator.



Math Reasoning

Analyze If the series were infinite, would it converge or diverge? Why?

Lesson Practice

(Ex 1) **a.** Find S_{10} for the geometric series $6 + (-12) + 24 + (-48) + \ldots$

(Ex 1) **b.** Find S_9 for the sequence $a_k = 400\left(\frac{1}{4}\right)^{k-1}$.

Determine if the geometric series converges or diverges.

(Ex 2)

c. $15 + (-25) + \frac{125}{3} + \left(-\frac{625}{9}\right) + \ldots$ **d.** $90 + 30 + 10 + \frac{10}{3} + \ldots$

(Ex 3) **e.** Find the sum of the geometric series $700 + 140 + 28 + \left(\frac{28}{5}\right) + \ldots$

(Ex 4) **f.** An employee's salary is structured so that he earns $27,700 in the first year and a 2.85% raise each year thereafter. How much can the employee expect to earn in total after 20 years with this company?

Practice Distributed and Integrated

(113) ***1. Verify** Show numerically that a convergent geometric series has a sum of 125 if the first term is 25 and the common ratio is $\frac{4}{5}$.

(28) **2.** Identify any excluded values, then simplify the expression $\frac{x^2 + 4x - 12}{x^2 + 6x}$.

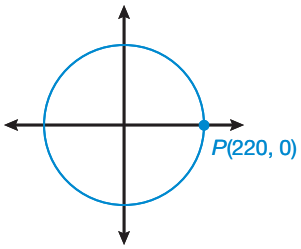
*3. (111) **Baseball Attendance** The season attendance at Baltimore Orioles home games for the years 1996-2005 can be modeled by $f(x) = 7.300x^3 - 116.411x^2 + 363.122x + 3400$, where x is the number of years since 1995 and $f(x)$ is in thousands. Describe the transformation $g(x) = f(x) + 100$ by writing the rule for $g(x)$ and explaining the change in the context of the problem.

4. (48) Simplify: $\dfrac{\dfrac{x}{x+2}}{2x + \dfrac{x}{5}}$.

5. (37) Add: $\dfrac{5}{2x^8} + \dfrac{7}{2x^8}$.

6. (31) Simplify: $\dfrac{x(y^3+6)}{4x+8} \cdot \dfrac{x+2}{x^2y^2} \div \dfrac{y}{x}$.

*7. (112) **Travel** The London Eye is a Ferris wheel with an approximate diameter of 440 feet, making one full rotation in 30 minutes. The diagram represents the London Eye centered on a coordinate grid. From this view, the Ferris wheel rotates counterclockwise. To the nearest tenth, what will be the coordinates of point P in one minute? Assume a constant rate. Show how you arrived at your answer.



8. (88) Given $A = \frac{1}{2}(b_1 + b_2)h$, find b_1.

*9. (112) **Error Analysis** Given that $\sin A = \frac{4}{5}$ and $90° < A < 270°$, a student found $\cos A$ as shown at the right. What is the error? What is the correct value of $\cos A$?

$x^2 + y^2 = r^2$
$x^2 + 4^2 = 5^2$
$x = \sqrt{25 - 16} = 3$
$\cos A = \frac{x}{r} = \frac{3}{5}$

10. (12) **Geometry** Write the equation for the area of a triangle as a joint variation.

11. (83) **Write** Write a quadratic equation whose roots are 4 and 1.

12. (89) **Model** Sketch a system of inequalities, which includes at least one quadratic inequality, such that the system has no solutions.

Find the mean, median, and mode for the following sets of data.

13. (25) 18, 20, 14, 15, 20, 17, 16

14. (25) 21, 23, 22, 25, 28, 31, 28

*15. (30) **Graphing Calculator** Graph the function $y = -2x^2 - 6x + 3$ using your graphing calculator. Determine the vertex and axis of symmetry.

16. (37) Find the LCM. $\dfrac{1}{2x^2 - 3x - 2} - \dfrac{1}{x^2 - 4}$

*17. (102) **Estimate** Estimate the solution to $\log_2 25x = \log_2 151$. Explain your method.

*18. (107) **Multi-Step** Let $f(x) = x^2 - 1$ and $g(x) = x^2 - 1$.

a. Find $\dfrac{x^5}{f(g(x))}$.

b. Find all asymptotes and holes for $\dfrac{x^5}{f(g(x))}$.

19. (Inv 8) Estimate the area under the curve $y = \frac{1}{2}x^2 + 4x + 1$ from $0 \le x \le 4$. Use 4 partitions.

*20. (113) **Multi-Step** A job-seeker has offers from two companies. Job A offers a first-year salary of $31,225 with a 1.85% raise every year thereafter. Job B offers a first-year salary of $28,995 with a 2.25% raise each year thereafter. With which job will he make more money all together after 15 years? What is the difference?

21. (49) Use the Binomial Theorem to expand $(n + 2m)^4$.

*22. (105) **Exercise** An athlete is on a 12-day jogging plan. On the plan, the athlete is to jog 3 miles on the first day, and on each day thereafter, jog 0.75 miles longer than the previous day. How many total miles will the athlete have jogged while on the plan?

23. (21) Solve the system $\begin{aligned} 10x - 2y &= 16 \\ 5x + 3y &= -12 \end{aligned}$ by substitution.

*24. (113) **Salaries** A salary is structured so that the employee earns a 1.5% raise each year after the first year. How much did the employee earn in the first year if his total after 10 years was $393,967.18?

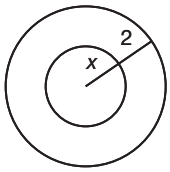
25. (90) Determine the period of $y = 3\tan(2x) + 6$.

26. (93) **Multiple Choice** A substance has a half-life of 16 years. Which equation can be solved to find the approximate number of years t that it will take for 2 grams of the substance to decay to 0.5 gram?

A $2 = e^{-0.5t}$ **B** $2 - 0.5 = e^{-16t}$

C $0.25 = e^{-0.04332t}$ **D** $0.5 = e^{-0.04332t}$

27. (94) **Probability** **a.** Write a simplified rational expression to represent the probability of a randomly selected point in the larger circle also being in the smaller circle.



b. Write and solve a rational inequality to represent the values of x for which the probability is less than or equal to 0.25.

*28. (113) **Multiple Choice** Which series converges?

A $2 + 4 + 8 + 16 + \ldots$ **B** $16 + 8 + 4 + 2 + \ldots$

C $3 + 15 + 75 + 375 + \ldots$ **D** $3 + (-15) + 75 + (-375) + \ldots$

29. (97) **Analyze** When finding the nth term of a geometric sequence, given any two terms, what must be true about the given terms in order for there to be two possible values for the common ratio? Why?

30. (99) **Analyze** Describe geometrically what the matrix equation below shows.

$$\begin{bmatrix} |\vec{A}|\cos(60) \\ |\vec{A}|\sin(60) \end{bmatrix} + \begin{bmatrix} |\vec{B}|\cos(30) \\ |\vec{B}|\sin(30) \end{bmatrix} = \begin{bmatrix} |\vec{A}|\cos(60) + |\vec{B}|\cos(30) \\ |\vec{A}|\sin(60) + |\vec{B}|\sin(30) \end{bmatrix}$$

LESSON 114

Identifying Conic Sections

Warm Up

1. (91) **Vocabulary** A _____ is the set of all points in a plane that are equidistant from a fixed point.

2. (19) Multiply $(x - 8)^2$.

3. (74) Find the discriminant of $x^2 + 5x - 12 = 0$.

New Concepts

Standard Forms for Conic Sections with Center (h, k)		
Circle	$(x-h)^2 + (y-k)^2 = r^2$	
	Horizontal Axis	**Vertical Axis**
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1;\ a > b$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1;\ a > b$
Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Parabola	$x - h = \frac{1}{\pm 4p}(y-k)^2$	$y - k = \frac{1}{\pm 4p}(x-h)^2$

Example 1 Identifying Conic Sections in Standard Form

Identify the conic section for each standard form equation.

a. $\frac{(x-3)^2}{36} + \frac{(y+3)^2}{16} = 1$

SOLUTION $\frac{(x-3)^2}{36} + \frac{(y+3)^2}{16} = 1$ can be written as $\frac{(x-3)^2}{6^2} + \frac{[y-(-3)]^2}{4^2} = 1$.
The equation is an ellipse with a horizontal major axis.

b. $x - 7 = \frac{1}{16}(y-1)^2$

SOLUTION $x - 7 = \frac{1}{16}(y-1)^2$ can be written as $x - 7 = \frac{1}{4(4)}(y-1)^2$.
The equation is a parabola with a horizontal axis of symmetry.

Hint

A parabola has only one squared term in its equation.

The equation for any conic section can be written in the general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. The coefficients A, B, and C can be used to identify conic sections.

Caution

A circle has both $B = 0$ and $A = C$.

Identifying Conic Sections in General Form	
The **general form of a conic section** is given by $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, and C are not all zero.	
Conic Section	**Coefficients A, B, and C**
Circle	$B^2 - 4AC < 0$, $B = 0$, and $A = C$
Ellipse	$B^2 - 4AC < 0$ and either $B \neq 0$ or $A \neq C$
Hyperbola	$B^2 - 4AC > 0$
Parabola	$B^2 - 4AC = 0$

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Example 2 Identifying Conic Sections in General Form

Identify the conic section for each general form equation.

a. $4x^2 + 9y^2 - 16x - 108y + 304 = 0$

SOLUTION

Step 1: Identify the values of A, B, and C.

$$A = 4,\ B = 0, \text{ and } C = 9.$$

Step 2: Find $B^2 - 4AC$.

$$0^2 - 4(4)(9) = -144$$

$B^2 - 4AC < 0$, so the conic section is either a circle or an ellipse.
Since $A \neq C$, the conic section is an ellipse.

Math Language

$B^2 - 4AC > 0$ is called the **discriminant** of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

b. $x^2 - 3x + 2y + 5 = 0$

SOLUTION

Step 1: Identify the values of A, B, and C.

$$A = 1,\ B = 0, \text{ and } C = 0.$$

Step 2: Find $B^2 - 4AC$.

$$0^2 - 4(1)(0) = 0$$

$B^2 - 4AC = 0$, so the conic section is a parabola.

If you know the equation of a conic section in standard form, you can write the equation in general form by expanding the binomials.

Example 3 Finding the General Form for a Conic Section

Write each equation in general form.

a. $y + 2 = \frac{1}{16}(x + 6)^2$

SOLUTION

$y + 2 = \frac{1}{16}(x^2 + 12x + 36)$	Expand the binomial.
$16y + 32 = x^2 + 12x + 36$	Multiply both sides by 16.
$x^2 + 12x - 16y + 4 = 0$	Write in general form.

b. $\frac{(y-4)^2}{25} - \frac{(x+2)^2}{20} = 1$

SOLUTION

$\frac{(y^2 - 8y + 16)}{25} - \frac{(x^2 + 4x + 4)}{20} = 1$	Expand the binomials.
$4(y^2 - 8y + 16) - 5(x^2 + 4x + 4) = 100$	Multiply both sides by 100.
$4y^2 - 32y + 64 - 5x^2 - 20x - 20 = 100$	Simplify.
$-5x^2 + 4y^2 - 20x - 32y - 56 = 0$	Write in general form.

Example 4 Application: Racing

An oval stockcar race track can be modeled by the equation $x^2 + 4y^2 + 2x - 16y - 19 = 0$. Write the equation in standard form.

SOLUTION The track is an oval, so the equation represents an ellipse. Use the method of completing the square to write the equation in standard form.

$x^2 + 2x + __ + 4y^2 - 16y + __ = 19$ Rearrange terms.

$x^2 + 2x + __ + 4(y^2 - 4y + __) = 19$ Factor 4 from the y^2 and y terms.

Complete each square:

$$x^2 + 2x + \left(\frac{2}{2}\right)^2 + 4\left[y^2 - 4y + \left(\frac{-4}{2}\right)^2\right] = 19 + \left(\frac{2}{2}\right)^2 + 4\left(\frac{-4}{2}\right)^2$$

$(x + 1)^2 + 4(y - 2)^2 = 36$ Factor and simplify.

$\frac{(x + 1)^2}{36} + \frac{(y - 2)^2}{9} = 1$ Divide each side by 36.

Check The equation is of the form of an ellipse: $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$.

Caution

The coefficients of the x^2 and y^2 terms must be 1 to use the completing the square method.

Lesson Practice

Identify the conic section for each standard form equation.

(Ex 1)

a. $x^2 + y^2 = 225$

b. $\frac{(x - 4)^2}{25} + \frac{y^2}{49} = 1$

Identify the conic section for each general form equation.

(Ex 2)

c. $5x^2 - 4y^2 - 40x - 16y - 36 = 0$

d. $3x^2 - 24x - y + 50 = 0$

Write each equation in general form.

e. (Ex 3) $x - 2 = \frac{1}{4}(y + 3)^2$

f. (Ex 3) $(x - 5)^2 + (y + 8)^2 = 64$

g. (Ex 4) The path of a circular Ferris wheel can be modeled by the equation $x^2 + y^2 - 30x - 30y - 175 = 0$. Write the equation in standard form.

Practice Distributed and Integrated

1. (32) Solve for matrix X. $\begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix} X = \begin{bmatrix} 3 & -1 & 0 \\ 6 & 8 & 4 \end{bmatrix}$

2. (90) Identify the values for which $y = \frac{1}{2}\tan(2x)$ is undefined.

Simplify each expression.

3. (40) $\sqrt[3]{-125}$

4. (40) $\sqrt[3]{12} \cdot \sqrt[3]{18}$

*5. (111) **Baseball Salaries** The total payroll of a baseball team for the years 1996–2005 can be modeled by $f(x) = 0.048x^3 + 0.605x^2 + 5.649x + 45$, where x is the number of years since 1995 and $f(x)$ is in millions of dollars. Describe the transformation $g(x) = f(x) - 20$ by writing the rule for $g(x)$ and explaining the change in the context of the problem.

6. (26) **Multi-Step** A line passed through points $(-2, 4)$ and $(0, 5)$. Write the equation of the line in slope-intercept form. Then, determine the x and y-intercepts.

7. (104) **Write** Explain what must be done to an equation to horizontally shift the graph to the right.

8. (100) **Physics** A satellite in circular orbit moving at speed v has an acceleration a of $\frac{v^2}{r}$, where r is the radius of the circular orbit. Write and graph the family of functions for calculating the acceleration for $v = 1, 2, 3, 4$, and 5. At what radius does the acceleration range from $a = 0.5$ to 12.5?

*9. (114) **Write** An equation of a conic section written in general form has $B^2 - 4AC < 0$. Explain why you need more information to identify the conic section.

*10. (114) **Carousel** A carousel's path can be modeled by the equation $x^2 + y^2 - 10x - 10y - 575 = 0$. Write the equation in standard form.

Evaluate.

11. (105) $\sum_{k=1}^{20}(0.75k - 5)$

12. (105) $\sum_{k=2}^{7}(8 + 5k)$

*13. (103) **Graphing Calculator** Graph $y = 6\cot(x)$ on a calculator. Determine its period and asymptotes.

14. (102) **Analyze** For what values of x is the equation $\log x^2 = 2 \log x$ true? Explain why the equations $\log x^2 = 2$ and $2 \log x = 2$ are not equivalent.

15. (97) **Coordinate Geometry** The points $(1, -5)$, $(2, -10)$, $(3, -20)$, and $(4, -40)$ represent the first four terms of a geometric sequence. What is the common ratio? What is the y-coordinate when the x-coordinate is 6?

*16. (114) **Multiple Choice** Which is the standard form of the equation $x^2 - 4x - 8y - 84 = 0$?

A $(x - 2)^2 = 8(y + 11)$ **B** $\frac{1}{8}(x - 2)^2 = y + 11$

C $8(x - 2)^2 = y + 11$ **D** $(x - 2)^2 = \frac{1}{8}(y + 11)$

17. (29) Solve the system of equations:
$$\begin{aligned} 4x - 8y + 2z &= 10 \\ -3x + y - 2z &= 6 \\ -2x + 4y - z &= 8 \end{aligned}$$

18. (106) **Business** The profits in thousands of dollars for a business over x years can be approximated by $f(x) = x^4 - 37x^3 + 406x^2 - 1068x - 1512$. The company broke even in the 18th year of business. What other years did they break even?

19. (93) **Multi-Step** Silver-101 has a half-life of 11.1 minutes.

a. Use the natural decay function $N(t) = N_0e^{-kt}$ to find the decay constant k for silver-101.

b. Write the natural decay function for silver-101 and solve for t to find how long it would take 5 grams to decay to 1 gram.

20. (101) **Write** Explain why a polynomial function that has exactly n distinct real zeros must have exactly $n - 1$ turning points.

21. (108) Let $f(x) = \frac{1}{x^2 - 1}$ and $g(x) = \csc(x)$. Find $f(g(x)) + 1$.

***22.** (92) Find a_1 of an arithmetic sequence that $a_9 = 60$ and $a_{13} = 8$.

***23.** (113) **International Landmarks** The first level of the Eiffel Tower is about 57 meters above the ground. Suppose a ball is dropped from this level and rebounds 30% of its previous height after each bounce.

a. Use summation notation to write an infinite geometric series to represent the total distance the ball travels after it initially hits the ground. Keep in mind the ball travels both down and up on each bounce.

b. Find the sum of the series from part a. Round to the nearest tenth.

24. (109) The distance between the two foci of a hyperbola is 26. The equations of the asymptotes are $y = \frac{5}{12}x - \frac{20}{3}$ and $y = -\frac{5}{12}x - \frac{10}{3}$. Write the equation of the hyperbola.

25. (88) **Geometry** The area of a circle is given by $A = \pi r^2$. Solve the formula for r. If the ratio of areas of two circles is 2:1, what is the ratio of the radii?

***26.** (113) **Error Analysis** Explain and correct the error a student made finding the sum of the infinite geometric series $5 + \frac{15}{2} + \frac{45}{4} + \ldots$.

Student's work: $S = \frac{5}{1 - \frac{3}{2}} = \frac{5}{-\frac{1}{2}} = -10$

27. (105) **Multiple Choice** Which series is not an arithmetic series?

A $\sum_{k=1}^{5} 16k$ **B** $\sum_{k=1}^{5} 16k + 2$ **C** $\sum_{k=1}^{5} 16^k$ **D** $\sum_{k=1}^{5} \frac{1}{16}k$

28. (49) 1 in 3 people that enter a post office wears glasses. At noon, 4 people enter. What is the probability that 2 of the 4 people are wearing glasses?

29. (98) **Error Analysis** A student writes the equation $16x^2 + 9y^2 = 144$ in standard form. What is his mistake?

$$16x^2 + 9y^2 = 144 \longrightarrow \frac{16}{144}x^2 + \frac{9}{144}y^2 = 1 \longrightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1 \longrightarrow \frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$$

***30.** (112) **Estimate** Without using a calculator, estimate the coordinates of the image point that results when the point (1, 1) is rotated 47° counterclockwise about the origin. Explain your reasoning.

LESSON 115

Finding Double-Angle and Half-Angle Identities

Warm Up

1. (63) **Vocabulary** A _____ is a unit of measure based on arc length.
2. (63) State the sign of sine, cosine, and tangent for each quadrant.
3. (112) What is the exact value of sin 105°?

New Concepts

The double-angle identities and half-angle identities are special cases of the sum and difference identities.

Double-Angle Identities		
$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta$	$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
	$\cos 2\theta = 2\cos^2\theta - 1$	
	$\cos 2\theta = 1 - 2\sin^2\theta$	

Example 1 Evaluate Expressions with Double-Angle Identities

Find $\sin 2\theta$ and $\cos 2\theta$ if $\sin\theta = \frac{1}{3}$ and $0° < \theta < 90°$.

SOLUTION

Step 1: The identity for $\sin 2\theta$ requires $\cos\theta$. Find $\cos\theta$.

$\cos\theta = \sqrt{1-\sin^2\theta}$ Pythagorean Identity

$= \sqrt{1-\left(\frac{1}{3}\right)^2}$ Substitute $\frac{1}{3}$ for $\sin\theta$.

$= \sqrt{\frac{8}{9}}$ Simplify.

$= \frac{2\sqrt{2}}{3}$

Step 2: Find $\sin 2\theta$.

$\sin 2\theta = 2\sin\theta\cos\theta$ Apply identity for $\sin 2\theta$.

$= 2\left(\frac{1}{3}\right)\left(\frac{2\sqrt{2}}{3}\right)$ Substitute $\frac{2\sqrt{2}}{3}$ for $\cos\theta$ and $\frac{1}{3}$ for $\sin\theta$.

$= \frac{4\sqrt{2}}{9}$ Simplify.

Step 3: Find $\cos 2\theta$.

$\cos 2\theta = 1 - 2\sin^2\theta$ Select an identity for $\cos 2\theta$.

$= 1 - 2\left(\frac{1}{3}\right)^2$ Substitute for $\sin\theta$.

$= \frac{7}{9}$ Simplify.

Hint

Use $\sin^2\theta + \cos^2\theta = 1$ to find the value of $\cos\theta$.

Online Connection
www.SaxonMathResources.com

Double-angle identities can be used to prove trigonometric identities.

Example 2 Prove Identities with Double-Angle Identities

Prove each identity.

a. $\tan\theta(\cos 2\theta + 1) = \sin 2\theta$

SOLUTION

$$\begin{aligned}
\tan\theta(\cos 2\theta + 1) &= \tan\theta((2\cos^2\theta - 1) + 1) && \text{Substitute } \cos 2\theta = 2\cos^2\theta - 1.\\
&= \tan\theta(2\cos^2\theta) && \text{Simplify.}\\
&= \frac{\sin\theta}{\cos\theta}(2\cos^2\theta) && \text{Substitute } \tan\theta = \frac{\sin\theta}{\cos\theta}.\\
&= 2\sin\theta(\cos\theta) && \text{Simplify.}\\
&= \sin 2\theta && \text{Substitute double angle identity } \sin 2\theta = 2\sin\theta(\cos\theta).
\end{aligned}$$

Hint

To prove an identity, modified only one side until it matches the other side.

b. $\cos 2\theta + 2\sin^2\theta = 1$

SOLUTION

$$\begin{aligned}
\cos 2\theta + 2\sin^2\theta &= (2\cos^2\theta - 1) + 2\sin^2\theta && \text{Substitute } \cos 2\theta = 2\cos^2\theta - 1.\\
&= 2(\cos^2\theta + \sin^2\theta) - 1 && \text{Regroup.}\\
&= 2(1) - 1 && \text{Simplify.}\\
&= 1
\end{aligned}$$

Half-angle identities can be used to calculate values that are not usually known.

Half-Angle Identities

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

Choose + or − depending on the location of $\frac{\theta}{2}$.

Example 3 Evaluate Expressions with Half-Angle Identities

Use half-angle identities to find the exact value of each trigonometric expression.

a. $\sin 105°$

SOLUTION Select the identity to use and choose + or − depending on the location of $\frac{\theta}{2}$. $\sin 105°$ is located in QII, so $\sin\frac{210}{2} = \sqrt{\frac{1-\cos 210°}{2}}$.

The cosine of 210° is $-\frac{\sqrt{3}}{2}$. Substitute and solve.

$$\sqrt{\frac{1-\left(\frac{-\sqrt{3}}{2}\right)}{2}} = \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

b. $\cos\frac{5\pi}{8}$

Caution

Remember cosine is negative in the second quadrant.

SOLUTION Select the identity to use and choose + or − depending on the location of $\frac{\theta}{2}$. The cosine of $\frac{5\pi}{8}$ is negative, as it is located in QII, so

$$\cos\frac{5\pi}{8}\ \cos\left(\frac{1}{2}\right)\left(\frac{5\pi}{4}\right) = -\sqrt{\frac{1+\cos\frac{5\pi}{4}}{2}}.$$

The value of $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$.

Substitute and solve. $\cos\frac{5\pi}{8} = -\sqrt{\frac{1+\cos\frac{5\pi}{4}}{2}} = -\sqrt{\frac{1+\frac{-\sqrt{2}}{2}}{2}}$

$$= -\sqrt{\frac{1}{2}+\frac{-\sqrt{2}}{4}} = \frac{-\sqrt{2-\sqrt{2}}}{2}$$

Example 4 Using the Pythagorean Theorem with Half-Angle Identities

Find $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ if $\sin\theta = \frac{3}{5}$ and $90° < \theta < 180°$.

SOLUTION

Step 1: Use the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$, to find $\cos\theta$. Since $90° < \theta < 180°$, $\cos\theta$ is negative.

$$\left(\frac{3}{5}\right)^2 + \cos^2\theta = 1 \qquad \cos^2\theta = \frac{16}{25} \qquad \cos\theta = -\frac{4}{5}$$

Hint

Multiply $90° < \theta < 180°$ by $\frac{1}{2}$. Then choose the correct sign of the identity for $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$.

Step 2: Evaluate $\sin\frac{\theta}{2}$. $\sin\frac{\theta}{2}$ is positive since $45° < \frac{\theta}{2} < 90°$.

$$\sin\frac{\theta}{2} = +\sqrt{\frac{1-\cos\theta}{2}} = \sin\frac{\theta}{2} = \sqrt{\frac{1-\left(-\frac{4}{5}\right)}{2}} \quad \text{Substitute } \cos\theta = -\tfrac{4}{5}.$$

$$= \sqrt{\left(\frac{9}{5}\right)\left(\frac{1}{2}\right)} = \sqrt{\frac{9}{10}} = \frac{3\sqrt{10}}{10} \quad \text{Simplify.}$$

Step 3: Evaluate $\cos\frac{\theta}{2}$. $\cos\frac{\theta}{2}$ is positive since $45° < \theta < 90°$.

$$\cos\frac{\theta}{2} = +\sqrt{\frac{1+\cos\theta}{2}} = \cos\frac{\theta}{2} = \sqrt{\frac{1+\left(-\frac{4}{5}\right)}{2}} \quad \text{Substitute } \cos\theta = -\tfrac{4}{5}.$$

$$= \sqrt{\left(\frac{1}{5}\right)\left(\frac{1}{2}\right)} = \sqrt{\frac{1}{10}} = \frac{\sqrt{10}}{10} \quad \text{Simplify.}$$

Step 4: Evaluate $\tan\frac{\theta}{2}$. Substitute $\cos\theta = -\frac{4}{5}$ to solve.

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\left(-\frac{4}{5}\right)}{1+\left(-\frac{4}{5}\right)}} = \sqrt{9} = 3 \quad \text{Simplify.}$$

Example 5 Application: Sports

The horizontal distance x traveled by an object launched from ground level with an initial speed v and angle θ is given by $x = \frac{1}{32}v^2 \sin 2\theta$. What horizontal distance will a football travel if it is kicked from ground level with an initial speed of 80 feet per second at an angle of 30°? Round your answer to the nearest hundredth.

SOLUTION

$$x = \frac{1}{32}v^2 \sin 2\theta$$

$$= \frac{1}{32}(80)^2 \sin 2(30°) \qquad \text{Substitute given values.}$$

$$= (200)(2(\sin 30)(\cos 30)) \qquad \text{Use } \sin 2\theta = 2\sin\theta\cos\theta.$$

$$= 400\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \qquad \text{Substitute values and simplify.}$$

$$= 100\sqrt{3} \approx 173.21 \text{ feet}$$

Lesson Practice

a. *(Ex 1)* Find $\sin 2\theta$ and $\cos 2\theta$ if $\sin\theta = \frac{1}{5}$ and $0° < \theta < 90°$.

b. *(Ex 2)* Prove the identity $2\cos\theta = 1 - \sin 2\theta \tan\theta$ using double-angle identities.

c. *(Ex 2)* Prove the identity $\cot\theta = \frac{\sin 2\theta}{1 - \cos 2\theta}$ using double-angle identities.

d. *(Ex 3)* Use half-angle identities to find the exact value of $\cos 75°$.

e. *(Ex 3)* Use half-angle identities to find the exact value of $\tan\frac{\pi}{12}$.

f. *(Ex 4)* Find $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ if $\cos\theta = -\frac{7}{9}$ and $180° < \theta < 270°$.

g. *(Ex 5)* The horizontal distance x traveled by a football kicked from ground level with an initial speed v and angle θ is given by $x = \frac{1}{32}v^2 \sin 2\theta$. What horizontal distance will a football travel if it is kicked from ground level with an initial speed of 100 feet per second at an angle of 45°?

Practice Distributed and Integrated

1. *(107)* **Analyze** In the rational function $f(x) = \frac{g(x)}{h(x)}$, if polynomial $g(x)$ is of degree $3m + 1$, what must the degree of $h(x)$ be for $f(x)$ to have a slant asymptote?

Factor completely.

2. *(23)* $x^2 - 9x - 10$

3. *(23)* $4ax + ax^2 - 5a$

4. *(Inv 10)* **Model** Write and graph the polar equation of a rose with 5 petals, each of length 4.

***5.** (115) **Soccer** The horizontal distance x traveled by an object kicked from ground level with an initial speed v and angle θ is given by $x = \frac{1}{32}v^2 \sin 2\theta$. A soccer player would like to determine the maximum distance her ball will travel depending on the angle of the ball leaving the ground. She knows she can kick with an initial velocity of 80 feet per second.

a. Find the distance traveled for an angle of 30°, 45°, 70°, and 90°.

b. Which angle maximizes the distance? Use the value of $\sin 2\theta$ to explain.

***6.** (108) **Coordinate Geometry** Graph $y_1 = 1 + \sec^2(x)\sin^2(x)$ and $y_2 = \tan^2(x)$. Use the information in the graphs to prove the following: $\sec^2(x)\sin^2(x) = \tan^2(x)$.

7. (55) Gisella spins a spinner with equal-sized sections numbered 1–6. In one spin, what is the likelihood that the spinner will stop on an even number?

Write the exponential equation in logarithmic form.

8. (64) $16^4 = 65536$

9. (64) $3^{14} = 4782969$

10. (96) **Geometry** Describe the geometric figure represented by the polar equation $r = 4$.

***11.** (114) **Recreation** A bungee jumper's path can be modeled by the equation $x^2 - 10x - 12y - 95 = 0$.

a. What conic section models the situation?

b. Write the equation in standard form.

Divide using long division.

12. (38) $4x^4 + 5x - 4$ by $x^2 - 3x + 2$

13. (38) $x^3 - 2$ by $x - 5$

14. (97) **Error Analysis** Explain and correct the error a student made in finding the 8^{th} term of the geometric sequence 2, −10, 50, −250, …

$$a_8 = 2(-5)^8$$
$$a_8 = 2(390{,}625) = 781{,}250$$

***15.** (115) **Justify** Use half-angle identities to find the exact value of cos 112.5°. Show your work.

16. (32) Use an inverse matrix to solve the linear system.

$$\begin{aligned} 4x - y &= 10 \\ -7x - 2y &= -25 \end{aligned}$$

17. (106) **Multiple Choice** A polynomial equation has roots of $1 + 2i$, 0, and $3i$. What is the minimum degree of the polynomial?

A 3 **B** 4 **C** 5 **D** 6

***18.** (109) Which of the following are the equations for the asymptotes of $\frac{(x-9)^2}{12^2} - \frac{(y-13)^2}{7^2} = 1$?

A $y = \frac{12}{7}x - \frac{17}{7}; y = -\frac{12}{7}x + \frac{199}{7}$ **B** $y = \frac{7}{12}x + \frac{31}{4}; y = -\frac{7}{12}x + \frac{73}{4}$

C $y = \frac{12}{7}x + \frac{17}{7}; y = -\frac{12}{7}x - \frac{199}{7}$ **D** $y = \frac{7}{12}x - \frac{31}{4}; y = -\frac{7}{12}x - \frac{73}{4}$

*19. (97) **Analyze** Without dividing, how can you tell if the common ratio of a geometric sequence is positive or negative?

20. (87) Convert $\log_8 (32x)^4$ to base e. Then evaluate when $x = 5$.

21. (89) Solve $4x^2 + 8x - 21 \geq 0$ algebraically.

22. (104) **Retirement Funds** A retirement system paid out an interest rate of 2.25% compounded annually from 1963 to 1965. The function $A(x) = P \cdot 1.0225^x$ can be used to calculate the amount in the account, $A(x)$, after x years with the original amount placed in the account, P. Use the function $f(x) = 1.0225^x$ to sketch a graph of the equation $A(x) = P \cdot 1.0225^x$ if the original amount in the account is \$15,000.

23. (93) **Population Growth** Based on census statistics, the population of the United States grew at an average annual rate of approximately 1.24% from 1990 to 2000, and the population in 2000 was 281,000,000 (rounded to the nearest million). If the population continues to increase at the same rate, in what year will it reach 400,000,000?

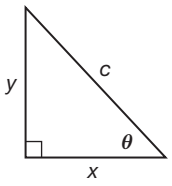
*24. (114) **Multi-Step** **a.** Identify the conic section for the equation $x^2 + (y - 2)^2 = 36$.

b. Write an equivalent equation in general form.

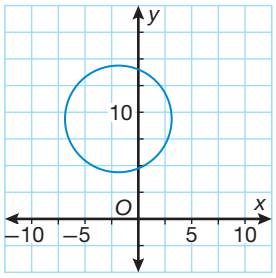
*25. (29) **Graphing Calculator** Use a graphing calculator to solve the system of equations.

$$4x + 5y + 3z = 15$$
$$x - 3y + 2z = -6$$
$$-x + 2y - z = 3$$

26. (108) **Algebra** Using the information in the figure below show algebraically that $1 + \cot^2(\theta) = \csc^2(\theta)$.



27. (91) **Error Analysis** Explain and correct the error a student made in graphing $(x - 2)^2 + (y + 9)^2 = 25$.



28. (33) **Editors** A publishing company needs 4 editors for the next big assignment. How many ways can the editors be chosen from a pool of 13 editors?

*29. (110) **Generalize** The function $y = a\log_b(cx)$ has an x-intercept at $x = \frac{1}{c}$. What is the x-intercept for $y = a\log_b(cx - d)$?

30. (47) Identify the domain, range, and asymptote(s) of $y = \left(\frac{7}{11}\right)^x + 3$.

LAB
14

Determining Regression Models

Graphing Calculator Lab (*Use with Lesson 116*)

Calculate a Regression Model

1. Enter the values from the table into L1 and L2.

Table 1	
L_1	L_2
6	2
5	3
1	3
4	4
9	9
7	5
11	5

Graphing Calculator

Refer to Calculator Lab 5 for clearing and entering data into list.

2. Turn on the diagnostics feature. Press 2nd 0 to open the **CATALOG** menu. Scroll down to **DiagnosticsOn** and press ENTER twice. Turn on plot feature.

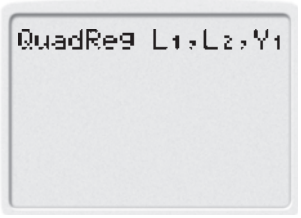
Graphing Calculator

Refer to Calculator Lab 5 for turning on the plot feature.

3. Calculate a regression model and transfer the equation to the "$y =$ screen".

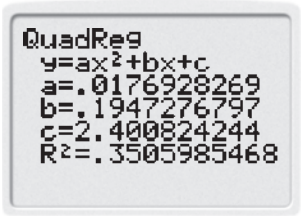
a. Press STAT, and then ▶ over to CALC.

b. Press ▼ to access the desired model: **[5:QuadReg], [6:CubicReg],** or **[7:QuartReg]**. For this example, chose **[5:QuadReg]** feature and press ENTER.



c. When **QuadReg** appears on your main screen, press 2nd 1 , to type in L1 and 2nd 2 , to type in L2. Press VARS ▶ to access the Y-VARS menu. Press ENTER to select **[1: Function].** Press ENTER to select **[1: Y1].**

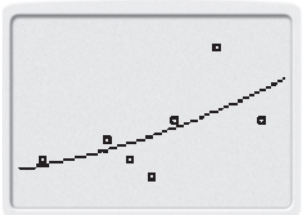
d. Once back on the main screen, press ENTER to calculate the Quadratic Regression. R^2, or r^2, value will be displayed along with the other statistic values.



4. Graph the equation and the scatter plot.

a. Make sure **Plot1** is customized and highlighted in the **Y=** menu so that data points appear in the graph with the regression line.

b. Press ZOOM 9 to select **[9: ZoomStat]** which graphs the regression line and the data points.



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Lab Practice

1. Using the data in L1 and L2 from Table 1, find the equation of the regression line and calculate R^2 value using a cubic regression. Plot the regression line along with the data points.

2. Using the data in L1 and L2 from Table 2, find the equation of the regression line and calculate R^2 value using a natural logarithmic regression.

Table 2	
L_1	L_2
5	1.7
2.7	1.2
32	3.6
17	2.9
12	2.3
25	2.9
47	3.8

LESSON
116

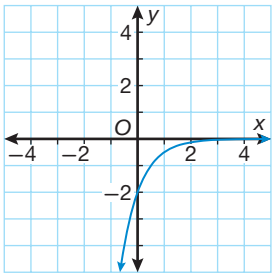
Finding Best Fit Models

Warm Up

1. (45) **Vocabulary** The _____ coefficient, r, tells how well a line fits a set of points and ranges from -1 to 1.

2. (97) Find the constant ratio in the geometric sequence $\left\{\frac{5}{6}, \frac{5}{8}, \frac{15}{32}, \frac{45}{128}\right\}$.

3. (47) Which function is shown in the graph?



A $y = -2\left(\frac{1}{4}\right)^x$

B $y = -2(4)^x$

C $y = -4\left(\frac{1}{2}\right)^x$

D $y = 4\left(\frac{1}{2}\right)^x$

4. (21) Solve the system of equations.

$3a + b = -5$

$2a - 6b = 30$

New Concepts

Recall that regression is the process of identifying a relationship between variables. In a linear regression, a line of best fit was used to describe and predict points in a scatter plot.

Best fit models are not always lines; they can also be curves. When a quadratic function best fits the points, the model is said to be a **quadratic model** and the regression is said to be a **quadratic regression.**

Math Language

Quadratic regression is a statistical method used to fit a quadratic model to a given set of data.

In a quadratic function, when the x-values are equally spaced, the y-values have constant nonzero second differences.

x-values	−2	−1	0	1	2
y-values	8	2	0	2	8

First differences: −6 −2 2 6

Second differences: 4 4 4

Example 1 Fitting Data with a Quadratic Model

Write a quadratic function that models the data shown in the table.

Year	1	2	3	4	5	6
Population	3	10	23	42	66	94

SOLUTION The second differences are almost constant.

First differences: 7 13 19 24 28

Second differences: 6 6 5 4

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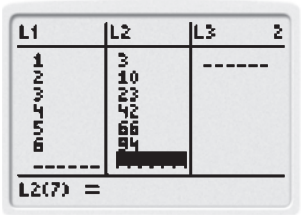
Math Language

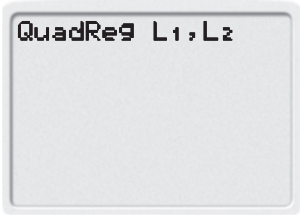
The R^2 value is called the **coefficient of determination.** The closer it is to 1, the better it fits the data.

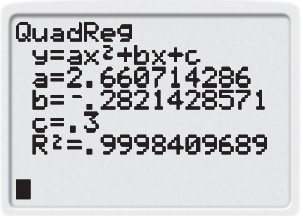
Graphing Calculator

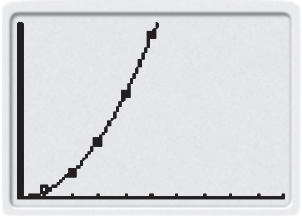
See Graphing Calculator Lab on page 804 for more help with regression models.

Enter the data into a graphing calculator and choose **QuadReg** for a quadratic regression from the STAT CALC menu. A quadratic model of the data is $y \approx 2.66x^2 - 0.28x + 0.3$. Graph the function and the data to see how well the model fits.









Some data are best modeled by polynomial models where the polynomial has a degree greater than two. To determine which polynomial model to use, either graph the given data, or look at the differences in the y-values, when given equally spaced x-values.

Model Differences in y-value	
Linear model	Constant first differences
Quadratic model	Constant second differences
Cubic model	Constant third differences
Quartic model	Constant fourth differences

Example 2 Fitting Data with a Polynomial Model

Write a cubic function that models the data shown in the table.

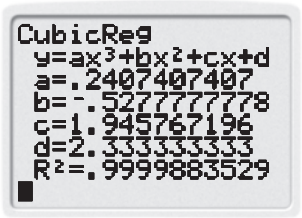
Year	1	2	3	4	5	6
Stock value	$4	$6	$10	$17	$29	$47

SOLUTION Notice that the third differences are almost constant.

First differences: 2 4 7 12 18

Second differences: 2 3 5 6

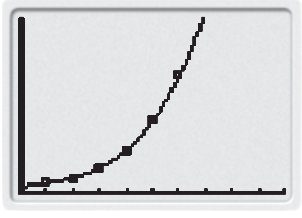
Third differences: 1 2 1



Math Reasoning

Predict Use the model to predict the stock value in Year 7.

Enter the data into a graphing calculator and choose **CubicReg** for a cubic regression from the STAT CALC menu. A cubic model is $y \approx 0.24x^3 - 0.53x^2 + 1.95x + 2.33$. The R^2 value and the graph show that the function models the data well.



Data that does not have constant differences may have constant or near constant ratios. For these, use an **exponential regression** to find an exponential model.

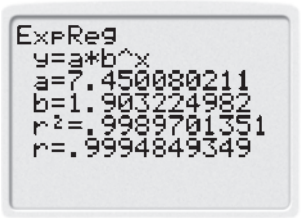
Example 3 Fitting Data with an Exponential Model

Write an exponential function that models the data shown in the table.

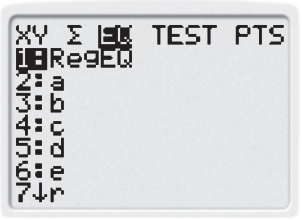
Day	1	2	3	4	5	6
Number of e-mails	14	28	49	98	196	343

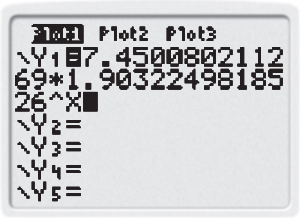
SOLUTION The ratios between consecutive y-values are 2, 1.75, 2, 2, and 1.75. Since they all either 2 or near 2, an exponential regression is appropriate.

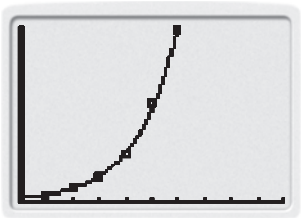
Enter the data into a graphing calculator and choose **ExpReg** to get an exponential function in the form $y = ab^x$. An exponential model of the data is $y \approx 7.45(1.9)^x$.



To enter any regression equation directly into Y1, press VARS, choose Statistics, and choose RegEQ from the EQ menu.







The R^2 value and the graph show that the function models the data well.

Some data are best modeled by a logarithmic function. For this data set use a **logarithmic regression** to find a logarithmic model.

Example 4 Fitting Data with a Logarithmic Model

Write a logarithmic function that models the data shown in the table.

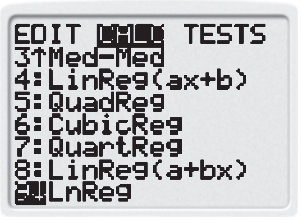
Number of minutes	1	2	3	4	5	6
Miles per hour	24	32	39	43	47	49

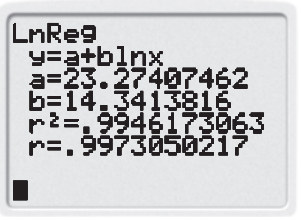
SOLUTION Notice that the y-values continue to increase, but the amounts of increase become less and less as the number of minutes increase.

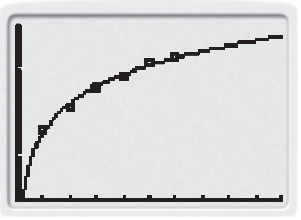
Enter the data into a graphing calculator and choose **LnReg** to get a natural logarithmic function in the form $y = a + b\ln(x)$. A logarithmic model of the data is $y \approx 23.27 + 14.34\ln(x)$.

Caution

Be sure to choose **LnReg** and not **LinReg** from the Calc menu.







The R^2 value and the graph show that the function models the data well.

Example 5 Application: Population

The population for a certain town for given years is shown in the table.

Year	1991	1992	1995	2000	2006	2007
Population	4680	4824	5716	6205	8944	8991

Find the model that best fits the data. Then use the model to estimate the population of the town in 2002.

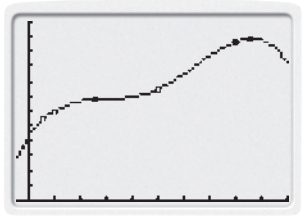
SOLUTION

1. Understand The x-values are not equally spaced. Use a graphing calculator to help decide which types of regressions to try. For simplicity, the x-values can be the number of years after the year 1990.

2. Plan The data points are somewhat close to forming a line, although a nonlinear model may better fit the data. Compare the R^2 values for different types of regression models to find which fit is best.

3. Solve The R^2 values are as follows, linear: ≈ 0.9604, quadratic: ≈ 0.9818, cubic: ≈ 0.9868, quartic: 0.9941, exponential: ≈ 0.9764, logarithmic: ≈ 0.8039.

The quartic model is the best fit. The function is $y \approx -0.474x^4 + 17.73x^3 - 205.718x^2 + 1014.48x + 3717.54$.



Find the value of the function when $x = 12$. This can be done by graphing the equation, choosing **value** from the Calc menu, and typing 12 for X =. This model estimates the population in 2002 to be about 7081.

4. Check Compare the answer with the numbers in the table to check for reasonableness. 2002 is between 2000 and 2006 and 7081 is between 6205 and 8944, so the answer is reasonable.

Math Reasoning

Analyze Would you use the quartic model to predict the population for the year 2012? Explain.

Lesson Practice

a. *(Ex 1)* Write a quadratic function that models the data shown in the table.

Temperature (°C)	−5	−4	−3	−2	−1	0
Number of Campers	8	12	15	19	19	20

b. *(Ex 2)* Write a quartic function that models the data shown in the table.

Week	10	20	30	40	50	60
Value of Stock	$42	$25	$9	$12	$27	$41

c. *(Ex 3)* Write an exponential function that models the data shown in the table.

Minutes	1	2	3	4	5	6
Number of Bacteria	115	168	250	371	547	808

d. (Ex 4) Write a logarithmic function that models the data shown in the table.

Years since 1990	3	6	9	12	15	18
Animal population	15	43	60	72	82	89

e. (Ex 5) The population for a certain town for given years is shown in the table.

Year	1993	1997	1999	2001	2005	2007
Population (millions)	0.85	1.25	1.28	1.31	1.39	1.42

Find the model that best fits the data. Use the year 1990 as the initial time. Then use the model to estimate the population of the town in 2003.

Practice Distributed and Integrated

***1.** (116) **Multi-Step** Write a cubic function to model the data in the table. Then estimate the value of y when x is 3. How does the estimate change when a quartic function is used instead?

x	−10	−5	0	5	10
y	1996	263	−12	−245	−2010

***2.** (113) **International Landmarks** The Santa Justa Elevator (Elevador de Santa Justa) in Lisbon, Portugal is 45 meters tall. Suppose a ball is dropped from the top of the elevator and rebounds 25% of its previous height after each bounce.

a. Use summation notation to write an infinite geometric series to represent the total distance the ball travels after it initially hits the ground. Keep in mind that the ball travels both down and up on each bounce.

b. Find the sum of the series from part a.

c. Find the total distance the ball travels by adding the distance traveled before the ball initially hit the ground.

3. (33) Determine the number of outcomes if you choose an even number or an odd number greater than 25 between the numbers 1 through 50.

4. (107) Find the roots of $y = x^9 - 8x^6 - x^3 + 8$.

***5.** (116) **Population** The table shows the approximate populations of the United States for certain decades.

Year	1930	1940	1960	1980	1990	2000
Population (millions)	1.23	1.32	1.79	2.27	2.49	2.81

a. List the following types of regressions in order from those of best fit to worst fit for the data: linear, quadratic, cubic, quartic, logarithmic, and exponential.

b. Use the first model listed for the answer in part a to estimate the population of the United States in 1950.

Simplify the expressions.

(40)
6. $\sqrt{15} \cdot \sqrt{5}$

7. $\sqrt{175} - \sqrt{63} + \sqrt{20}$

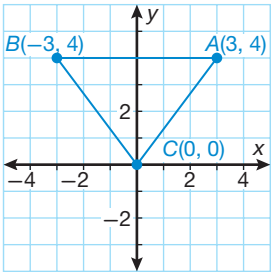
8. (92) The last term of a finite arithmetic sequence is 33 and the common difference is -9. Find the 5th term of the sequence given that there are 21 terms in all.

***9.** (115) **Error Analysis** A student claims that $\cot\theta\cos\theta - \sin\theta = \cos 2\theta \csc\theta$ is NOT an identity based on the work shown below. What is the student's error?

$$\begin{aligned}\cot\theta\cos\theta - \sin\theta &\stackrel{?}{=} \cos 2\theta \csc\theta \\ &\stackrel{?}{=} \cos^2\theta - \sin^2\theta\left(\frac{1}{\sin\theta}\right) \\ &\stackrel{?}{=} \cos^2\theta - \sin\theta\end{aligned}$$

10. (31) Divide and then evaluate for $x = 7$: $\frac{x+9}{3x} \div \frac{3x+27}{x}$.

***11.** (112) **Coordinate Geometry** Find the coordinates of the images of points A, B, and C after a 25° rotation counterclockwise about the origin. Round answers to the nearest hundredth. Show how you arrived at your answers.



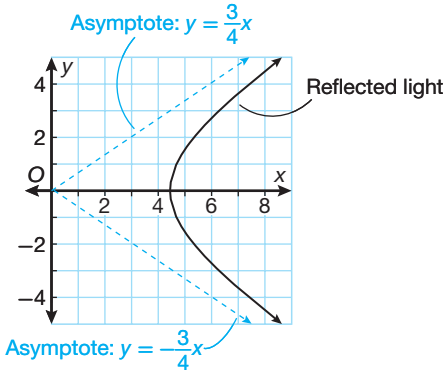
Determine if the given points are solutions of the inequalities.

12. (39) $(3, -3)$; $y \le \frac{2}{3}x - 5$

13. (39) $(-2, -6)$; $y < 9x - 3$

***14.** (111) **Analyze** Choosing from translations, stretches, compressions, and reflections, which transformation(s) can result in a polynomial function which has a different description of end behavior than that of the original polynomial function? Justify your answer.

15. (109) **Optics** A diagram of a hyperbolic mirror is shown. The property of a hyperbolic mirror is that if you shine a beam of light from the mirror, the light is reflected toward the focal point. For this mirror, where should you place a view lens to see the converging light?



***16.** (114) **Write** Tell why the graph of the equation $4x^2 + 4xy + y^2 - 12x + 8y + 36 = 0$ is not a parabola.

Find the roots of the equations.

17. (106) $y = 12x^5 - 49x^4 + 49x^3 - 96x^2 + 392x - 392$

18. (106) $y = x^{14} - x^{12} - x^{11} + x^9 - x^5 + x^3 + x^2 - 1$

19. (100) **Multiple Choice** Which of the following rational functions includes a hole at $x = -5$?

A $y = \dfrac{x^2 - 25}{x^2 - 12x + 35}$ **B** $y = \dfrac{x^2 - 25}{x^2 - 14x + 48}$

C $y = \dfrac{x^2 - 25}{x^2 - x - 30}$ **D** $y = \dfrac{x^2 - 25}{x^2 + x - 30}$

20. (91) **Geometry** Write the equation of a circle if its center is located at $(5, -1)$ and its circumference is 20π units.

21. (101) **Error Analysis** A student described the end behavior of $f(x) = -x^2 + 2x^3 + x^4$ as shown below. What error did the student likely make? Describe the end behavior correctly.

$$\text{As } x \longrightarrow +\infty, f(x) \longrightarrow -\infty, \text{ and as } x \longrightarrow -\infty, f(x) \longrightarrow -\infty.$$

***22.** (116) **Graphing Calculator** Perform a quadratic regression for the data in the table. Give the R^2 value. Use the function to predict the y-value when x is 6.

x	1.5	2.6	3.5	4.5	8
y	−4.3	−3.05	−2.11	−0.95	0

23. (106) **Forestry** The volume of a giant sequoia, in cubic feet, can be approximated by the function $y = 0.485x^3 - 362x^2 + 89{,}889x - 7{,}379{,}874$, where x is the height of the tree, between 220 feet and 280 feet inclusive. What are the possible heights of a tree whose volume is 45,000 cubic feet? Hint: Set the Window of your graphing calculator to the range of heights for which the function is given.

***24.** (115) **Multiple Choice** Which of the following is the exact value of $\text{Sin}\,\frac{\theta}{2}$ if $\text{Cos}\,\theta = -\frac{1}{4}$ and $90° < \theta < 180°$.

A $-\dfrac{\sqrt{10}}{4}$ **B** $\dfrac{\sqrt{10}}{4}$ **C** $-\dfrac{\sqrt{6}}{4}$ **D** $\dfrac{\sqrt{6}}{4}$

25. (94) **Money** The members of a bowling team are sharing the cost of a $125 gift for their team leader. Write and solve an inequality to show the numbers of members whose participation would bring the cost per member to less than $7.50.

26. (104) **Write** Explain in complete sentences what must be done to an equation to vertically compress the graph.

27. (50) Find an equation for the inverse of $y = 4x^2 + 24$.

Factor completely.

(61)

28. $x^3 + 7x^2 + 7x - 15$ **29.** $x^4 + 3x^3 - 25x^2 + 9x - 84$

***30.** (105) **Formulate** Substitute the formula for the nth term of an arithmetic sequence into the formula for the sum of the first n terms of an arithmetic series. In what situation could this new formula be used?

LESSON
117

Solving Systems of Nonlinear Equations

Warm Up

1. (114) **Vocabulary** Of the four conic sections, only a _____ has only one squared term when written in standard form.

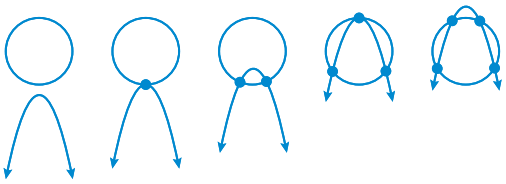
2. (114) True or False: $9x^2 + 25y^2 = 900$ is the equation of a hyperbola.

3. (109) Sketch the graph of $25x^2 - 9y^2 = 225$.

New Concepts

When at least one equation in a system of equations is nonlinear, the system is a **nonlinear system of equations.** As with a system of linear equations, the solution set is the point, or points, where the graphs of the equations intersect. These coordinates are the values that make all the equations in the system true.

When one or both of the equations are equations of conic sections, there can 0, 1, 2, 3, or 4 points of intersection. Possible intersections between a parabola and a circle are shown to the right.



Math Reasoning

Model Sketch another way a circle and a parabola can have no points of intersection.

Methods for solving nonlinear systems are the same as for linear systems: by substitution, by elimination, and by graphing.

Example 1 Solving a Nonlinear System by Substitution

Solve $\begin{aligned} x^2 + y^2 &= 13 \\ x + y - 5 &= 0 \end{aligned}$ by using the substitution method.

SOLUTION The solution is the intersection point(s) of a circle and a line. There could be 0, 1, or 2 solutions.

Step 1: Solve the second equation for either x or y: $x = 5 - y$.

Step 2: Substitute $5 - y$ for x in the first equation and solve for y.

$$\begin{aligned} x^2 + y^2 &= 13 \\ (5 - y)^2 + y^2 &= 13 \\ 25 - 10y + y^2 + y^2 &= 13 \\ 12 - 10y + 2y^2 &= 0 \\ 2y^2 - 10y + 12 &= 0 \\ 2(y^2 - 5y + 6) &= 0 \\ 2(y - 2)(y - 3) &= 0 \\ y = 2 \text{ or } y &= 3 \end{aligned}$$

Step 3: Substitute both 2 and 3 for y and solve for x.

When $y = 2$: $x + y - 5 = 0$

$$\begin{aligned} x + 2 - 5 &= 0 \\ x - 3 &= 0 \\ x &= 3 \end{aligned}$$

When $y = 3$: $x + y - 5 = 0$

$$\begin{aligned} x + 3 - 5 &= 0 \\ x - 2 &= 0 \\ x &= 2 \end{aligned}$$

Step 4: Write the solutions: (2, 3) and (3, 2).

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Example 2 Solving a Nonlinear System by Elimination

Solve $\begin{aligned} x^2 - y^2 &= 7 \\ x^2 + y^2 &= 25 \end{aligned}$ by using the elimination method.

SOLUTION The solution is the intersection point(s) of a hyperbola and a circle. There could be up to 4 solutions.

Step 1: Eliminate either x^2 or y^2. Since the y^2-terms are already opposites, it makes sense to eliminate those.

$$\begin{aligned} x^2 - y^2 &= 7 \\ +x^2 + y^2 &= 25 \\ \hline 2x^2 &= 32 \\ x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$

Align the equations.

Add the equations vertically.

Divide both sides by 2.

Take the square root of each side.

Step 2: Substitute both -4 and 4 for x and solve for y.

When $x = -4$: $x^2 + y^2 = 25$

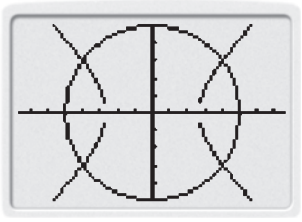
$$\begin{aligned} (-4)^2 + y^2 &= 25 \\ 16 + y^2 &= 25 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

When $x = 4$: $x^2 + y^2 = 25$

$$\begin{aligned} (4)^2 + y^2 &= 25 \\ 16 + y^2 &= 25 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

Step 3: Write the solutions: $(-4, -3)$, $(-4, 3)$, $(4, -3)$, $(4, 3)$.

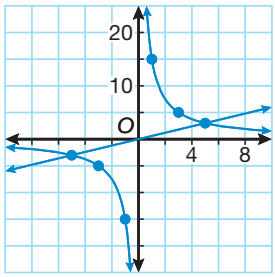
The graphs of the equations support four solutions.



Example 3 Solving a Nonlinear System by Graphing

Solve $\begin{aligned} 5y &= 3x \\ xy &= 15 \end{aligned}$ by using the graphing method.

SOLUTION Graph each equation and find the points of intersection.



Rewrite the first equation as $y = \frac{3}{5}x$.

Plot the point with the y-intercept, $(0, 0)$, and move up 3, right 5 to plot anther point.

Rewrite the second equation as $y = \frac{15}{x}$.

x	-5	-3	-1	1	3	5
$y = \frac{15}{x}$	-3	-5	-15	15	5	3

The graphs intersect at $(-5, -3)$ and $(5, 3)$.

Math Reasoning

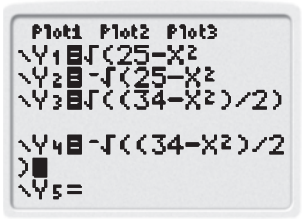
Justify How does $B^2 - 4AC$ show that the graph of the second equation will be a hyperbola?

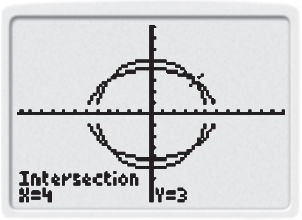
Example 4 Solving a System by Using a Graphing Calculator

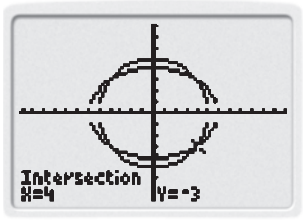
Solve $\begin{aligned} x^2 + y^2 &= 25 \\ x^2 + 2y^2 &= 34 \end{aligned}$ by using a graphing calculator.

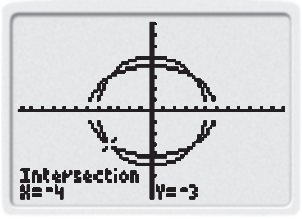
SOLUTION Solve each equation for y.

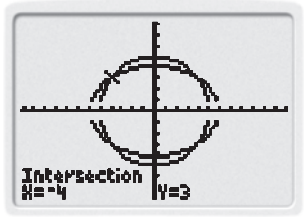
$$x^2 + y^2 = 25 \qquad x^2 + 2y^2 = 34$$
$$y^2 = 25 - x^2 \qquad 2y^2 = 34 - x^2$$
$$y = \pm\sqrt{25 - x^2} \qquad y^2 = \frac{34 - x^2}{2}$$
$$y = \pm\sqrt{\frac{34 - x^2}{2}}$$











The solutions are $(4, 3)$, $(4, -3)$, $(-4, -3)$, and $(-4, 3)$.

Graphing Calculator

Watch the equation at the top of the screen to see which graph the cursor is on. Use the arrow keys to move among the graphs.

Example 5 Application: Event Planning

A planner is designing a rollerblading and bicycling exhibit. She laid a coordinate grid over the available space. She would like the rollerblading arena to be in the area enclosed by $2x^2 + 3y^2 = 50$ and the biking paths along the curves modeled by $y^2 - x^2 = 25$. Will the biking paths intersect the arena? If so, where?

SOLUTION Solve for y.

$$\begin{aligned} 2x^2 + 3y^2 &= 50 \\ +\ -x^2 + y^2 &= 25 \end{aligned} \longrightarrow \begin{aligned} 2x^2 + 3y^2 &= 50 \\ +\ -2x^2 + 2y^2 &= 50 \end{aligned}$$
$$5y^2 = 100$$
$$y^2 = 20$$
$$y = \pm\sqrt{20}$$

Solve for x.

$$y^2 - x^2 = 25 \qquad y^2 - x^2 = 25$$
$$\left(\sqrt{20}\right)^2 - x^2 = 25 \qquad \left(-\sqrt{20}\right)^2 - x^2 = 25$$
$$20 - x^2 = 25 \qquad 20 - x^2 = 25$$
$$-x^2 = 5 \qquad -x^2 = 5$$
$$x^2 = -5 \qquad x^2 = -5$$

For both, the values of x are not real, so the system has no solutions. The bike paths will not intersect the arena.

Lesson Practice

a. (Ex 1) Solve $\begin{cases} x^2 - y^2 = 48 \\ x = 7y \end{cases}$ by using the substitution method.

b. (Ex 2) Solve $\begin{cases} x^2 + y^2 = 16 \\ x^2 - 2y^2 = 1 \end{cases}$ by using the elimination method.

c. (Ex 3) Solve $\begin{cases} y + 2 = 2x \\ xy = 24 \end{cases}$ by using the graphing method.

d. (Ex 4) Solve $\begin{cases} y^2 - x^2 = 36 \\ 2x + y = -1.5 \end{cases}$ by using a graphing calculator.

e. (Ex 5) Two event planners are responsible for the design of a boating exhibit. They have each laid a coordinate grid over the lake that will be used. One planner would like boaters to be able to make laps around the curve modeled by $9x^2 + 16y^2 = 144$ and the other would like to hold a speedboat race along the line modeled by $3x + 4y = 12$. As is, will the path of the speedboaters intersect the path of the boaters making laps? If so, where?

Practice Distributed and Integrated

***1.** (109) **Generalize** For a hyperbola of the form $ax^2 + by^2 + cx + dy + e = 0$, what must be true about a and b if the orientation of the hyperbola is vertical.

2. (88) Given $A = \frac{1}{2}d_1d_2$ find d_1.

3. (47) **Savings** $2,000 is deposited into a savings account. How much money will be in the account in 30 years if there is an 8% interest rate compounded quarterly?

***4.** (111) **Multiple Choice** The graph of $g(x)$ is the graph of $f(x) = 3x^4 - 4$ shifted 2 units left. Which is the correct rule for $g(x)$?

A $g(x) = 3(x + 2)^4 - 4$ **B** $g(x) = 3(x - 2)^4 - 4$

C $g(x) = 3x^4 - 2$ **D** $g(x) = 3x^4 - 6$

5. (106) **Probability** A function is randomly selected from the list below and graphed.

$P(x) = x^3 + 2x^2 - 13x + 10$, $Q(x) = x^3 - 2x^2 + x$, $R(x) = 3x^3 + 48x$,
$S(x) = 2x^3 - 32x$, $T(x) = x^3 - 7x^2 + 4x - 28$

What is the probability that the graph has exactly one x-intercept? What are the odds in favor?

Simplify the logarithms.

6. (72) $\log_7 5^3$

7. (72) $\log_5 42 - \log_5 7$

***8.** (116) **Justify** Tell why it is incorrect to call the R^2 value the *correlation coefficient.*

9. (95) Determine if $P(x) = 3x^3 - 4x^2 - 28x - 16$ has a zero remainder when divided by $x + 2$. Determine $Q(x)$.

*10. (22) **Graphing Calculator** Graph $y = -x^2 + 3$ on your calculator. Classify the function as discrete or continuous and identify the range.

11. (92) Find a_6 of an arithmetic sequence given that $a_3 = 55$ and $a_{20} = 140$.

*12. (117) **Recreation** A boy's remote-controlled airplane is making loops in a park and the path can be modeled by $4x^2 + y^2 = 25$. A girl's remote-controlled plane is making a straight line that can be modeled by $2x + y = 7$. Is there any chance that the two planes will collide? If so, where?

13. (94) **Analyze** Explain why you must consider two cases when solving a rational inequality by using the LCD.

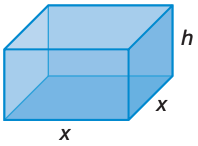
14. (63) **Physics** As an airplane comes in for a landing, the pilot wants to maintain a constant angle relative to the ground. If the pilot wants to maintain an angle between 30 and 40 degrees, define the bounds on x for the tangent function $y = \tan x$ in radians that monitor the plane's orientation.

15. (105) **Coordinate Geometry** For the sequence $\{2, 4, 6, 8, 10\}$, graph the ordered pairs (n, a_n) and (n, S_n). How do the graphs compare?

16. (98) **Multiple Choice** Which of the following points is not on the ellipse $\frac{x^2}{2^2} + \frac{y^2}{5^2} = 1$?

A $(0, 5)$ **B** $(2, 0)$ **C** $(5, 0)$ **D** $(-2, 0)$

17. (101) **Multi-Step** A manufacturer wants to design an open box with a square base and a total surface area of 200 square inches.



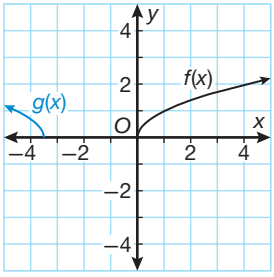
a. Write an equation that relates x, h, and 200. Solve it for h.

b. Write a function $V(x)$ for the volume of the box.

c. Graph $V(x)$ on a graphing calculator. Find the maximum possible volume of the box and the corresponding dimensions.

*18. (117) **Analyze** What is the maximum number of intersection points that a circle and a line can have?

19. (104) **Formulate** In the coordinate plane, functions f and g are graphed. If g is a translation of $f = \sqrt{x}$, find an equation for g.



*20. (117) **Multi-Step** Consider the system $\begin{aligned} x^2 - y^2 &= 15 \\ 9x^2 + 16y^2 &= 160 \end{aligned}$.

a. What figures do the graphs of each equation make?

b. At most, how many solutions could there be?

c. Solve the system by elimination.

d. **Verify** If you were to verify your solutions by using a graphing calculator, what would be the equations you would enter?

21. *(36)* Find the equation of the line that passes through the point (1, 7) and is perpendicular to the line $y = -\frac{1}{2}x + 4$.

22. *(Inv 10)* Describe the graph of a conic section whose eccentricity is 2 and vertical distance from its focus or center is 2 and write the general equation in polar form.

23. *(45)* Use the data in the table to write the equation for the line of best fit.

Level	2	3	4	5	6
Cars Sold	8	10	13	18	25

***24.** *(115)* **Entertainment** The horizontal distance x traveled by an object launched from ground level with an initial speed v and angle θ is given by $x = \frac{1}{32}v^2 \sin 2\theta$. A firework with a shell size of 36 inches has an initial velocity of 481 feet per second. What horizontal distance will a firework travel if it is launched from ground level at an angle of 85°?

Solve.

25. *(102)* $\log_5 625$

26. *(102)* $\log_5 (5x + 9) = \log_5 6x$

27. *(110)* **Error Analysis** Two students are simplifying the logarithmic function below, but they get different results. Which student made the mistake?

$$y = \log(9x^2 + 30x + 25)$$

Student A	Student B
$y = \log(9x^2 + 30x + 25)$ $= \log((3x + 5)(3x + 6))$ $= \log(3x + 5) + \log(3x + 6)$	$y = \log(9x^2 + 30x + 25)$ $= \log(9x^2 + 2 \cdot 3 \cdot 5x + 5^2)$ $= \log(3x + 5)^2$ $= 2\log(3x + 5)$

28. *(101)* Describe the end behavior of the graph of the polynomial function $f(x) = -2x^3 + 7x - 4$.

***29.** *(116)* **Finance** The table shows stock values of a certain stock for 5 different days.

Day	3	6	8	14	16
Value of Stock	$38	$21	$15	$34	$52

Use a sketch of the scatter plot to determine if a quartic or logarithmic model would better fit the data. Then write the function of the chosen regression model.

***30.** *(105)* Find S_{18} for 2, 9, 16, 23, 30, …

LESSON
118

Recognizing Misleading Data

Warm Up

1. (73) **Vocabulary** In _____ sampling, every individual has a known probability, greater than 0, of being selected.

2. (73) True or False: In simple random sampling, every person and every possible group has an equal chance of being chosen.

3. (73) A reporter wants to know what teens think of a proposed curfew and asks every 10th teen leaving a convenience store if they believe the curfew is best for all the citizens of the county. Identify the sample and the population.

New Concepts

Statistical claims are not always accurate. These inaccuracies can come from poor sampling methods, poorly worded questions, distorted data displays, and the misuse of data.

Recall that a biased sample is one that is not representative of its population. Someone making a claim about a population based on a biased sample may or may not know that their sample is biased. Biased samples can be the result of using a sample that is not a probability sample, such as a volunteer or convenience sample.

Hint

In a **probability sample,** every member of the population has a known, nonzero, chance of being selected.

Inaccurate data may also be the result of using a sample that is too small. A simple random sample that chooses only 5 people from a population of 10,000 is likely to be biased.

Example 1 Identifying Misleading Data through Sampling Methods

A reporter for a school newspaper asks students to go to the school website and report which school sport is their favorite so that she can write a story featuring the history and popularity of that particular sport. Why are the results likely to be misleading?

SOLUTION Although every student may have the opportunity to use a school computer, it is a voluntary response sample and therefore likely to be biased, with the reported sport being more popular than what it actually is. In addition, students may be more likely to choose a sport that is in-season. Therefore, baseball may be most popular if the poll is conducted in the spring, or football if the poll is conducted in the fall. Also, students may pick the sport with the most successful team, regardless of whether that sport is their favorite.

A representative sample of a large enough size does not guarantee that the data collected from that sample will be accurate. The question can be worded in a way that leads people towards a certain response.

Measurement data, such as for height or weight, is most accurate when the surveyor takes the measurement. This is because people tend to round numerical answers to the nearest 5 or 10. People may also lie about personal characteristics; this factor cannot be controlled except by measuring on the spot.

Online Connection
www.SaxonMathResources.com

Math Language

A **closed question** has a limited number of answers such as: yes/no, true/false, or a multiple-choice option.

Survey questions which are multiple-choice are biased when they do not give every possible answer as a choice, leaving people to either skip the question or choose an answer that does not reflect their true feelings.

Example 2 Identifying Misleading Data through Questioning

The manager of an apartment complex posts the following survey in every tenant's mailbox.

Next month, maintenance workers must enter homes for an annual inspection. You must be home during the inspection. We value your time and would like your input on making the inspection schedule. Which is most convenient for you?

a. *Weekdays, before noon*

b. *Weekdays, between noon and 5 pm*

c. *Weekdays, between 5 pm and 9 pm*

How can the results of this survey be misleading?

SOLUTION A tenant who works during the day and goes to school at night cannot easily be home during the week, so their most convenient time is on the weekend, but this is not given as an option. Therefore, they may not respond to the survey, or they may unhappily answer one of the given options. This could give management the impression that their tenants are okay with being home during the week, which may not be the case. Leaving the question open for tenants to write in a time frame would give the most accurate results.

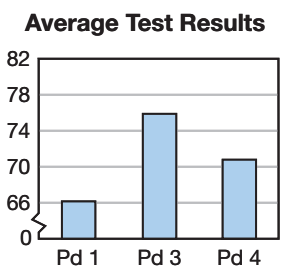
Accurate data can be made to appear misleading by the way it is displayed. For example, a bar graph that does not begin at zero can make small differences between categories appear greater than they are.

When art is used instead of bars in a bar graph, the size of the art may make the data appear misleading because the reader will be drawn to the area of the figure, and not necessarily the actual numbers to which the pictures extend. For example, suppose one category has a value twice as much as another. Only the height of the bar or figure used should be doubled, but if the width is doubled as well, it will give the misleading impression that the value is four times as large as the other.

Example 3 Identifying Misleading Data through Data Displays

Explain how the graph could be misleading.

SOLUTION Because the vertical axis does not begin at 0, the difference in the heights of the bars is misleading. Without looking at the numbers, it would appear that the average score for Period 4 is twice that of Period 1 and that the average score for Period 3 is greater than three times that of Period 1. In actuality, the difference between the greatest and lowest score is only 10 points.



An outlier is a value much lower or higher than the numbers in the rest of the data set. A value less than $Q_1 - 1.5(\text{IQR})$ or greater than $Q_3 + 1.5(\text{IQR})$ is considered to be an outlier. Recall that an outlier can greatly affect the mean of a data set. When a data set includes an outlier, reporting the median will be less misleading than reporting the mean. An alternative is to report the mean without the outlier, but this should be noted, especially if the sample size is small. Likewise, reporting the range of a data set that includes an outlier could be misleading as well.

A common misuse of statistics is to assume that correlation is causation. A correlation between two sets of data means that there is a relationship between the data sets. For example, student athletes may have a higher grade point average than the rest of the student body. This does not mean that one can report that playing sports causes students to do better in school. There could be any number of reasons, or combinations of reasons, for the correlation.

Example 4 Identifying Misleading Uses of Data

The prices of the houses sold in one week in a certain county are \$162,000, \$185,000, \$159,000, \$205,000, and \$2,000,000. A statement in the newspaper reads, *"Average home prices last week in the county were over half a million dollars."* Why is the statement misleading? What would be a better statement?

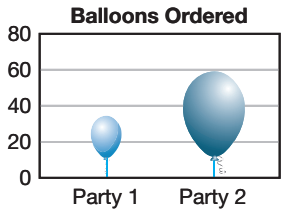
SOLUTION The statement is misleading because only one of the homes sold for an amount greater than a half million dollars. It would be less misleading to report the statistic as the median. Another option is to report that the average sale price of 4 out of the 5 homes sold last week was \$177,750.

Lesson Practice

(Ex 1) **a.** A sales person for a new lotion gave samples of the lotion to four dermatologists and asked them if they would recommend it to their patients. Three of the dermatologists said they would. The sales person wrote a brochure stating that 3 out of 4 dermatologists recommend the lotion. Why is the claim misleading?

(Ex 2) **b.** A student wants to know the average height difference between boys and girls in the eleventh grade. In 11th-grade homerooms, the student walks up and down the rows, asking other students to state their height, in inches. How can the results of this survey be misleading?

(Ex 3) **c.** Explain how the graph could be misleading.



(Ex 4) **d.** A study showed that individuals who visit a dentist twice a year tend to make more money than individuals who visit less than twice a year. Can it be concluded that visiting a dentist causes a person to make more money? Explain.

Practice Distributed and Integrated

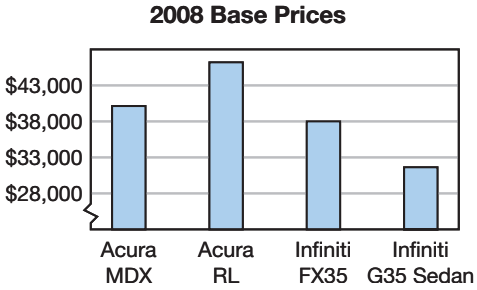
1. (105) **Savings** A teen deposits money into a savings account. The first week she deposits $20. Each week thereafter, she deposits $10 more than the previous week. Show how to use an arithmetic series to find the total amount of money in the account after 16 weeks.

Find the zeros of the functions.

2. (99) $y = 36x^4 - 27x^3 - 13x^2 + 3x + 1$

3. (99) $y = 80x^4 - 64x^3 - 21x^2 + 4x + 1$

***4.** (118) **Automobiles** Explain why the graph is misleading. Who might have made the graph and why?



5. (115) **Multi-Step** If $\tan\theta = \frac{5}{12}$ and $0° < \theta < 90°$.

a. What is $\cos\theta$?

b. What is the exact value of $\sin\frac{\theta}{2}$?

6. (89) **Geometry** The perimeter of a rectangle is 82 feet. Write and solve an inequality to find the width, rounded to the nearest tenth, for which the rectangle has an area less than 250 square feet.

7. (39) Determine if $(-6, 10)$ is a solution of the inequality $y > \frac{4}{3}x + 11$.

8. (107) **Analyze** The graph of the rational function $f(x) = \frac{g(x)}{h(x)}$ has holes at $x = \pm c$, for constant c. Define a new polynomial using $j(x)$ that is equivalent to $h(x)$ and takes the values of c into account.

***9.** (116) **Probability** Suppose for the data set $\{(1, 4), (4, 10), (6, 11), (10, 12), (12, 13)\}$ that one of the following regression models is randomly chosen to predict a value: linear, quadratic, cubic, quartic, exponential, and logarithmic. What is the probability that the R^2 value for the chosen model is less than 0.95?

10. (106) **Analyze** Can a sixth-degree polynomial function have exactly six irrational roots? Why or why not?

11. (115) **Write** Find $\sin 2\theta$ and $\cos 2\theta$ if $\cos\theta = -\frac{1}{4}$ and $90° < \theta < 180°$. Did you need to find the value of $\sin\theta$ to find these values? Explain.

***12.** (116) **Postage** The cost of a US first class stamp for given years are shown in the table.

Year	1895	1914	1947	1966	1973	1980	1982	1991	2001	2006	2007
Cost, in cents	2	2	3	5	8	15	20	29	34	39	41

a. Find an exponential function to model the data. Use the number of years before or after 1900 for x. Use the model to predict the cost of a stamp in 2025.

b. Predict the cost of a stamp in 2025 when a quartic model is used instead.

c. **Justify** Which price do you think is more reliable? Why?

13. *(114)* **Error Analysis** Find and correct the error a student made in stating that the graph of the equation $\frac{(x-4)^2}{25} + \frac{(y-1)^2}{8} = 1$ is a hyperbola.

***14.** *(118)* **Multiple Choice** A data set has an outlier. Which statement is false?

A Reporting the mean is likely to be misleading.

B Reporting the median is likely to be misleading.

C Reporting the range is likely to be misleading.

D Reporting either the mean or range is likely to be misleading.

15. *(113)* **Write** What is a limit and when does a geometric series have a limit?

16. *(101)* Identify the leading coefficient, degree, and describe the end behavior of $f(x) = x^4 - x^3 - 4x^2 + 4$.

***17.** *(117)* **Multiple Choice** Which could not have three points of intersection?

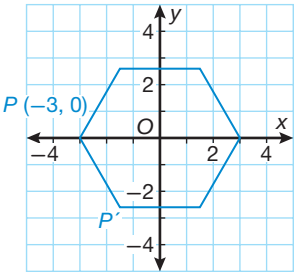
A circle and line

B hyperbola and circle

C circle and ellipse

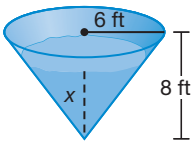
D two parabolas

18. *(112)* **Furniture** The diagram represents a table top in the shape of a regular hexagon centered on a coordinate grid. To the nearest hundredth, what are the coordinates of point P'? Show how you arrived at your answer. (Hint: Use a rotation matrix.)



***19.** *(118)* **Justify** A pollster suggests the following question to be on a ballot. 'Do you support a new traffic light on First Street and Walnut Street which engineers claim will reduce accidents by up to 65%?' Another pollster says that this question is biased and must be changed. Explain why this pollster is correct in saying the question is biased.

***20.** *(111)* **Multi-Step** This conical container has a radius of 6 feet and a height of 8 feet.



a. Write a function $V(x)$ that gives the volume of water in the container when the height of the water is x feet.

b. Describe the transformation $g(x) = V(2x)$ by writing the rule for $g(x)$ and explaining the change in the context of the problem.

Divide using synthetic division.

21. *(51)* $x^4 - 11x^3 + 14x^2 + 80x$ by $x + 2$

22. *(51)* $x^3 - 57x + 56$ by $x - 1$

23. *(106)* Solve the equation by finding all the roots: $x^4 - 2x^3 - 14x^2 - 2x - 15 = 0$.

24. *(90)* Let $f(x) = \tan(x)$ and $g(x) = 3x - 2$. Find the period of $f(g(x))$.

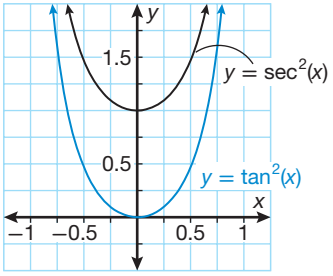
***25.** *(30)* **Graphing Calculator** Graph the function $y = -4x^2 + 8x + 2$. Determine the vertex and axis of symmetry.

*26. (117) **Error Analysis** Explain and correct the error(s) a student made in solving the system $\begin{cases} x = y + 2 \\ x^2 - y^2 = 16 \end{cases}$.

$$\begin{aligned} (y+2)^2 - y^2 &= 16 & x &= y + 2 \\ y^2 + 4y + 4 - y^2 &= 16 & x &= 3 + 2 \\ 4y + 4 &= 16 & x &= 5 \\ y &= 3 & &\text{The solution is } (3, 5). \end{aligned}$$

*27. (110) Use the Rational Root Theorem to find the roots of $y = 2x^7 - x^6 - 15x^5 + 18x^4 - 2x^3 + x^2 + 15x - 18$.

28. (108) **Coordinate Geometry** The graphs of $y_1 = \tan^2(x)$ and $y_2 = \sec^2(x)$ are shown at right. The graph of y_1 has a y-intercept of 0, and the graph of y_2 has a y-intercept of 1. Use the graphs to prove the trigonometric identity, $1 + \tan^2(x) = \sec^2(x)$.



29. (33) A card is drawn at random from a standard deck. How many outcomes are in the event of drawing a diamond or a 4?

30. (42) Evaluate 4! • 6!

LESSON 119

Solving Trigonometric Equations

Warm Up

1. (28) **Vocabulary** An _____ is an equation that is true for all values of the variable.
2. (23) What is the only positive solution of $x^2 - 9 = 8x$?
3. (108) Complete the trigonometric identity $\sin^2 x + \cos^2 x =$ _____.
4. (108) What is the reciprocal trigonometric identity for $\sin \theta$?

New Concepts

When solving trigonometric equations, use the same methods for solving algebraic equations and apply the inverse trigonometric functions.

Example 1 Solving Trigonometric Equations with Infinitely Many Solutions Algebraically

Find all the solutions of $4\cos\theta - 1 = 3\cos\theta$.

SOLUTION

Solve for θ such that $0° \le \theta \le 360°$ over the principal values of the cosine.

$4\cos\theta - 1 = 3\cos\theta$	Subtract $3\cos\theta$ from both sides.
$\cos\theta - 1 = 0$	Add 1 to both sides.
$\cos\theta = 1$	Apply the inverse cosine.
$\theta = \cos^{-1}(1), 360°$	$\theta = 0°$ when $0° \le \theta \le 360°$
$\theta = 0° + 360°n$ when n is any integer	

Graphing Calculator

Check your answers by graphing $y = 4\cos\theta - 1$ and $y = 3\cos\theta$ on the same screen over the interval $0° \le x \le 360°$, then find any points of intersection.

Example 2 Solving Trigonometric Equations in Quadratic Form

Find the exact solutions for each equation.

a. $2\sin^2 x - 2\sin x = 1 - 3\sin x$ for $0° \le x \le 360°$

SOLUTION

$2\sin^2 x - 2\sin x = 1 - 3\sin x$	Subtract 1 from both sides. Add $3\sin x$ to both sides.
$2\sin^2 x + \sin x - 1 = 0$	Factor by comparing to $2z^2 + z - 1 = 0$ where $z = \sin x$.
$(2\sin x - 1)(\sin x + 1) = 0$	Apply the Zero Product Property. Apply the inverse sine to each equation.

$2\sin x - 1 = 0$ or $\sin x + 1 = 0$

$\sin x = \frac{1}{2}$ $\quad$ $\sin x = -1$

$x = 30°, 150°$ $\quad$ $x = 270°$

$2\sin^2 x - 2\sin x = 1 - 3\sin x$ when $x = 30°, 150°$ and $270°$

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b. $\tan^2 z - 3\tan z = 5$ for $-\frac{\pi}{2} < z < \frac{\pi}{2}$

SOLUTION Use the Quadratic Formula.

$\tan^2 z - 3\tan z = 5$ Subtract 5 from both sides.

$\tan^2 z - 3\tan z - 5 = 0$ Substitute 1 for *a*, −3 for *b*, and −5 for *c*, into the Quadratic Formula and Simplify.

$$\tan z = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)}$$

$$\tan z = \frac{3 \pm \sqrt{29}}{2}$$ Apply the inverse tangent to each equation. Use a calculator.

$$z = \text{Tan}^{-1}\left(\frac{3 + \sqrt{29}}{2}\right)$$

$$\approx 1.34 \text{ radians or } z = \text{Tan}^{-1}\left(\frac{3 - \sqrt{29}}{2}\right) \approx -0.87 \text{ radians}$$

$\tan^2 z - 3\tan z = 5$ when $z = 1.34$ and -0.87 radians

Hint

Note that the range of the sine and cosine functions is −1 to 1, but the range of the tangent function is all real numbers. So, the domain of the inverse tangent function includes numbers less than −1 and greater than 1.

When solving equations involving more than one function, trigonometric identities often can be used to write the equation with only one trigonometric function.

Example 3 Solving Trigonometric Equations with Trigonometric Identities

Use trigonometric identities to find the exact solutions.

$2\sin^2\theta + 2 = 3 - \cos\theta$ for $0 \le \theta < 2\pi$.

SOLUTION

$$2\sin^2\theta + 2 = 3 - \cos\theta$$

$2(1 - \cos^2\theta) + 2 = 3 - \cos\theta$ Use the trig identity $\sin^2\theta + \cos^2\theta = 1$ to replace $\sin^2\theta$

$-2\cos^2\theta + \cos\theta + 1 = 0$ Distribute and collect terms on the left side.

$2\cos^2\theta - \cos\theta - 1 = 0$ Divide both sides by −1.

$(2\cos\theta + 1)(\cos\theta - 1) = 0.$ Factor terms.

$2\cos\theta + 1 = 0$ or $\cos\theta - 1 = 0$ Apply Zero Product Property.

$\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$ Solve for θ, when $0 \le \theta < 2\pi$

$\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or $\theta = 0$

Hint

Your goal is to have only one trigonometric function in the resulting equation.

Graphing Calculator Tip

When entering equations, Theta (θ) will display on most calculators only if the calculator is in polar mode. Do NOT use polar mode. Instead, simply note that the variable is x.

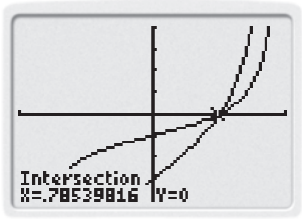
Example 4 Solving Trigonometric Equations Using a Graphing Calculator

Find all the solutions of $3\tan\theta - 3 = \tan\theta - 1$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ using a graphing calculator.

SOLUTION

Graph $y = 3\tan\theta - 3$ and $y = \tan\theta - 1$ in the same viewing window for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Find the intersection of the two graphs using the calc menu on your calculator.



Between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ the graphs only intersect at $\theta \approx 0.7854$. Therefore the only solution to $3\tan\theta - 3 = \tan\theta - 1$ for $-\frac{\pi}{2} \le \theta < \frac{\pi}{2}$ is $\theta \approx 0.7854$.

Example 5 Application: Migratory Populations

The number of goldfinches in a coastal county of New England varies cyclically over the course of the year as the birds migrate. The approximate number of goldfinches in the county can be modeled by $P(t) = 250\cos\left(\frac{\pi}{6}(t+6)\right) + 250$, where P is the number of birds and t is number of months. In which month does the population first reach 375 birds?

SOLUTION

1. **Understand** The number of birds during each month can be found by substituting the approximate value (1–12) for t.

2. **Plan** To find the month or months in which the population reaches 375 birds, you need to substitute 375 for $P(t)$ and solve for t.

3. **Solve**

$$250\cos\left(\frac{\pi}{6}(t+6)\right) + 250 = 375$$ Substitute 375 for $P(t)$.

$$250\cos\left(\frac{\pi}{6}(t+6)\right) = 125$$ Subtract 250 from both sides.

$$\cos\left(\frac{\pi}{6}(t+6)\right) = \frac{1}{2}$$ Divide each side by 250.

$$\frac{\pi}{6}(t+6) = \cos^{-1}\left(\frac{1}{2}\right)$$ Apply the inverse cosine.

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}\left(\text{or } \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots\right)$$

$$\frac{\pi}{6}(t+6) = \frac{\pi}{3} \Rightarrow t = -4$$ Test $\frac{\pi}{3}$. not a solution because $1 \le t \le 12$.

$$\frac{\pi}{6}(t+6) = \frac{5\pi}{3} \Rightarrow t = 4$$ Test $\frac{5\pi}{3}$. $t = 4$ is the solution

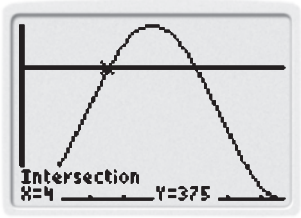
The population first reaches 375 birds in April.

Hint

$\cos\theta = \frac{1}{2}$ for $\theta = \frac{\pi}{3}$, $\frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}\ldots$ therefore $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and also $\frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}\ldots$

4. Check Check your answer using a graphing calculator.

Graph: $250 \cos\left(\frac{\pi}{6}(t+6)\right) + 250 = 375$



Lesson Practice

a. (Ex 1) Find all the solutions of $8 \sin\theta + 1 = 4 \sin\theta - 1$ algebraically.

b. (Ex 2) Solve $4\cos^2 x = 4\cos x - 1$ for $0° \le x < 360°$.

c. (Ex 2) Solve $3 \tan^2 z - \tan z - 3 = 0$ for $-\frac{\pi}{2} < z < \frac{\pi}{2}$.

d. (Ex 3) Use trigonometric identities to solve $1 - \cos^2\theta + 3 \sin\theta = \sin\theta - 1$ for $0° \le \theta < 360°$.

e. (Ex 4) Find all the solutions of $\sin\theta - \frac{1}{2} = 2 \sin\theta + \frac{1}{2}$ for $0 \le \theta \le 2\pi$ using a graphing calculator.

f. (Ex 5) As in Example 5, in which month of the year does the bird population first reach 125 birds?

Practice Distributed and Integrated

1. (106) **Analyze** A polynomial equation has a root of $2 + 3i$ and this root has a multiplicity of 2. What is the least degree that the polynomial can be? Why?

***2.** (119) **Write** Describe the difference between an equation and an identity.

3. (93) If a private school's enrollment is 600 students and it increases 5% each year, how long will it take to get 800 students?

4. (114) **Multi-Step** The boundary of the set of possible points for the epicenter of an earthquake recorded at an observation station can be represented by the equation $x^2 + y^2 - 8x - 40y - 209 = 0$.

a. What conic section represents the situation?

b. Write the equation in standard form.

5. (42) Find the number of combinations of 16 objects taken 12 at a time.

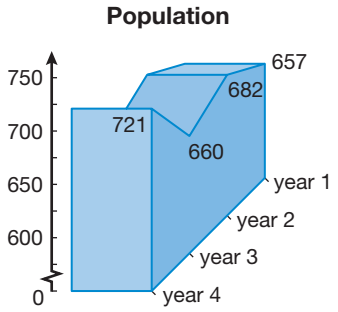
***6.** (117) **Graphing Calculator** Solve $\begin{aligned} 25x^2 + 4.3y^2 &= 89 \\ x^2 + y^2 &= 9.8 \end{aligned}$. Round to the nearest hundredth.

***7.** (119) **Solar Eclipses** For a solar eclipse to be possible, the moon must be near the line between the earth and the sun within 3 days of a new moon. Suppose the cycle of the moon can be modeled by $M(t) = \cos\left(\frac{\pi}{14}t\right)$, where t is the number of days from now. The minimum value of $M(t)$ represents a new moon. How many days from now is the next new moon? When $P(t) = 2\sin^2\left(\frac{\pi}{14}t\right) + \cos\left(\frac{\pi}{14}t\right) - \frac{1}{2}$ equals $M(t)$, the moon is on the line between earth and the sun. Is a solar eclipse possible before the next new moon? Why or why not? Explain.

*8. (118) **Write** When surveying, what are the pros and cons of open questions and closed questions?

9. (97) Find the common ratio of the geometric sequence and use it to find the next three terms. 14, 2, $\frac{2}{7}$…

*10. (118) **Data Analysis** What could be misleading about the graph shown?



11. (Inv 11) Use De Moivre's Theorem to evaluate the complex number $(6 - 7i)^3$ to express the result in trigonometric form.

*12. (117) **Model** Sketch diagrams to show how two parabolas can intersect in 0, 1, 2, 3, or 4 points.

*13. (119) **Multiple Choice** For what values $0 \le \theta < 2\pi$ does $\sin 2\theta = \sin \theta$?

A $\theta = \frac{1}{2}$ **B** $\theta = 0, \frac{1}{2}$ **C** $\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ **D** $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

14. (116) **Population** The year when the world population reached each billion mark is shown in the table.

Population (billions)	1	2	3	4	5	6
Year	1800	1927	1960	1974	1987	1999

Find a logarithmic function to model the data and use it to predict when the world population will reach 8 billion.

15. (113) **Finance** The monthly rent for an apartment is $850 the first year. Every year thereafter, the rent increases by 2%. Write a geometric series in summation notation to represent the total rent paid after 10 years. Then find the sum.

*16. (116) **Multiple Choice** Which data set is best modeled by an exponential function?

A {(1, −16), (2, −12), (3, −8), (4, −4), (5, 0)}

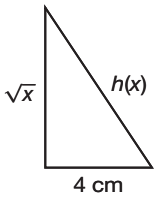
B {(1, −16), (2, −8), (3, −4), (4, −2), (5, −1)}

C {(1, −16), (2, 0), (3, 16), (4, 32), (5, 48)}

D {(1, −16), (2, −64), (3, −144), (4, −256), (5, −400)}

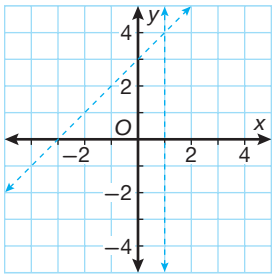
17. (115) **Manufacturing** The horizontal distance x traveled by a golf ball struck from ground level with an initial speed v and angle θ is given by $x = \frac{1}{32}v^2 \sin 2\theta$. The rule determined by the United States Golf Association states that the initial velocity of a golf ball cannot exceed 255 feet per second when tested under specified conditions. If a ball is manufactured with the highest allowed initial velocity, use the equation to determine the horizontal distance of the ball if it is hit at an angle of 45°.

18. (104) **Geometry** The Pythagorean Theorem states that the length of the hypotenuse of a right triangle is equal to the square root of the sum of the squared lengths of the legs of the triangle. The equation $h(x) = \sqrt{x + 16}$ can be used to model the possible lengths of the hypotenuse and the square of the other leg for the right triangle shown.



Use $f(x) = \sqrt{x}$ to find the graph of $h(x) = \sqrt{x + 16}$. Then use the graph of $h(x)$ to find two possibilities for the lengths of the other leg and the hypotenuse.

19. (107) **Formulate** The graph of a rational function includes the asymptote shown to the right. Write an equation that fits these parameters.



20. (110) **Multi-Step** Let $f(x) = 12\log(4x)$ and $g(x) = \frac{x+2}{(x^4-1)}$. Find the x-intercepts of $f(g(x))$.

21. (87) **Temperature** Using Newton's Law of Cooling $T = e^{(kt+C)} + R$, determine the amount of time needed to cool a cup of water (80°C) to 50°C in a room temperature of 30°C. Use $k = -0.200064$ and $C = 4.0073$.

22. (102) **Error Analysis** A student solved $\log 2x + \log 4 = 3$ as shown. What is the error? What is the correct solution?

$$\log 2x + \log 4 = 3 \rightarrow \log(2x + 4) = 3 \rightarrow 2x + 4 = 1000 \rightarrow x = 498$$

23. (105) The common difference in an arithmetic series is 8 and the first term is −160. Find the sum of the first 40 terms.

Find the roots of the equations.

24. (100) $y = 6x^4 - 4x^3 - 8x^2 + 4x + 2$ **25.** (100) $y = 15x^4 - 16x^3 - 56x^2 + 64x - 16$

Write the expression in simplest form.

26. (40) $5\sqrt[3]{48} - \sqrt[3]{750}$ **27.** (40) $\frac{\sqrt[4]{36} \cdot \sqrt[4]{9}}{\sqrt[4]{4}}$

***28.** (119) **Multi-Step** Consider the equation $3\tan^2 z + 2\tan z - 1 = 0$.

a. What values of $-90° \le z \le 90°$ satisfy the equation?

b. For how many of the values of z that satisfy the equation does $\sin z = \cos z$?

***29.** (118) **Error Analysis** A student with a part-time job at a movie theater uses ticket stubs he found on the floor to collect data about the types of movies people watch on different days. He says that he takes a simple random sample of the stubs he finds, so his sample cannot be biased. Find the error in the student's reasoning.

30. (45) The data in the table is represented by an exponential function. Use the data to solve for the correlation coefficient, r.

Time (in years)	1	2	3	4	5
Bunnies	30	100	335	1100	3650

INVESTIGATION

12

Using Mathematical Induction

A mathematical proof is true for all values that relate to the proof. This is what makes a mathematical proof a powerful tool. One type of mathematical proof is **mathematical induction**.

In an arithmetic sequence the value of each term a_k can be calculated based on its position k in the sequence. For the sequence of consecutive integers the formula for each term is $a_k = k$. The sum of consecutive integers 1 to n can be calculated using this formula,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

1. Show that this formula works for $n = 10$, 20, and 30.
2. Why is the formula a better way of calculating the result?

While you can show that this formula works for *any* value of n, a mathematical proof needs to show that this formula works for *every* value of n. Proof by mathematical induction is a three-step process.

Step 1: Show that the base case is true.

3. **Verify** Show that the base case $n = 1$ is true.

Step 2: Assume the case is true for an arbitrary number of terms, represented by m.

4. Rewrite the formula using m instead of n.

Step 3: Show that the formula is true for $m + 1$.

5. Add the next term a_{m+1} to the formula from question 4 and simplify the expression.
6. **Verify** Show that the new expression from question 5 is the result of the summation formula for $m + 1$.
7. When all three steps are true, then the mathematical proof is complete through the process of induction. Is $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ true?

Suppose that you want to generate the formula for adding consecutive even numbers. For example, the sum of consecutive even integers is shown for the first six even numbers:

$$0 = 0$$
$$0 + 2 = 2$$
$$0 + 2 + 4 = 6$$
$$0 + 2 + 4 + 6 = 12$$
$$0 + 2 + 4 + 6 + 8 = 20$$
$$0 + 2 + 4 + 6 + 8 + 10 = 30$$

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8. Write a formula that can be used to generate even numbers only for any integer value of k.
9. Write the summation expression that uses Σ and the expression from question 8 that can be used to find the sum of consecutive even integers.
10. Find the general form that does not use Σ that can be used to find the sum of n consecutive even integers for any value of n.

The sum of consecutive even integers from 0 to n can be calculated using this formula,

$$\sum_{k=1}^{n} 2k = n(n+1).$$

Hint

For Problem 12c, substitute $m + 1$ in the original formula $\sum_{k=0}^{n} 2k$.

11. Show that this formula works for $n = 10$, 20, and 30.
12. To show that this formula works for every value of n, use mathematical induction.
    a. Show that this formula works for the base case of $n = 1$.
    b. Assume that the formula works for value m, and rewrite the formula.
    c. Show that the formula still works for $m + 1$.

Investigation Practice

Suppose that you want to generate the formula for adding the set of consecutive odd integers from 1 to n.

a. Generate the first five sums of this series.
b. Write a formula that can be used to generate odd numbers only for any value of k.
c. Write the summation expression that uses Σ and the expression from question **b** that can be used to find the sum of consecutive odd integers.
d. Find the general form that does not use Σ that can be used to find the sum of n consecutive odd integers for any value of n.

Use mathematical induction to prove the formula from question a.

e. Show that this formula works for the base case of $n = 1$.
f. Assume that the formula works for value m, and rewrite the formula.
g. Show that the formula still works for $m + 1$.

Use mathematical induction to prove or disprove the following formulas. For formulas that are incorrect, determine if there is a correct formula.

h. The sum of consecutive multiples of 3

$$\sum_{k=1}^{n} 3k = \frac{3n(n+1)}{2}.$$

i. The sum of consecutive multiples of 5:

$$\sum_{k=1}^{n} 5k = \frac{n(n+1)}{5}.$$

j. Generate the formula for finding the sum of the consecutive numbers that are divisible by both 3 and 5. Use mathematical induction to prove the formula.

APPENDIX LESSON 1

Changes in Measure

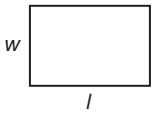
New Concepts

Use the formulas for area, surface area, and volume to determine how changes in measure affect other measures.

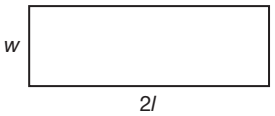
Example 1 Determining How Changes in Measure Affect Area

a. The length of a rectangle is doubled. What is the change in the area of the rectangle?

SOLUTION



The area of the rectangle before the length is doubled is $A = lw$.



The area of the rectangle after the length is doubled is $A = (2l)w$. Using the Associative Property of Multiplication, you can express the area as:

$$A = 2(lw)$$

When the length of a rectangle is doubled, the area of the rectangle is also doubled.

b. The length and width of a rectangle are each doubled. What is the change in the area of the rectangle?

SOLUTION The area of the rectangle before the length and width are doubled is $A = lw$. The area of the rectangle after the length and width are doubled is $A = (2l)(2w)$. Using the Commutative and Associative Properties of Multiplication, the area can be expressed as:

$$A = 4(lw)$$

When the length and width of a rectangle are doubled, the area of the rectangle is four times the original area.

Hint

Remember, the Associative Property of Multiplication states that for any real numbers, a, b, and c:

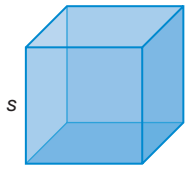
$a(bc) = (ab)c$

Predict

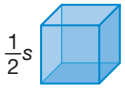
What do you think the change in area will be if the length and width are tripled?

Example 2 Determining How Changes in Measure Affect Surface Area and Volume

a. The length of the side of a cube is halved. What is the change in the surface area and volume of the cube?



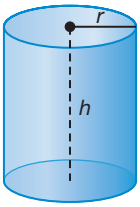
SOLUTION



The surface area of the cube before the side is halved is $SA = 6s^2$.
The volume of the cube before the side is halved is $V = s^3$.

Surface Area	Volume
The surface area of the cube after the side is halved is: $SA = 6\left(\frac{1}{2}s\right)^2 = 6\left(\frac{1}{4}s^2\right) = \frac{1}{4}(6s^2)$ When the side of a cube is halved, the surface area is $\left(\frac{1}{2}\right)^2$ or $\frac{1}{4}$ of the original surface area.	The volume of the cube after the side is halved is: $V = \left(\frac{1}{2}s\right)^3 = \left(\frac{1}{8}s^3\right) = \frac{1}{8}(s^3)$ When the side of a cube is halved, the volume is $\left(\frac{1}{2}\right)^3$ or $\frac{1}{8}$ of the original volume.

b. The height of a cylinder is tripled. What is the change in the volume of the cylinder?



SOLUTION

The volume of the cylinder before the height is tripled is $V = \pi r^2 h$.

The volume of the cylinder after the height is tripled is:

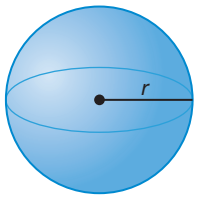
$$V = \pi r^2(3h) = 3(\pi r^2 h).$$

When the height of a cylinder is tripled, the volume is tripled.

Generalize

If the height of a cylinder is multiplied by any positive real number, how does the volume of the cylinder change?

c. The diameter of a sphere is doubled. What is the change in the surface area of the sphere?



SOLUTION

The surface area of the sphere before the diameter is doubled is $SA = 4\pi r^2$.

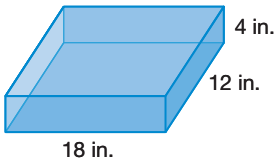
When the diameter of a sphere is doubled, the radius is also doubled.

The surface area of the sphere after the diameter is doubled is:

$$4\pi(2r)^2 = 4\pi(4r^2) = 16\pi r^2 = 4(4\pi r^2)$$

When the diameter of a sphere is doubled, the surface area is quadrupled, or 4 times the original surface area.

d. If each dimension of a rectangular prism is halved, what will be the change in the volume of the prism?



SOLUTION

The volume of the rectangular prism before the changes in measure is $V = (18)(12)(4) = 864$ cubic inches.

The volume of the rectangular prism after the changes in measure is $V = \left(\frac{18}{2}\right)\left(\frac{12}{2}\right)\left(\frac{4}{2}\right) = (9)(6)(2) = 108$ cubic inches.

The new volume is $\left(\frac{1}{2}\right)^3$ or $\frac{1}{8}$ the original volume.

Example 3 Application: Sports

The diameter of a softball is about 1.5 times the diameter of a baseball. What is the relationship between the volumes of the balls?

SOLUTION Softballs and baseballs are spheres. The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. If the diameter of a softball is 1.5 times the diameter of a baseball, then the radius of the softball is also 1.5 times the radius of the baseball.

Let r represent the radius of the baseball and $1.5r$ represent the radius of the softball. Then the volume of the softball is:

$$V = \frac{4}{3}\pi(1.5r)^3 = \frac{4}{3}\pi(3.375r^3) = 3.375\left(\frac{4}{3}\pi r^3\right)$$

The volume of the softball is $(1.5)^3$ or 3.375 times the volume of the baseball.

Generalize

If the diameter of one sphere is x times the diameter of another sphere, then the radius of the first is also x times the radius of the second.

Lesson Practice

(Ex 1) **a.** The height of a triangle is doubled. What is the change in the area of the triangle?

(Ex 1) **b.** The bases and height of a trapezoid are tripled. What is the change in the area of the trapezoid?

(Ex 2) **c.** The length of each side of a cube is divided by 5. What is the change in the surface area and volume of the cube?

(Ex 2) **d.** The height of a cone is quadrupled. What is the change in the volume of the cone?

(Ex 2) **e.** The radius of a sphere is tripled. What is the change in the volume of the sphere?

(Ex 2) **f.** A rectangular prism is 12 feet long, 8.2 feet wide, and 4 feet high. The length and height are halved. What is the change in the volume of the prism?

(Ex 3) **g.** The diameter of one circular pool is twice the diameter of a second circular pool. The height of both pools is 4 feet. What is the relationship between the volumes of the pools?

APPENDIX LESSON 1

Changes in Measure

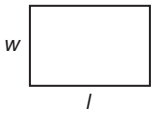
New Concepts

Use the formulas for area, surface area, and volume to determine how changes in measure affect other measures.

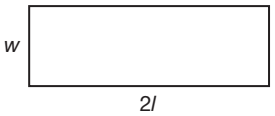
Example 1 Determining How Changes in Measure Affect Area

a. The length of a rectangle is doubled. What is the change in the area of the rectangle?

SOLUTION



The area of the rectangle before the length is doubled is $A = lw$.



The area of the rectangle after the length is doubled is $A = (2l)w$. Using the Associative Property of Multiplication, you can express the area as:

$$A = 2(lw)$$

When the length of a rectangle is doubled, the area of the rectangle is also doubled.

b. The length and width of a rectangle are each doubled. What is the change in the area of the rectangle?

SOLUTION The area of the rectangle before the length and width are doubled is $A = lw$. The area of the rectangle after the length and width are doubled is $A = (2l)(2w)$. Using the Commutative and Associative Properties of Multiplication, the area can be expressed as:

$$A = 4(lw)$$

When the length and width of a rectangle are doubled, the area of the rectangle is four times the original area.

Hint

Remember, the Associative Property of Multiplication states that for any real numbers, a, b, and c:

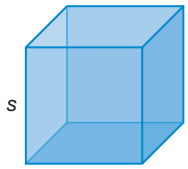
$a(bc) = (ab)c$

Predict

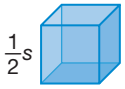
What do you think the change in area will be if the length and width are tripled?

Example 2 Determining How Changes in Measure Affect Surface Area and Volume

a. The length of the side of a cube is halved. What is the change in the surface area and volume of the cube?



SOLUTION



The surface area of the cube before the side is halved is $SA = 6s^2$.
The volume of the cube before the side is halved is $V = s^3$.

Surface Area	Volume
The surface area of the cube after the side is halved is: $SA = 6\left(\frac{1}{2}s\right)^2 = 6\left(\frac{1}{4}s^2\right) = \frac{1}{4}(6s^2)$ When the side of a cube is halved, the surface area is $\left(\frac{1}{2}\right)^2$ or $\frac{1}{4}$ of the original surface area.	The volume of the cube after the side is halved is: $V = \left(\frac{1}{2}s\right)^3 = \left(\frac{1}{8}s^3\right) = \frac{1}{8}(s^3)$ When the side of a cube is halved, the volume is $\left(\frac{1}{2}\right)^3$ or $\frac{1}{8}$ of the original volume.

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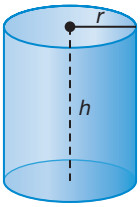

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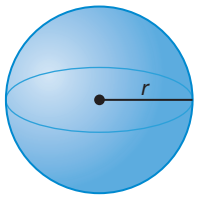

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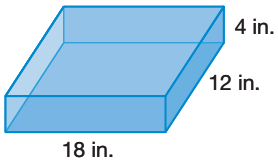

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(Ex 3) **g.** The diameter of one circular pool is twice the diameter of a second circular pool. The height of both pools is 4 feet. What is the relationship between the volumes of the pools?

APPENDIX LESSON 2

Computer Spreadsheets

New Concepts

An **electronic spreadsheet** is used to organize and manipulate data. Formulas and functions can be used to perform arithmetic and logical operations on the data in a spreadsheet.

Spreadsheets contain rows and columns of information, just as tables and matrices do. Rows are labeled with numbers and columns are labeled with letters. The intersection of a row and a column is called a **cell.** Each cell has a unique label known as the **cell address.** For example, the cell in the first row and first column of the spreadsheet has the address A1. The cell in the fifth row and third column has the address C5. To enter data in a cell, click on the cell and type in the data.

Connect

Remember, array elements are named using their row and column. Spreadsheet elements, called cells, are also named using their row and column.

Example 1 Creating a Computer Spreadsheet

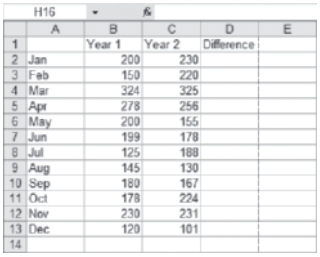
For two consecutive years, a cyclist recorded the number of miles he rode during training each month. His recorded information is shown below. Store the information in a computer spreadsheet.

Handwritten Record of Miles Biked

Miles Biked		
Month	**Year 1**	**Year 2**
Jan	200	230
Feb	150	220
Mar	324	325
Apr	278	256
May	200	155
Jun	199	178
Jul	125	188
Aug	145	130
Sep	180	167
Oct	178	224
Nov	230	231
Dec	120	101

SOLUTION

Enter the data into a spreadsheet. If a mistake is made while entering the data, retype the data by moving out of the cell and then back into it, or edit the data in the formula bar.



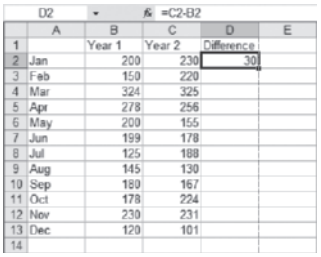
	A	B	C	D	E
1		Year 1	Year 2	Difference	
2	Jan	200	230		
3	Feb	150	220		
4	Mar	324	325		
5	Apr	278	256		
6	May	200	155		
7	Jun	199	178		
8	Jul	125	188		
9	Aug	145	130		
10	Sep	180	167		
11	Oct	178	224		
12	Nov	230	231		
13	Dec	120	101		
14					

Formulas and functions are used to manipulate data in the spreadsheet. A formula is like an expression. It can be used to perform arithmetic operations on the data in the spreadsheet.

Example 2 Using a Formula in a Spreadsheet

Using the spreadsheet created in Example 1, find the difference between the distances the cyclist biked each month from year 1 to year 2.

SOLUTION Create a formula to calculate the difference. To subtract the number of miles in January of year 1 from the number of miles in January of year 2, type the formula =C2−B2 in cell D2.



	A	B	C	D	E
1		Year 1	Year 2	Difference	
2	Jan	200	230	30	
3	Feb	150	220		
4	Mar	324	325		
5	Apr	278	256		
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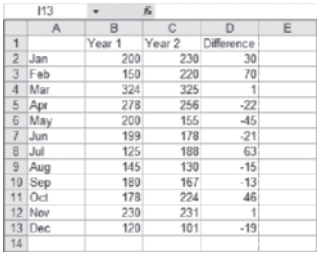
8	Jul	125	188		
9	Aug	145	130		
10	Sep	180	167		
11	Oct	178	224		
12	Nov	230	231		
13	Dec	120	101		
14					

Now copy the formula in cell D2 to the remaining cells. The spreadsheet will automatically adjust the formula to use the data in the corresponding row.

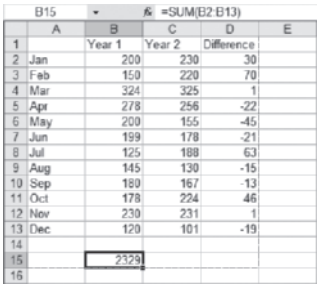


	A	B	C	D	E
1		Year 1	Year 2	Difference	
2	Jan	200	230	30	
3	Feb	150	220	70	
4	Mar	324	325	1	
5	Apr	278	256	-22	
6	May	200	155	-45	
7	Jun	199	178	-21	
8	Jul	125	188	63	
9	Aug	145	130	-15	
10	Sep	180	167	-13	
11	Oct	178	224	46	
12	Nov	230	231	1	
13	Dec	120	101	-19	
14					

Example 3 Using a Function in a Spreadsheet

a. Using the spreadsheet from Example 2, find the total number of miles the cyclist biked in year 1 and the total he biked in year 2.

SOLUTION Use a function to find the totals. A function is a program that performs a specific, commonly used task. To find the totals for each year, use the SUM() function. Type =SUM(B2:B13) in cell B15.



	A	B	C	D	E
1		Year 1	Year 2	Difference	
2	Jan	200	230	30	
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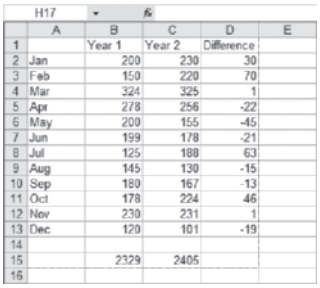
7	Jun	199	178	-21	
8	Jul	125	188	63	
9	Aug	145	130	-15	
10	Sep	180	167	-13	
11	Oct	178	224	46	
12	Nov	230	231	1	
13	Dec	120	101	-19	
14					
15		2329			
16					

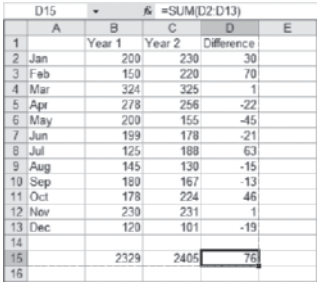
Copy the formula in cell B15 to cell C15, and the spreadsheet will automatically adjust the function to use the data in the corresponding column.



	A	B	C	D	E
1		Year 1	Year 2	Difference	
2	Jan	200	230	30	
3	Feb	150	220	70	
4	Mar	324	325	1	
5	Apr	278	256	-22	
6	May	200	155	-45	
7	Jun	199	178	-21	
8	Jul	125	188	63	
9	Aug	145	130	-15	
10	Sep	180	167	-13	
11	Oct	178	224	46	
12	Nov	230	231	1	
13	Dec	120	101	-19	
14					
15		2329	2405		
16					

b. Find the sum of the difference column.

SOLUTION Copy the formula in cell B15 to cell D15.



	A	B	C	D	E
1		Year 1	Year 2	Difference	
2	Jan	200	230	30	
3	Feb	150	220	70	
4	Mar	324	325	1	
5	Apr	278	256	-22	
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7	Jun	199	178	-21	
8	Jul	125	188	63	
9	Aug	145	130	-15	
10	Sep	180	167	-13	
11	Oct	178	224	46	
12	Nov	230	231	1	
13	Dec	120	101	-19	
14					
15		2329	2405	76	
16					

Analyze

What formula could also be used to find the sum of cells B2 through B13?

Analyze

What formula could also be used to find the sum of the difference column?

Example 4 Using a Spreadsheet to Answer Questions About Data

Use the spreadsheet created in Example 3 to answer each question.

a. During which months in year 2 did the cyclist bike a greater number of miles than he did in the corresponding months in year 1?

SOLUTION Look in the difference column and find the months whose numbers are positive.

	A	B	C	D	E
1		Year 1	Year 2	Difference	
2	Jan	200	230	30	
3	Feb	150	220	70	
4	Mar	324	325	1	
5	Apr	278	256	-22	
6	May	200	155	-45	
7	Jun	199	178	-21	
8	Jul	125	188	63	
9	Aug	145	130	-15	
10	Sep	180	167	-13	
11	Oct	178	224	46	
12	Nov	230	231	1	
13	Dec	120	101	-19	
14					
15		2329	2405	76	
16					

The cyclist biked a greater number of miles during the months of January, February, March, July, October, and November in year 2 than in year 1.

b. How many more miles did the cyclist bike in year 2 than in year 1?

SOLUTION Look in the sum of the difference column.

	A	B	C	D	E
1		Year 1	Year 2	Difference	
2	Jan	200	230	30	
3	Feb	150	220	70	
4	Mar	324	325	1	
5	Apr	278	256	-22	
6	May	200	155	-45	
7	Jun	199	178	-21	
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11	Oct	178	224	46	
12	Nov	230	231	1	
13	Dec	120	101	-19	
14					
15		2329	2405	76	
16					

The cyclist biked 76 more miles in year 2 than in year 1.

Example 5 Application: Graphs

Using the spreadsheet from Example 3, create a bar graph to display the data for year 1.

SOLUTION Select the data from cell A2 through B13.

	A	B	C	D	E
1		Year 1	Year 2	Difference	
2	Jan	200	230	30	
3	Feb	150	220	70	
4	Mar	324	325	1	
5	Apr	278	256	-22	
6	May	200	155	-45	
7	Jun	199	178	-21	
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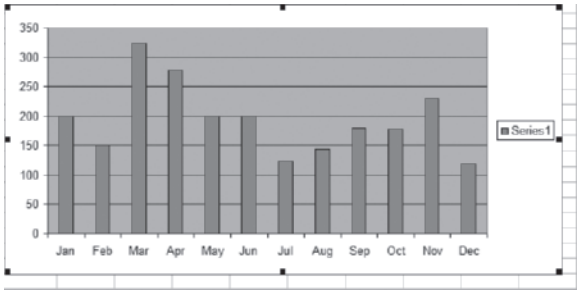
12	Nov	230	231	1	
13	Dec	120	101	-19	
14					
15		2329	2405	76	
16					

Select the Chart Wizard button from the toolbar and choose Column as the chart type and Clustered Column as the chart subtype. Click the Finish button.



Lesson Practice

(Ex 1 and 3) **a.** The 2007 ticket prices for a major league franchise are shown. The prices of each kind of ticket will be increased by 5% for the 2008 season. Create a spreadsheet showing the 2007 and 2008 ticket prices.

Table of Ticket Prices

	Ticket Prices 2007
Terrace Box	$30
Upper Reserved	$10
Upper Box	$20
Club Box	$55
Field Box	$40
Lower Box	$40

(Ex 2) **b.** Add a column to the spreadsheet to display the amount of increase in the cost of ticket prices from 2007 to 2008.

(Ex 4) **c.** How much would it cost to purchase two terrace box seat tickets in 2008?

(Ex 5) **d.** Create a double bar graph to display the ticket information for both years.

APPENDIX LESSON 3

Precision and Accuracy

New Concepts

The *accuracy* of a measurement refers to how close it is to the actual or accepted value.

Precision can refer to any of the following:

- how tightly together a set of measurements is clustered
- how close a single measurement is to the mean of a set
- the interval of values implied by the last significant digit in a number

Relative error is a measure of accuracy.

$$\text{Relative error} = \frac{|\text{Measured value} - \text{Accepted value}|}{\text{Accepted value}}$$

Relative deviation is a measure of precision.

$$\text{Relative deviation} = \frac{|\text{Measured value} - \text{Mean value of the set}|}{\text{Mean value of the set}}$$

Example 1 Finding a Measure of Accuracy

a. A standard 50-gram weight is placed on a balance. The balance reads 49.2 grams. What is the relative error?

SOLUTION

$$\text{Relative error} = \frac{|49.2 - 50|}{50}$$

$$= \frac{0.8}{50}$$ The absolute error is 0.8 gram.

$$= 0.016, \text{ or } 1.6\%$$

b. A standard 2000-gram weight is placed on a balance. The balance reads 2006 grams. What is the relative error?

SOLUTION

$$\text{Relative error} = \frac{|2006 - 2000|}{2000}$$

$$= \frac{6}{2000}$$ The absolute error is 6 grams.

$$= 0.003, \text{ or } 0.3\%$$

The following points in Example 1 are worth noting:

- The absolute error is greater in part b, but the relative error is greater in part a.
- A gram is a measure of mass, not weight. But in this context the distinction is not important.

- The actual weight of a "standard 50-gram weight" might not be 50 grams. (It might be closer to 50.001 grams.) This illustrates why accuracy refers to an actual or *accepted* value.

The simplest method of gauging the precision of a set of measurements is to compare the ranges. The range is the difference between the greatest and least values in the set.

Example 2 Comparing Precision in Sets of Data

All the measurements below are in grams. Using range, which set shows a greater degree of precision?

Set A: 14.3, 14.0, 14.3, 14.5

Set B: 22.9, 22.9, 23.2, 22.8

SOLUTION

The range of set A is $14.5 - 14.0 = 0.5$.

The range of set B is $23.2 - 22.8 = 0.4$.

Set B shows a greater degree of precision.

Example 3 Finding a Measure of Precision

Eight different balances are used to weigh the same object. The results, in grams, are listed below.

19.8 19.9 19.9 20.0 20.1 20.3 20.3 20.3

Then a student weighs the same object, obtaining 19.9 grams. What is the relative deviation associated with the student's result?

SOLUTION

$$\text{Mean value} = \frac{19.8 + 19.9 + 19.9 + 20.0 + 20.1 + 20.3 + 20.3 + 20.3}{8}$$

$$= 20.075$$

$$\text{Relative deviation} = \frac{|19.9 - 20.075|}{20.075}$$

$$= \frac{0.175}{20.075}$$ The absolute deviation is 0.175 gram.

$$\approx 0.009, \text{ or } 0.9\%$$

The last *significant digit* in a decimal number is the rightmost digit. When a quantity is measured, the last significant digit in the reported measurement implies (indicates) the degree of precision. For example, a reported length of 16 cm implies that the measured length is closer to 16 cm than to either 15 cm or 17 cm. But a reported length of 16.0 cm implies that the measured length is closer to 16.0 cm than to either 15.9 cm or 16.1 cm.

Example 4 Finding an Implied Interval of Values

For each reported measurement, state the implied interval of values that contains the measured value x.

a. Reported length 16 cm

SOLUTION

The last significant digit is 6, which is in the ones place.

$16 - 0.5 = 15.5,\ 16 + 0.5 = 16.5$ Subtract and add half of 1.

$15.5 \text{ cm} \leq x < 16.5 \text{ cm}$ State the implied interval of values that contains the measured value x.

To understand why the inequality symbols $\leq$ and $<$ are used, note that when 15.5 is rounded to the nearest whole number, the result is 16, and when 16.5 is rounded to the nearest whole number, the result is 17.

b. Reported weight 3.24 kg

SOLUTION

The last significant digit is 4, which is in the hundredths place.

$3.24 - 0.005 = 3.235,\ 3.24 + 0.005 = 3.245$ Subtract and add half of 0.01.

$3.235 \text{ kg} \leq x < 3.245 \text{ kg}$ State the implied interval of values that contains the measured value x.

Example 5 Determining if a Measurement is Within Tolerance Limits

Use the stated tolerance limits for each manufactured product to determine if the given measurement is acceptable.

a. The length of a bolt is required to be $3.2 \text{ cm} \pm 2\%$. Is 3.1 cm acceptable?

SOLUTION

$0.02 \times 3.2 = 0.064$ Find 2% of 3.2.

$3.136 \leq x \leq 3.264$ State the interval of values that are acceptable.

$3.136 \leq 3.1 \leq 3.264$ is not true, so 3.1 cm is not acceptable.

b. The diameter of a pipe is required to be $\frac{3}{4}$ in. $\pm\frac{1}{32}$ in. Is $\frac{49}{64}$ in. acceptable?

SOLUTION

$\frac{3}{4} - \frac{1}{32} = \frac{48}{64} - \frac{2}{64} = \frac{46}{64}$ and $\frac{3}{4} + \frac{1}{32} = \frac{48}{64} + \frac{2}{64} = \frac{50}{64}$.

$\frac{46}{64} \leq x \leq \frac{50}{64}$ State the interval of values that are acceptable.

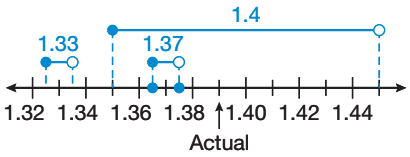
$\frac{46}{64} \leq \frac{49}{64} \leq \frac{50}{64}$ is true, so $\frac{49}{64}$ in. is acceptable.

Example 6 Use Accuracy and Precision to Compare Measurements

Suppose the actual length of an object is known to be 1.39 m, and these three different measurements are reported for the object: 1.33 m, 1.37 m, and 1.4 m. Compare the reported measurements, discussing accuracy and precision.

SOLUTION

The implied interval of values for each measurement is shown in the diagram.



Accuracy: The most accurate is 1.37 because its interval is closest to 1.39. To compare 1.33 and 1.4, choose the endpoint of each corresponding interval that is farthest from 1.39, and then compare those endpoints. Comparing 1.325 and 1.45, you can see that 1.45 is closer to 1.39. So, 1.4 is the next most accurate. The least accurate is 1.33.

Precision: 1.33 and 1.37 are equally precise because the last significant digit for each number is in the same place. The least precise is 1.4.

Lesson Practice

(Ex 1) **a.** A standard 200-gram weight is placed on a balance. The balance reads 201.2 grams. What is the relative error?

(Ex 1) **b.** A timer is set for 3 minutes. It rings after 2 minutes 58 seconds has elapsed. What is the relative error?

(Ex 2) **c.** Two groups measure the temperature of a liquid 5 times, with the results shown below. Using range, which group has greater precision?

Group A: 85.6°C, 85.4°C, 85.0°C, 86.2°C, 86.2°C
Group B: 86.0°C, 86.4°C, 85.0°C, 86.2°C, 85.8°C

(Ex 3) **d.** Some researchers weigh a fossil. The mean weight is 10.1 grams. What is the relative deviation associated with a measured weight of 10.0 grams?

(Ex 4) **e.** For a reported measurement of 124 g, what is the implied interval of values that contains the measured value x?

(Ex 4) **f.** For a reported measurement of 124.0 g, what is the implied interval of values that contains the measured value x?

(Ex 5) **g.** A serving of cereal is required to be 56 g $\pm$ 5%. Is 58 g acceptable?

(Ex 5) **h.** The thickness of a sheet of plastic is required to be $\frac{5}{8}$ in. $\pm\frac{1}{64}$ in. Is $\frac{39}{64}$ in. acceptable?

(Ex 6) **i.** Suppose the actual weight of an object is known to be 35.2 g, and these measured weights are reported for the object: 35 g, 34.8 g, and 35.5 g. Compare the reported measurements, discussing accuracy and precision.

APPENDIX LESSON 4

Predictions

New Concepts

A *prediction* is a statement about something that is not known. Two methods of making predictions are extrapolation and interpolation.

- *Extrapolation* is the process of obtaining a value that corresponds to a value that is outside of the known data set.
- *Interpolation* is the process of obtaining a value that corresponds to a value that is between known values in the data set.

A simple way to extrapolate and interpolate is to use proportions.

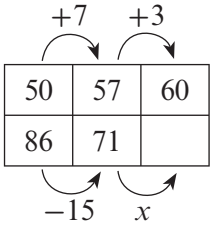
Example 1 Using Proportions to Extrapolate and Interpolate

The table shows data that relates heating cost to average temperature.

Average temperature (°F)	40	44	50	57
Monthly heating bill ($)	126	102	86	71

a. Extrapolate to predict the monthly heating bill for a month with an average temperature of 60°F.

SOLUTION



Use the differences +7 and −15 from the last interval in which both values are known. Use the differences +3 and x for the next interval.

$$\frac{7}{-15} = \frac{3}{x}$$

Write and solve a proportion.

$$x \approx -6.43$$

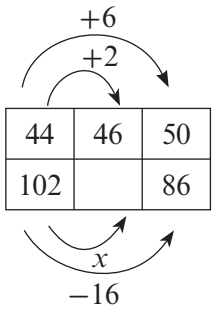
$$71 - 6.43 = 64.57$$

Use the value of x to compute the predicted value.

The predicted monthly heating bill is $64.57.

b. Interpolate to predict the monthly heating bill for a month with an average temperature of 46°F.

SOLUTION



Use the differences +6 and −16 from the interval with known endpoints that contains the unknown value. Use the differences +2 and x that correspond to the unknown value.

$\frac{6}{-16} = \frac{2}{x}$ Write and solve a proportion.

$x \approx -5.33$

$102 - 5.33 = 96.67$ Use the value of x to compute the predicted value.

The predicted monthly heating bill is $96.67.

Using a proportion to extrapolate or interpolate is appropriate for a data set that is linear or nearly linear. It is usually better to use a linear regression model because it consideres all of the data, not just certain intervals. *Regression* is a process of finding an equation that models a set of data.

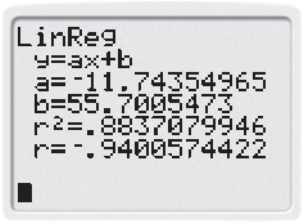
Example 2 Using a Linear Regression Model to Predict

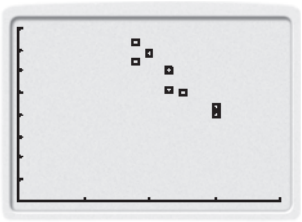
The table shows data that relates gas mileage and engine size for several cars. Predict the mileage for a car with an engine size of 3.5 liters.

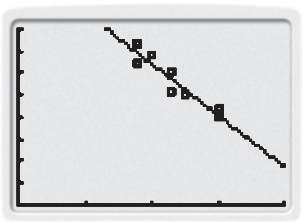
Engine size (liters)	1.8	2.3	3.0	2.3	3.0	2.5	2.0	1.8
Mileage (mi/gal)	37	26	22	30	20	25	34	32

SOLUTION Enter the engine sizes into list **L1** and the mileages into list **L2** on a graphing calculator. Press STAT, then choose **CALC,** and then **4: LinReg** to obtain a linear regression model. The linear regression model, or equation of the line of best fit, is $y \approx -11.74x + 55.70$, where x represents engine size in liters and y represents mileage in miles per gallon.

The correlation coefficient is $r \approx -0.94$. Make a scatter plot of the data and graph the equation.







$y \approx -11.74x + 55.70$ Substitute 3.5 for x in the equation.

$y \approx -11.74(3.5) + 55.70$

$y \approx 14.61$

The predicted mileage for a car with an engine size of 3.5 liters is approximately 14.6 miles per gallon.

You can obtain different regression models for any data set. Some of the regression models available on a graphing calculator are linear, quadratic, cubic, quartic, natural logarithmic, and exponential. After entering data into lists, press STAT, then choose **CALC,** and then scroll down. Choose different models and compare the values of the coefficients of determination, denoted by either r^2 or R^2.

Example 3 Choosing a Regression Model to Predict

The table shows the opening value of a stock index on the first day of trading in various years. Use a regression model to estimate the value on the first day of trading in 2004. State whether extrapolation or interpolation was used.

Year	Price ($)	Year	Price ($)
1996	616	2001	1320
1997	741	2002	1148
1998	970	2005	1212
1999	1229	2006	1248
2000	1469	2007	1418

SOLUTION

Let x represent the number of years since 1995. So $x = 1$ represents 1996, $x = 2$ represents 1997, and so on, with $x = 12$ representing 2007. Enter the x-values into list **L1** and the prices into **L2.** Obtain the following regression models and verify the values of the coefficients of determination, r^2 and R^2.

Linear $r^2 \approx 0.48$ Quadratic $R^2 \approx 0.74$ Cubic $R^2 \approx 0.87$

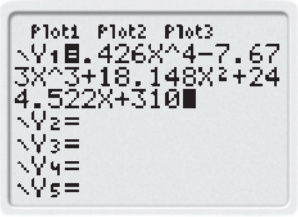
Quartic $R^2 \approx 0.90$ Natural logarithmic $r^2 \approx 0.70$ Exponential $r^2 \approx 0.50$

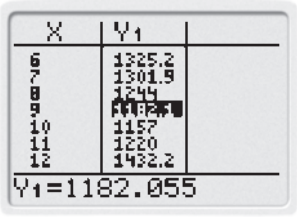
The quartic model seems to be the most appropriate choice because R^2 is closest to 1.

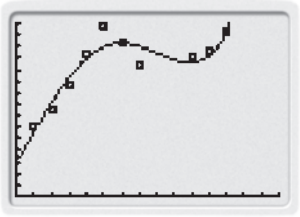
Press Y=, and then enter the quartic function model.

Use the TABLE feature to find the function value for $x = 9$, which represents 2004.

Graph the function. Make a scatter plot of the data.







Using a quartic model, the estimated value on the first day of trading in 2004 is $1182. Interpolation was used because the estimated value corresponds to 2004, which is between known values in the data set.

Lesson Practice

a. *(Ex 1)* The table shows rug prices. Extrapolate to predict the price of a rug with an area of 180 ft^2. Interpolate to predict the price of a rug with an area of 35 ft^2.

Area of Rug (ft^2)	24	54	80	108
Price ($)	99	199	309	399

b. *(Ex 2)* The table shows data that relates number of gallons of gasoline used and several driving distances. Write the best-fit linear regression model for the data. Use the model to predict the number of miles that will be driven if 4 gallons are used. State whether extrapolation or interpolation was used.

Gallons used	3.8	1.2	2.1	6.7	2.3	2.5	5.9
Miles driven	120	27	50	236	56	66	195

c. *(Ex 3)* The table shows the student enrollment on the opening day at a school for several years. Using $x = 1$ for 1998, write the best regression model for the data. Use the model to predict the enrollment on opening day of 2009. State whether extrapolation or interpolation was used.

Year	**Students**	**Year**	**Students**
1998	990	2002	1076
1999	1108	2005	1030
2000	1220	2006	986
2001	1184	2007	986

APPENDIX LESSON 5

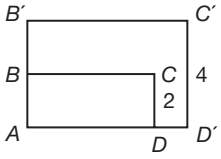
Scale Factor

New Concepts

A **dilation** is a transformation that changes the size of a figure by enlarging or reducing the figure. The figure is dilated about a fixed point called the **center of dilation.** The figure before the dilation is called the **pre-image** and the dilated figure is called the **image.** The **scale factor** of a dilation is the ratio of a side length of the image to the corresponding side length of the pre-image. If the scale factor is between 0 and 1, the dilation is a reduction. If the scale factor is greater than 1, the dilation is an enlargement.

Math Language

A **transformation** changes the size or position of a figure. Other common transformations include translations, rotations, and reflections.

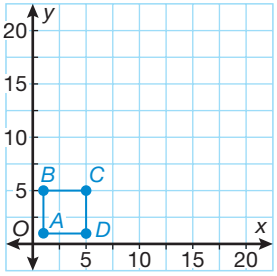


$A'B'C'D'$ is an enlargement of $ABCD$. The center of dilation is point A and the scale factor is the ratio $\frac{4}{2} = 2$.

You can perform dilations on the coordinate plane. When the origin is the center of dilation, multiply the scale factor, t by the coordinates of the pre-image to find the coordinates of the image.

Example 1 Drawing a Dilation with a Scale Factor Greater than 1

a. Draw a dilation of square $ABCD$ whose center of dilation is $(0, 0)$. Use a scale factor of 4.



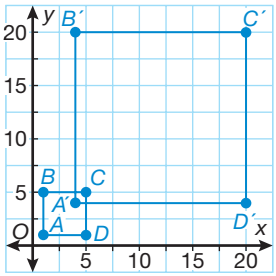
SOLUTION Multiply the coordinates of each vertex by the scale factor, 4. Then draw the image.

$A(1, 1)$ maps to $A'(4 \cdot 1, 4 \cdot 1) = A'(4, 4)$

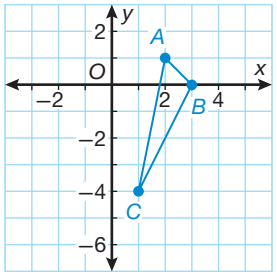
$B(1, 5)$ maps to $B'(4 \cdot 1, 4 \cdot 5) = B'(4, 20)$

$C(5, 5)$ maps to $C'(4 \cdot 5, 4 \cdot 5) = C'(20, 20)$

$D(5, 1)$ maps to $D'(4 \cdot 5, 4 \cdot 1) = D'(20, 4)$



b. Draw a dilation of triangle ABC whose center of dilation is $(0, 0)$. Use a scale factor of $\frac{3}{2}$.



Hint

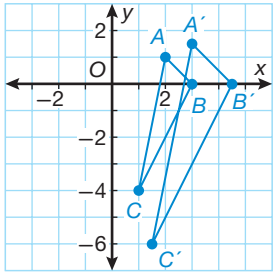
Use the scale factor to first determine if a dilation is a reduction or an enlargement.

SOLUTION Multiply the coordinates of each vertex by the scale factor, $\frac{3}{2}$. Then draw the image.

$A(2, 1)$ maps to $A'\left(\frac{3}{2} \cdot 2, \frac{3}{2} \cdot 1\right) = A'\left(3, \frac{3}{2}\right)$

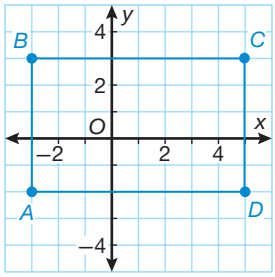
$B(3, 0)$ maps to $B'\left(\frac{3}{2} \cdot 3, \frac{3}{2} \cdot 0\right) = B'\left(\frac{9}{2}, 0\right)$

$C(1, -4)$ maps to $C'\left(\frac{3}{2} \cdot 1, \frac{3}{2} \cdot (-4)\right) = C'\left(\frac{3}{2}, -6\right)$



Example 2 Drawing a Dilation with a Scale Factor Between 0 and 1

Draw a dilation of rectangle $ABCD$ whose center of dilation is $(0, 0)$. Use a scale factor of $\frac{1}{2}$.



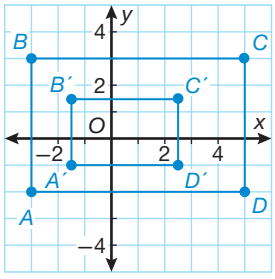
SOLUTION Multiply the coordinates of each vertex by the scale factor, $\frac{1}{2}$. Then draw the image.

$A(-3, -2)$ maps to $A'\left(\frac{1}{2} \cdot -3, \frac{1}{2} \cdot -2\right) = A'\left(-\frac{3}{2}, -1\right)$

$B(-3, 3)$ maps to $B'\left(\frac{1}{2} \cdot -3, \frac{1}{2} \cdot 3\right) = B'\left(-\frac{3}{2}, \frac{3}{2}\right)$

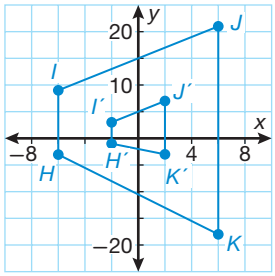
$C(5, 3)$ maps to $C'\left(\frac{1}{2} \cdot 5, \frac{1}{2} \cdot 3\right) = C'\left(\frac{5}{2}, \frac{3}{2}\right)$

$D(5, -2)$ maps to $D'\left(\frac{1}{2} \cdot 5, \frac{1}{2} \cdot -2\right) = D'\left(\frac{5}{2}, -1\right)$



Example 3 Finding a Scale Factor

Trapezoid $H'I'J'K'$ is a dilation of trapezoid $HIJK$. Find the scale factor.



SOLUTION The dilation is a reduction so the scale factor is between 0 and 1. The length of $\overline{H'I'}$ is 4 and the length of its corresponding side on the pre-image $\overline{HI}$ is 12. The scale factor is $\frac{4}{12} = \frac{1}{3}$.

A dilation is a similarity transformation. The pre-image and image of a dilation are similar figures.

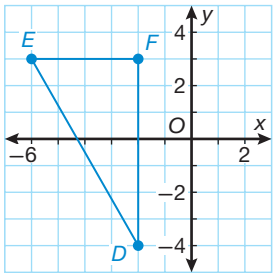
Two figures are similar if and only if the ratios of the lengths of their corresponding sides are equal and the corresponding angles are equal.

Caution

Translations, reflections, and rotations preserve the size of the pre-image. Dilations do not.

Example 4 Finding Coordinates of Vertices of Similar Triangles

Find the coordinates of the vertices of a triangle which is similar to triangle DEF where the ratios of the lengths of the corresponding sides of the triangles are $\frac{5}{2}$. Draw the similar triangle on the coordinate plane.



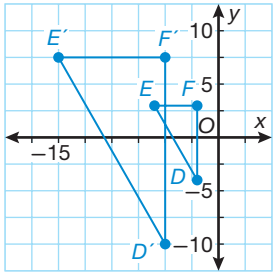
SOLUTION Find the dilation of triangle DEF using a scale factor of $\frac{5}{2}$.

$D(-2, -4)$ maps to $D'\left(\frac{5}{2} \cdot (-2), \frac{5}{2} \cdot (-4)\right) = D'(-5, -10)$

$E(-6, 3)$ maps to $E'\left(\frac{5}{2} \cdot (-6), \frac{5}{2} \cdot 3\right) = E'\left(-15, \frac{15}{2}\right)$

$F(-2, 3)$ maps to $F'\left(\frac{5}{2} \cdot (-2), \frac{5}{2} \cdot 3\right) = F'\left(-5, \frac{15}{2}\right)$

The coordinates of the triangle similar to triangle DEF are $D'(-5, -10)$, $E'\left(-15, \frac{15}{2}\right)$, and $F'\left(-5, \frac{15}{2}\right)$



Example 5 Application: Copy Machines

You want to reduce a photograph that is 8 inches by 8 inches to fit on the page of a newspaper. The photograph on the newspaper needs to be 6 inches by 6 inches. What scale factor should be used to reduce the photograph? What percentage should be used when reducing the photograph on the copy machine?

SOLUTION You are reducing a square photograph. The scale factor for the reduction is the ratio of a side length of the image to the corresponding side length of the pre-image or $\frac{6}{8}$.

To find the percentage used, write the ratio as a percent.

$$\frac{6}{8} = 75\%$$

You need to make a copy that is 75% of the original size.

Generalize

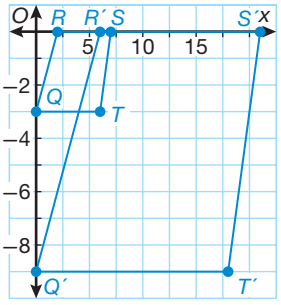
When making an enlargement on a copy machine, the percentage used will always be greater than what percent?

Lesson Practice

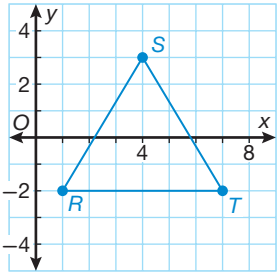
(Ex 1) **a.** Draw a dilation of a triangle with vertices $X(0, 1)$, $Y(-3, 5)$, and $Z(2, 3)$ whose center of dilation is $(0, 0)$. Use a scale factor of 1.2.

(Ex 2) **b.** Draw a dilation of a square with vertices $A(-4, 0)$, $B(0, 4)$, $C(4, 0)$, and $D(0, -4)$ whose center of dilation is $(0, 0)$. Use a scale factor of $\frac{1}{4}$

(Ex 3) **c.** Parallelogram $Q'R'S'T'$ is a dilation of $QRST$. Find the scale factor.



(Ex 4) **d.** Find the coordinates of the vertices of a triangle that is similar to triangle RST where the ratios of the lengths of the corresponding sides of the triangles are $\frac{1}{2}$.



(Ex 5) **e.** You want to enlarge a photograph that is 3 inches by 5 inches to fit on the title page of a yearbook. The photograph in the yearbook needs to be 7.5 inches by 12.5 inches. What scale factor should be used to enlarge the photograph?

What percentage should be used when enlarging the photograph on the copy machine?

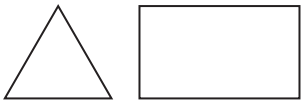
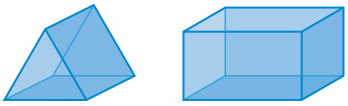
APPENDIX LESSON 6

Regions and Solids

New Concepts

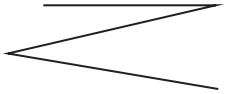
Polygons are closed figures made up of line segments called sides. **Polyhedra** are **solids** made up of polygons, where each polygon is called a **face** and the sides of the faces are called **edges.**

A polygon is a two-dimensional figure and a polyhedron is a three-dimensional figure.

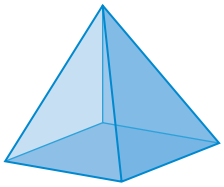
Polygons	Polyhedra
	

Example 1 Identifying Polygons and Polyhedra

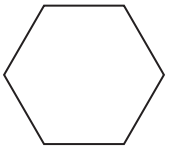
Tell if each figure is a polygon, a polyhedron, or neither.

a. 

SOLUTION The figure is two-dimensional and made up of segments, but the figure is not closed. It is neither a polygon nor a polyhedron.

b. 

SOLUTION The figure is a three-dimensional solid made up of polygons. The figure is a polyhedron.

c. 

SOLUTION The figure is a closed two-dimensional figure and all the sides are segments. The figure is a polygon.

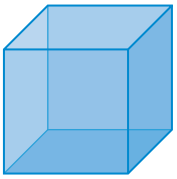
d.

SOLUTION The figure is a three-dimensional solid, but the faces are not polygons. It is neither a polygon nor a polyhedron.

Example 2 Identifying Faces and Edges

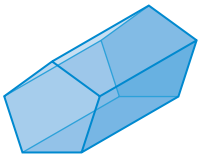
Give the number of faces and edges of each polyhedron. Identify the polygons that make up the faces.

a.



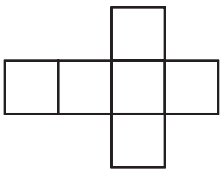
SOLUTION There are 6 faces and 12 edges. Every face is a square.

b.



SOLUTION There are 7 faces and 15 edges. Two of the faces are pentagons and five of the faces are rectangles.

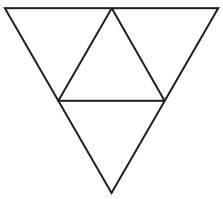
A **net** is a pattern that, when folded, forms a solid. The solid may or may not be a polyhedron. One possible net for a cube is shown to the right.



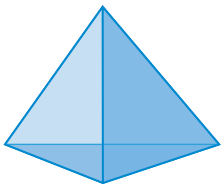
Example 3 Identifying Solids from Nets

Draw the solid from its net.

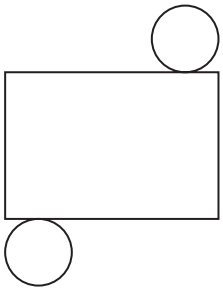
a.



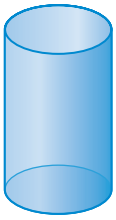
SOLUTION The middle triangle is the base, or bottom, of the figure. Fold the remaining faces up and the figure is a pyramid.



b.



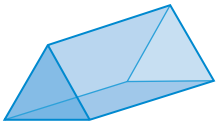
SOLUTION Bend the rectangle to join the left and right sides together. Fold the circles to form the top and bottom of a cylinder.



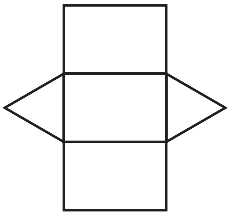
Example 4 Drawing Nets for Solids

Draw a net for the solid.

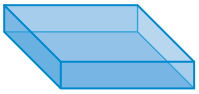
a.



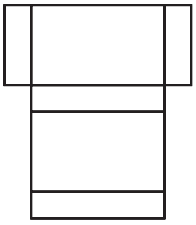
SOLUTION The solid is a polyhedron made up of three congruent rectangles and two congruent triangles.



b.



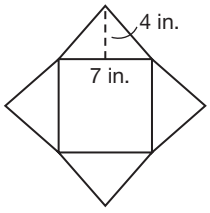
SOLUTION The solid is a rectangular prism. There are three pairs of congruent rectangles (top and bottom, front and back, left and right).



The surface area of a polyhedron is the sum of the areas of its faces.

Example 5 Finding Surface Area by Using a Net

Find the surface area of the square pyramid shown by the net.



SOLUTION

Area of the square base: $s^2 = 7^2 = 49$

Area of each triangular face: $\frac{1}{2}bh = \frac{1}{2}(7)(4) = 14$

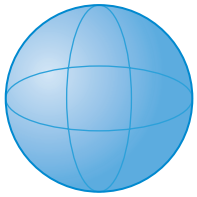
Sum of areas: $49 + 4(14) = 105$

The surface area is 105 square inches.

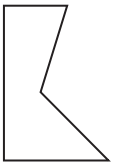
Lesson Practice

Tell if each figure is a polygon, a polyhedron, or neither.

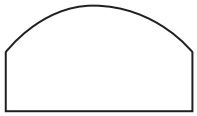
a. *(Ex 1)*



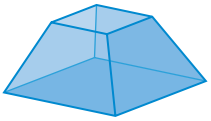
b. *(Ex 1)*



c. *(Ex 1)*

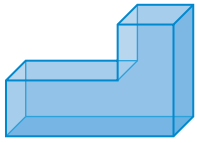


d. *(Ex 1)*

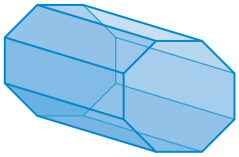


Give the number of faces and edges of each polyhedron. Identify the polygons that make up the faces.

e. *(Ex 2)*

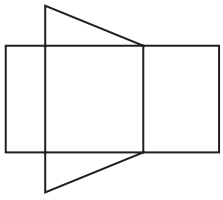


f. *(Ex 2)*

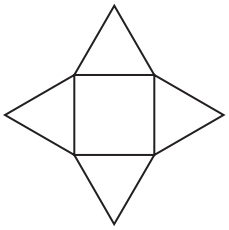


Draw the solid from its net.

g. *(Ex 3)*

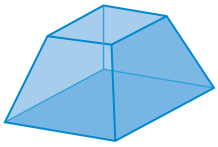


h. *(Ex 3)*

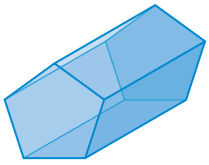


Draw a net for the solid.

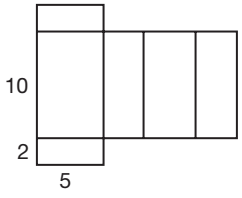
i. *(Ex 4)*



j. *(Ex 4)*



k. *(Ex 5)* Find the surface area of the rectangular prism shown by the net.



APPENDIX LESSON 7

Apply Scientific Notation

New Concepts

Scientific notation is a way of expressing very small or very large numbers in a shorthand way by using powers of 10. Numbers are written as a product, where the first factor is a decimal greater than or equal to 1 but less than 10. The second factor is a power of 10, and the exponent on the power is an integer.

Standard Notation	Scientific Notation
12,850,000,000	1.285×10^{10}
30,000,000,000,000,000	3×10^{16}
0.000000000098	9.8×10^{-11}
0.00000005002	5.002×10^{-8}

Notice that for numbers greater than 1, the exponent is positive, and for numbers between 0 and 1, the exponent is negative.

To convert from standard to scientific notation, move the decimal point after the first nonzero digit. Then count the number of places the decimal point moved. This number will be used in writing the exponent on the power of 10. The exponent is positive if the original number is greater than 1, and negative if the original number is between 0 and 1.

To convert from scientific to standard notation, move the decimal point the number of places indicated by the exponent on the power of 10. If the exponent is positive, move the decimal point to the right; if it is negative, move the decimal point to the left.

Example 1 Converting Between Standard Notation and Scientific Notation

a. Write 0.000000000239 in scientific notation.

SOLUTION Place the decimal point after the first nonzero digit to form the first factor: 2.39. Count the number of places from this place to the current decimal point: 10. Because the count was to the left, the exponent is negative: -10. The answer is 2.39×10^{-10}.

b. Write 7,300,000 in scientific notation.

SOLUTION Place the decimal point after the first nonzero digit to form the first factor: 7.3. Count the number of places from this place to the current decimal point: 6. Because the count was to the right, the exponent is positive: 6. The answer is 7.3×10^{6}.

c. Write 1.043×10^{15} in standard notation.

SOLUTION Move the decimal point 15 places to the right, adding zeros as placeholders: 1,043,000,000,000,000.

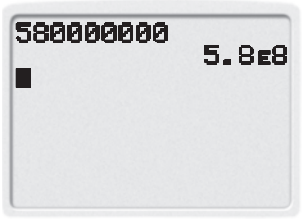
d. Write 7×10^{-11} in standard notation.

SOLUTION Move the decimal point 11 places to the left, adding zeros as placeholders. Remember that the $7 = 7.0$, so the number is 0.00000000007.

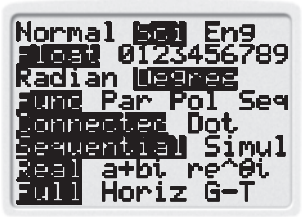
Both scientific and graphing calculators allow the user to enter numbers in scientific notation and to see results in scientific notation.

On a scientific calculator, look for the SCI mode. In this mode, the power of 10 may be displayed as an exponent to the right of the first factor.

On a graphing calculator, press the Mode key to view and change modes. The options Normal, Scientific (Sci), and Engineering (Eng) are in the first row.

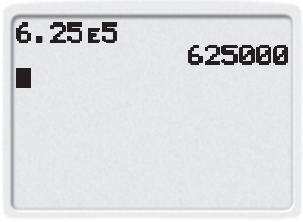


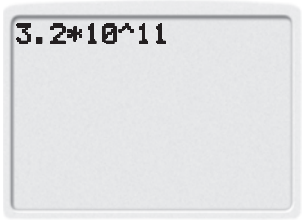
In Sci mode, when a number is entered in standard notation and the Enter key is pressed, the number is returned in scientific notation.



The E represents the "×10" part of the scientific notation expression.

A number can be entered in scientific notation, in any mode. Use the EE key to represent the "×10" part of the expression. This is above the comma key. Notice that although EE is pressed, only E appears on the screen. Scientific notation can also be entered on a calculator by using the exponent key.

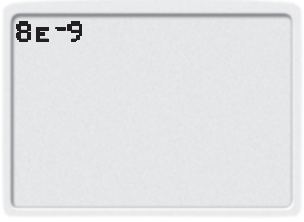




On a scientific calculator, look for an EE or EXP key to use. Otherwise, refer to the owner's manual.

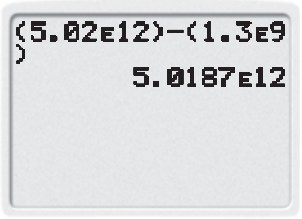
Example 2 Expressing Scientific Notation on a Graphing Calculator

a. What number, in standard notation, is displayed on the screen?



SOLUTION The number is 8×10^{-9}, which is equivalent to 0.000000008.

b. Write both the expression entered, and the answer, in standard notation.



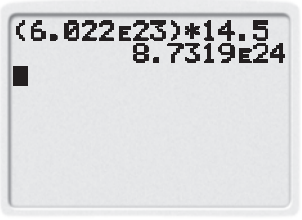
SOLUTION The expression entered was $(5.02 \times 10^{12}) - (1.3 \times 10^{9})$, or 5,020,000,000,000 − 1,300,000,000. The solution is 5.0187×10^{12}, which is 5,018,700,000,000.

Scientific notation is so named because of its role in the sciences.

Example 3 Application: Science

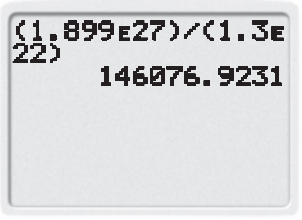
a. A mole is a standard unit in chemistry, defined to be 6.022×10^{23} molecules of a substance. How many molecules are in 14.5 moles of a given substance?

SOLUTION Multiply the number of molecules in a mole by the number of moles.



There are 8.7319×10^{24} molecules in the substance.

b. Pluto has a mass of about 1.3×10^{22} kilograms. Jupiter has a mass of about 1.899×10^{27} kilograms. About how many times greater is the mass of Jupiter than the mass of Pluto?



SOLUTION Divide the mass of Jupiter by the mass of Pluto.

The mass of Jupiter is about 146,000, or 1.46×10^{5}, times that of Pluto.

c. About twenty-nine percent, or 1.48×10^{8} square kilometers, of the earth's surface is covered in water. Approximate the total surface area of the earth. Write the answer in both scientific and standard notation.

SOLUTION

1. Understand Think of the situation as a percent problem, where the part is given and the whole is what is being asked for.

2. Plan Write a percent sentence and convert it to an equation.

1.48×10^{8} is 29% of the earth's surface.

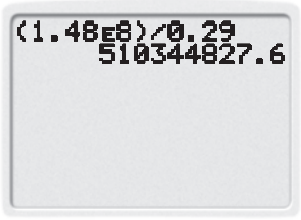
$$1.48 \times 10^{8} = 0.29x$$

3. Solve Solve for x.

$$1.48 \times 10^8 = 0.29x$$

$$\frac{1.48 \times 10^8}{0.29} = \frac{0.29x}{0.29}$$ Divide both sides by 0.29.

$$510344827.6 = x$$ Use a calculator.



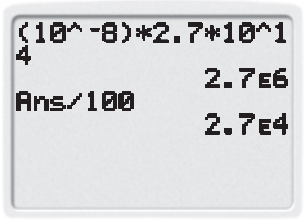
The surface area of the earth is about 510,000,000, or 5.1×10^8 square kilometers.

4. Check Find 29% of 510,000,000. It is 147,900,000 square kilometers, a number very close to that given for the amount of the earth's surface covered by water.

d. On average, an atom is about 10^{-8} centimeters in size. About how many meters long would 2.7×10^{14} atoms be, if placed in a row side by side?

SOLUTION Multiply the average size of an atom by the number of atoms. This gives the length in centimeters. To convert to meters, divide by 100.

The row would be 27,000 meters long.



Lesson Practice

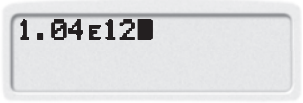
(Ex 1) **a.** Write 0.00000004 in scientific notation.

(Ex 1) **b.** Write 354,000,000,000,000 in scientific notation.

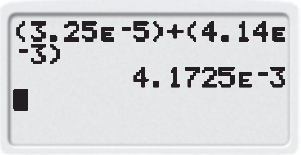
(Ex 1) **c.** Write 9.9×10^{-5} in standard notation.

(Ex 1) **d.** Write 6.02×10^9 in standard notation.

(Ex 2) **e.** What number, in standard notation, is displayed on the screen?



(Ex 2) **f.** Write both the expression entered, and the answer, in standard notation.



(Ex 3) **g.** A mole is defined to be 6.022×10^{23} molecules of a substance. How many molecules are in 8.15 moles of a given substance?

(Ex 3) **h.** Earth has a mass of about 5.974×10^{24} kilograms. Neptune has a mass of about 1.024×10^{26} kilograms. About how many times greater is the mass of Neptune than the mass of Earth?

(Ex 3) **i.** The diameter of Venus is about 1.2×10^4 kilometers, which is about 95% the diameter of Earth. Approximate the diameter of Earth. Write the answer in both scientific and standard notation.

(Ex 3) **j.** On average, an atom is about 10^{-10} meters in size. About how many kilometers long would 5×10^{10} atoms be, if placed in a row side by side?

Skills Bank

Estimation

Skills Bank Lesson 1

An **estimate** can be used to determine whether an answer is reasonable.

One way to estimate is to round some or all of the numbers to their greatest non-zero place value, then do the operation.

Example 1 **Estimating by Rounding to the Greatest Non-Zero Place Value**

Find a good estimate.

a. $48{,}304 + 2{,}349$

SOLUTION

$48{,}304 + 2{,}349 \approx 50{,}000 + 2{,}000 = 52{,}000$

A good estimate is **about 52,000.**

b. $628 \times 4{,}791$

SOLUTION

$628 \times 4{,}791 \approx 600 \times 5{,}000 = 3{,}000{,}000$

A good estimate is **about 3,000,000.**

Another way to estimate is to round the numbers to nearby numbers that make the operation easy to do. These numbers are called compatible numbers.

Example 2 **Estimating Using Compatible Numbers**

Find a good estimate.

a. $478 + 619$

SOLUTION

Rounding each number to the nearest 25 makes it easy to add.

$478 + 619 \approx 475 + 625 = 400 + 600 + 100 = 1{,}100$

A good estimate is **about 1,100.**

b. $3827 \div 595$

SOLUTION

Round to numbers that are easy to divide and will leave no remainder.

$3827 \div 595 \approx 3600 \div 600 = 6$

A good estimate is **about 6.**

Skills Bank Practice

Estimate.

a. 38×82

b. $8{,}320 - 94$

c. $0.078 \div 2$

d. $0.042 + 0.78$

e. $618 \cdot 68$

f. $3958 \div 492$

g. $906 + 378$

h. 439×87

i. 4023×50

j. $9387 - 1959$

k. $8{,}374 + 3{,}305 + 91$

l. $948 - 298$

m. $402 \div 95$

n. $6{,}306 \div 928$

o. $38 \times 5{,}820$

p. $4{,}503 - 581$

q. 298×682

r. $4.982 - 0.593$

Mental Math

Skills Bank Lesson 2

Use these mental math strategies to help you add, subtract, multiply, and divide.

Compensation: When adding or subtracting, change one number, then make up for it later.

Example 1 Using Compensation to Add

Add: $47 + 28$

SOLUTION

$47 + 28$	Change 47 to 50 by adding 3.
$50 + 28 = 78$	Add.
$78 - 3 = 75$	Subtract 3 from the sum.

Equal Additions: When subtracting, move each number up or down the number line by the same amount.

Example 2 Using Equal Additions to Subtract

Subtract: $91 - 25$

SOLUTION

Add 5 to each number.

$91 - 25 = 96 - 30 = 66$

Real Number Properties: Use the Associative, Commutative, and/or Distributive Properties.

Example 3 Using Real Number Properties with Mental Math

a. Commutative and Associative Properties:
$6 \times 9 \times 5 = (6 \times 5) \times 9 = 30 \times 9 = 270$

b. Distributive Property:
$4 \times 27 = (4 \times 20) + (4 \times 7) = 80 + 28 = 108$

c. Associative Property:
$(15 + 17) + 13 = 15 + (17 + 13) = 15 + 30 = 45$

Skills Bank Practice

Use mental math to evaluate.

a. $42 + 19 + 8$	**b.** 8×71	**c.** $63 - 28$	**d.** $75 + 17$
e. $6 \times 12 \times 5$	**f.** $514 - 298$	**g.** $3.2 + 2.5 + 4.5$	**h.** 4×241
i. 7×81	**j.** $45 + 92$	**k.** $138 - 29$	**l.** $32 + 78$
m. $949 + 111$	**n.** 7×26	**o.** $14 + 91 + 6$	**p.** $2 \times 18 \times 5$
q. $482 - 197$	**r.** $4 \times 3 \times 15$	**s.** $77 + 48$	**t.** $2 \times 7 \times 25 \times 2$
u. $57 + 245$	**v.** 8×32	**w.** $92 - 47$	**x.** 14×7

Exponents

Skills Bank Lesson 3

A **power** is an expression that shows repeated multiplication. In a power, an **exponent** shows how many times a **base** is used as a factor, as shown below:

$$\text{Base} \longrightarrow a^2 \longleftarrow \text{Exponent}$$
$$= a \cdot a$$

Example 1

Simplify.

3^4

SOLUTION

$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$

You can use the following rules to simplify expressions with exponents.

$$a^0 = 1 \qquad a^1 = a \qquad -a^n = -(a^n)$$

When n is even, $(-a)^n = a^n$.

Using the Rules of Exponents

Example 2

Simplify.

5^0

SOLUTION

$5^0 = 1$

Example 3

Simplify.

-4^2

SOLUTION

$-4^2 = -(4^2)$

$= -(4 \cdot 4) = -16$

Example 4

Simplify.

$(-3)^3$

SOLUTION

$(-3)^3 = (-3) \cdot (-3)^2$

$= (-3) \cdot 3^2 = (-3) \cdot 3 \cdot 3$

$= (-3) \cdot 9 = -27$

Skills Bank Practice

Simplify each expression.

a. 3^2 **b.** 9^0 **c.** 17^1 **d.** 5^4

e. -1^5 **f.** 0^8 **g.** $(-4)^3$ **h.** -8^1

i. $(-2)^6$ **j.** -5^0 **k.** $(3x)^2$ **l.** $(-3)^3 + 4^2$

m. $(-3)^3 - (-3)^2$ **n.** $(-x)^4 + (2x)^2$ **o.** $-7^0 + (-2)^2$

p. $(-9)^1 + (-9)^0$ **q.** $7^3 + (-7)^3$ **r.** $7 - 8^1$

Expand each expression.

s. 11^3 **t.** -6^7

u. $(-13)^5$ **v.** $-(-2)^3$

w. 4^8 **x.** $(-5)^2$

Operations with Decimals

Skills Bank Lesson 4

Operations with Decimals	
To **add** or **subtract:**	Align decimal points. Bring the decimal point directly down into the answer.
To **multiply:**	Multiply as with whole numbers. The number of decimal places in the product is the same as the total number of decimal places in the factors.
To **divide:**	Multiply dividend and divisor by the power of 10 that makes the divisor a whole number. Divide as with whole numbers, placing the decimal point in the quotient directly above the decimal point in the dividend.

Example 1 Adding Decimals

Add: $3.25 + 6.1$

SOLUTION

$$\begin{array}{r} 3.25 \\ +\ 6.1 \\ \hline 9.35 \end{array}$$

Example 2 Subtracting Decimals

Subtract: $4.083 - 2.96$

SOLUTION

$$\begin{array}{r} {\scriptstyle 3\ 10} \\ 4.\not{0}\ 8\ 3 \\ -\ 2.9\ 6 \\ \hline 1.1\ 2\ 3 \end{array}$$

Example 3 Multiplying Decimals

Multiply: 6.23×0.8

SOLUTION

$$\begin{array}{r} 6.23 \leftarrow \text{2 decimal places} \\ \times\ 0.8 \leftarrow \text{1 decimal place} \\ \hline 4.984 \leftarrow \text{3 decimal places} \end{array}$$

Example 4 Dividing Decimals

Divide: $0.1922 \div 0.62$

SOLUTION

$$0.62\overline{)0.1922} \longrightarrow$$

$$(0.62 \times 100)\overline{)(0.1922 \times 100)}$$

$$\begin{array}{r} 0.31 \\ 62\overline{)19.22} \\ -186 \\ \hline 62 \\ -62 \\ \hline 0 \end{array}$$

Skills Bank Practice

Simplify.

a. $4.6 + 3.92$	**b.** $(2.5)(1.5)$	**c.** $2.4 \div 0.08$	**d.** $3.05 - 1.6$
e. 4.9×2.27	**f.** $3.6 + 4.12$	**g.** $0.105 - 0.06$	**h.** $0.054 \div 0.36$
i. 0.2×3.8	**j.** $2.4 + 8.03$	**k.** $60 \div 0.04$	**l.** 5.25×8
m. $0.98 + 0.35$	**n.** $4.074 \div 1.4$	**o.** 3.6×0.4	**p.** $6.52 - 2.74$
q. $4.872 - 0.084$	**r.** 32.4×18.9	**s.** $2.334 \div 0.04$	**t.** $3.1 + 4.82$

Compare and Order Rational Numbers

Skills Bank Lesson 5

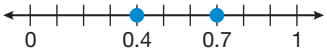
To compare and order rational numbers, convert them to the same form (fraction, decimal, percent). Graph each number on a number line.

Example 1 **Compare $\frac{2}{5}$ and 0.7.**

Step 1: Rewrite the fraction as a decimal to solve:

$\frac{2}{5} = 2 \div 5 = 0.4$

Step 2: Graph each of the numbers on a number line

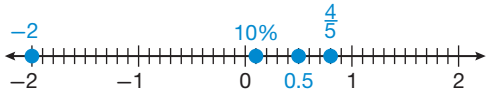


Step 3: Compare using $<$, $>$, or $=$.

Since 0.4 is further on the left of the number line than 0.7, $0.4 < 0.7$ which means $\frac{2}{5} < 0.7$

Example 2 **Write 0.5, -2, $\frac{4}{5}$, and 10% in order from least to greatest.**

Order the numbers from least to greatest using a number line.



In order from least to greatest, the numbers are **-2, 10%, 0.5, $\frac{4}{5}$.**

Skills Bank Practice

Compare using $<$, $>$, or $=$.

a. $\frac{1}{2} \square 0.3$ **b.** $0.65 \square \frac{13}{20}$ **c.** $-3 \square -\frac{45}{20}$ **d.** $\frac{2}{5} \square 25\%$

e. $17.5\% \square \frac{7}{40}$ **f.** $\frac{6}{5} \square 1.5$ **g.** $\frac{3}{4} \square \frac{5}{8}$ **h.** $1 \square 100\%$

i. $-\frac{15}{7} \square -2$ **j.** $1.27 \square \frac{37}{13}$ **k.** $\frac{23}{40} \square 0.575$ **l.** $0.87 \square 75\%$

m. $\frac{2}{17} \square 17\%$ **n.** $23\% \square \frac{5}{28}$ **o.** $-\frac{14}{23} \square -0.45$ **p.** $0.034 \square 34\%$

Write in order from least to greatest.

q. $0.6, 1.4, 30\%, \frac{15}{21}$

r. $-5, -\frac{27}{7}, -0.01, -0.45$

s. $0.7, 40\%, \frac{3}{5}, \frac{1}{8}$

t. $0.56, 0.65, 55\%, \frac{2}{3}$

u. $1.7, 50\%, \frac{46}{90}, 0.05$

v. $-1.5, -\frac{63}{20}, -\frac{57}{40}, -2.3$

Operations with Fractions

Skills Bank Lesson 6

A **fraction** names part of a whole.

Operations with Fractions	
To **add** or **subtract:**	Write equivalent fractions with a common denominator. Add or subtract the numerators. The denominator of the sum or difference is the same as the common denominator.
To **multiply:**	The numerator of the product is the product of the numerators. The denominator of the product is the product of the denominators.
To **divide:**	To divide by a fraction, multiply by its reciprocal.

Example 1 Adding Fractions

Add: $\frac{1}{2} + \frac{4}{5}$

SOLUTION

$$\frac{1}{2} + \frac{4}{5} = \frac{5}{10} + \frac{8}{10} = \frac{13}{10} = 1\frac{3}{10}$$

Example 2 Subtracting Fractions

Subtract: $\frac{3}{8} - \frac{1}{6}$

SOLUTION

$$\frac{3}{8} - \frac{1}{6} = \frac{9}{24} - \frac{4}{24} = \frac{5}{24}$$

Example 3 Multiplying Fractions

Multiply: $\frac{2}{5} \times \frac{3}{8}$

SOLUTION

$$\frac{2}{5} \times \frac{3}{8} = \frac{2 \times 3}{5 \times 8} = \frac{6}{40} = \frac{3}{20}$$

Example 4 Dividing Fractions

Divide: $\frac{3}{7} \div \frac{4}{5}$

SOLUTION

$$\frac{3}{7} \div \frac{4}{5} = \frac{3}{7} \times \frac{5}{4} = \frac{3 \times 5}{7 \times 4} = \frac{15}{28}$$

Skills Bank Practice

Simplify. Write the answer in lowest terms.

a. $\frac{2}{5} + \frac{1}{10}$ **b.** $\frac{3}{8} - \frac{1}{5}$ **c.** $\frac{1}{6} \cdot \frac{4}{5}$

d. $\frac{5}{6} \div \frac{1}{8}$ **e.** $\frac{7}{9} - \frac{3}{5}$ **f.** $\frac{11}{12} \times \frac{3}{55}$

g. $\frac{7}{15} + \frac{2}{3}$ **h.** $\frac{12}{17} \div \frac{3}{8}$ **i.** $\frac{7}{8} \times \frac{1}{3}$

j. $6 \div \frac{3}{4}$ **k.** $\frac{5}{8} + \frac{4}{5}$ **l.** $6\frac{1}{8} - 2\frac{5}{6}$

m. $1\frac{2}{3} \times 6\frac{1}{8}$ **n.** $7\frac{3}{8} \div 6$ **o.** $10 - 3\frac{7}{12}$

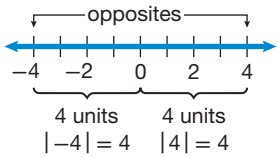
p. $5\frac{1}{4} + 8\frac{7}{8}$ **q.** $\frac{8}{19} - \frac{7}{38} + \frac{3}{19}$ **r.** $\frac{14}{26} \cdot \frac{13}{49} \cdot \frac{7}{10}$

s. $2\frac{2}{5} + 9\frac{1}{5} - 7\frac{3}{5}$

Negative Numbers and Operations with Integers

Skills Bank Lesson 7

A **negative number** is less than zero. The **absolute value** of a number is its distance from zero. **Opposite** numbers lie the same distance from zero, but in different directions.



To add numbers with the same sign, add their absolute values. The sign of the sum is the same as the sign of the addends. **Examples:** $4 + 9 = 13$ $-8 + -2 = -10$
To add numbers with different signs, subtract their absolute values. The sign of the sum is the same as the sign of the number with greater absolute value. **Examples:** $4 + (-7) = -3$ $-9 + 4 = -5$
To subtract a number, add its opposite. **Examples:** $6 - 9 = 6 + (-9) = -3$ $7 - (-6) = 7 + (+6) = 13$ $-5 - 8 = -5 + (-8) = -13$
To multiply two numbers, multiply their absolute values. The product of two numbers with the same sign is positive. The product of two numbers with different signs is negative. **Examples:** $-8 \times -3 = 24$ $9 \times -2 = -18$
To divide two numbers, divide their absolute values. The quotient of two numbers with the same sign is positive. The quotient of two numbers with different signs is negative. **Examples:** $-40 \div -5 = 8$ $10 \div -2 = -5$

Skills Bank Practice

Simplify.

a. $5 + 12$	**b.** $48 \div -6$	**c.** $7(-5)$	**d.** $1.6 \cdot -2$
e. $32 \div 4$	**f.** $4.3 + (-8.1)$	**g.** $4 - 9$	**h.** $(-2) \cdot (-9)$
i. $-42 + -12$	**j.** -12×4	**k.** $-540 \div -0.9$	**l.** $100 - (-4)$
m. -10×-14	**n.** $-81 \div 9$	**o.** $-6 + 13$	**p.** $16 - (-3)$

Ratios, Proportions, Percents

Skills Bank Lesson 8

A **ratio** is a comparison of two numbers by division. A **proportion** shows that two ratios are equal. In a proportion, **cross-products** are equal. For example, $\frac{2}{3} = \frac{4}{6}$ and the cross-products, $2 \cdot 6 = 3 \cdot 4 = 12$.

Example 1 Solving a Proportion

Solve for x. $\frac{8}{13} = \frac{x}{117}$

SOLUTION Cross-multiply to find the cross-products. Then, solve for x.

$$\frac{8}{13} = \frac{x}{117}$$

$$(13)x = (8)(117)$$

$$13x = 936$$

$$x = 72$$

A **percent** is a ratio that compares a number to 100.

Example 2 Solving Problems with Percents

a. What percent of 112 is 44.8?

SOLUTION Write and solve a proportion:

$$\frac{44.8}{112} = \frac{x}{100}$$

$$112x = 4480$$

$$x = 40$$

44.8 is 40% of 112.

b. What is 80% of 60

SOLUTION Write and solve a proportion:

$$\frac{x}{60} = \frac{80}{100}$$

$$100x = 4800$$

$$48 = x$$

48 is 80% of 60.

Skills Bank Practice

a. Write a proportion to find 5% of 10.

b. Solve for x. $\frac{19}{x} = \frac{57}{12}$

c. What is 11% of 14?

d. What percent of 80 is 3?

e. Find the cross-product to determine if the proportion is true or false. $\frac{7}{9} = \frac{84}{108}$

f. What percent of 300 is 81?

g. What is 25% of 88?

h. Solve for x. $\frac{23}{x} = \frac{46}{50}$

Time, Rate, Distance

Skills Bank Lesson 9

When an object moves in a straight path with constant speed, the distance (d) it travels is the product of the rate (r) and the time traveled (t). Use the formula $\boldsymbol{d = rt}$ to solve problems.

Example 1 Using $d = rt$

A cyclist travels 16.5 miles in 1.5 hours. What is her average rate of speed?

SOLUTION

$$d = rt$$

$$16.5 = r \cdot 1.5$$

$$\frac{16.5}{1.5} = r$$

$$11 = r$$

The cyclist's average rate of speed is 11 miles per hour.

Example 2 Using $d = rt$ to Solve Problems

Two trains leave a station traveling in opposite directions. The slower train travels 10 mph slower than the faster train. If the trains are 325 miles apart after 2.5 hours, what is the speed of each train?

SOLUTION

Use a table to organize the information.

	r	t	d
Faster Train	r	2.5	$2.5r$
Slower Train	$r - 10$	2.5	$2.5(r - 10)$

Write an equation: $2.5r + 2.5(r - 10) = 325$

$$2.5r + 2.5r - 25 = 325$$

$$5r - 25 = 325$$

$$5r = 350$$

$$r = 70$$

The faster train's rate is 70 mph. The slower train's rate is $70 - 10$, or 60 mph.

Skills Bank Practice

a. Find d when $r = 7$ mph and $t = 6$ hours.

b. Find t when $r = 3.5$ mph and $d = 34.3$ miles.

c. Mark drove 312 miles to the beach. The trip took 6.25 hours. What was Mark's average rate of speed? Round to the nearest whole number.

d. Two runners leave a gym at 10:00 AM and run in opposite directions. One runs at 6 mph and the other runs at 5 mph. How far apart will the runners be at 10:30 AM?

e. A runner training for a marathon leaves home running at 9 mph. At a certain point, he turns around and runs along the same route back home at 6 mph. If the entire run lasts 2.5 hours, how far has the runner run altogether?

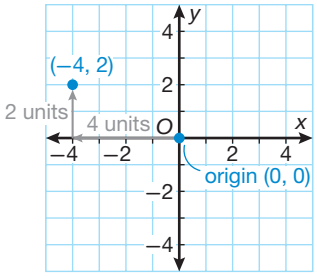
Coordinate Plane/Ordered Pairs

Skills Bank Lesson 10

The **coordinate plane** is formed by the x-axis and the y-axis. Their point of intersection is called the **origin.** Each location on the coordinate plane can be described using an **ordered pair,** which describes its location relative to the origin.

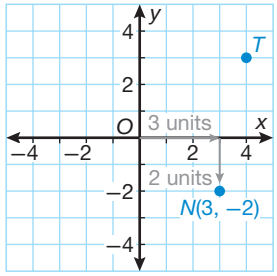
An ordered pair is written in the form (x, y), where x is the point's ***x*-coordinate** and y is its ***y*-coordinate.**

For the point $(-4, 2)$, the coordinates indicate that the point is located 4 units to the left of the origin, and 2 units up from the origin.



Example: Graph the point $N(3, -2)$.

- Start at the origin.
- Move 3 units to the right because the x-coordinate is $+3$.
- Move 2 units down because the y-coordinate is -2.
- Draw a point and label it N.



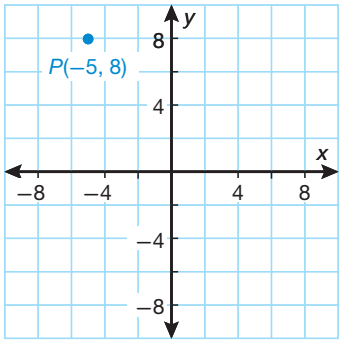
Example: Give the coordinates of T

- Start at the origin.
- Count to the right 4 units. The x-coordinate is 4.
- Count up 3 units. The y-coordinate is 3.
- The coordinates of T are $(4, 3)$.

Skills Bank Practice

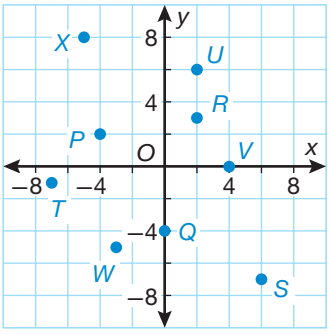
Graph each point on the coordinate plane like point $P(-5, 8)$ shown.

a. $A(2, 0)$
b. $B(-3, 1)$
c. $C(2, -4)$
d. $D(0, 2)$
e. $E(5, 7)$
f. $F(6, -1)$
g. $G(-3, -2)$
h. $H(8, 6)$
i. $I(2, 7)$
j. $J(-5, -6)$
k. $K(5, -2)$
l. $L(-6, -2)$
m. $M(4, 1)$
n. $N(-6, 0)$
o. $O(1, 4)$



Name the coordinates of each point.

p. P
q. Q
r. R
s. S
t. T
u. U
v. V
w. W
x. X

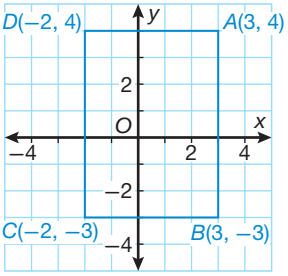


SKILLS BANK

Plane Figures and Coordinate Geometry

Skills Bank Lesson 11

Graph a plane figure on the coordinate plane by plotting and labeling its vertices.



Example 1 Identifying a Figure Using its Vertices

Describe polygon $ABCD$ with $A(3, 4)$, $B(3, -3)$, $C(-2, -3)$, and $D(-2, 4)$.

SOLUTION

Graph each point, then connect the vertices. Polygon $ABCD$ is a rectangle.

The **midpoint** of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is located at $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

The **distance,** d, between points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 2 Classifying a Triangle

Given $A(7, 1)$, $B(3, -4)$, and $C(-1, 1)$, classify ΔABC as scalene, isosceles, or equilateral.

SOLUTION

Use the distance formula to find the length of each side.

$$AB = \sqrt{(3 - 7)^2 + (-4 - 1)^2} = \sqrt{(-4)^2 + (-5)^2} = \sqrt{16 + 25} = \sqrt{41}$$

$$BC = \sqrt{(-1 - 3)^2 + (1 - (-4))^2} = \sqrt{(-4)^2 + (5)^2} = \sqrt{16 + 25} = \sqrt{41}$$

$$AC = \sqrt{(-1 - 7)^2 + (1 - 1)^2} = \sqrt{(-8)^2 + (0)^2} = \sqrt{64 + 0} = \sqrt{64} = 8$$

The triangle has two congruent sides. So, ΔABC is isosceles.

Skills Bank Practice

For a and b, graph the polygon with the given vertices. Describe the polygon.

a. $(-4, 3)$, $(4, -5)$, and $(4, 3)$

b. $(-4, -2)$, $(2, 2)$, $(5, 0)$, $(-1, -4)$

c. A square has vertices at $(-2, -2)$, $(1, 1)$ and $(4, -2)$. Name the coordinates of its fourth vertex.

d. A segment has endpoints at $(0, -4)$ and $(3, -7)$. Find the coordinates of the midpoint. Find the length of the segment.

e. Given $A(4, 2)$, $B(0, 9)$, and $C(-4, 2)$, classify ΔABC as scalene, isosceles, or equilateral.

f. A quadrilateral has vertices at $(-1, 1)$, $(2, 6)$, $(5, 1)$, and $(2, -4)$. Show that the quadrilateral is a rhombus.

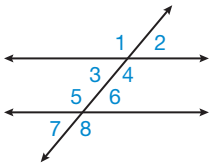
g. A circle has diameter with endpoints $(-8, 1)$ and $(-3, 13)$. Find the center of the circle. Find the area of the circle.

h. Given $R(4, 3)$, $S(0, 9)$, and $T(-4, 2)$, find the length of the segment with one endpoint at T and the other at the midpoint of $\overline{RS}$.

Parallel Lines and Transversals

Skills Bank Lesson 12

When a **transversal** intersects parallel lines, certain pairs of angles are congruent.



Corresponding angles lie in the same position relative to the parallel lines and the transversal. For example, $\angle 5 \cong \angle 1$.

Corresponding angles have the same measure. For example, $m\angle 5 = m\angle 1$.

Alternate interior angles lie between the parallel lines and on opposite sides of the transversal. For example, $\angle 3 \cong \angle 6$.

Alternate interior angles have the same measure. For example, $m\angle 3 = m\angle 6$.

Alternate exterior angles lie outside the parallel lines and on opposite sides of the transversal. For example, $\angle 1 \cong \angle 8$.

Alternate interior angles have the same measure. For example, $m\angle 1 = m\angle 8$.

Example 1 Identifying Angle Relationships

Use the diagram above to idenitfy the relationship between the given angles.

a. $\angle 2$ and $\angle 6$

SOLUTION

The angles lie to the right of the transversal, above the lines.

$\angle 2$ and $\angle 6$ are corresponding angles.

b. $\angle 7$ and $\angle 2$

SOLUTION

The angles lie outside the parallel lines, on opposite sides of the transversal.

$\angle 7$ and $\angle 2$ are alternate exterior angles.

c. $\angle 5$ and $\angle 4$

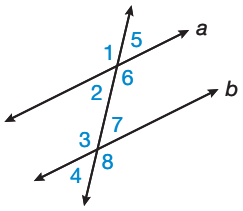
SOLUTION

The angles lie inside the parallel lines, on opposite sides of the transversal.

$\angle 7$ and $\angle 2$ are alternate interior angles.

Skills Bank Practice

Use the diagram at right. Lines *a* and *b* are parallel.



a. $m\angle 2 = m\angle$ _____ because they are corresponding angles.

b. $m\angle 8 = m\angle$ _____ because they are alternate exterior angles.

c. $m\angle 3 = m\angle$ _____ because they are alternate interior angles.

d. $m\angle 4 = m\angle 5$ because they are _____ angles

e. $m\angle 2 = m\angle 7$ because they are _____ angles

f. $m\angle 5 = m\angle 7$ because they are _____ angles

g. If $m\angle 3 = 135°$, find the measure of each numbered angle.

Angle Measurement

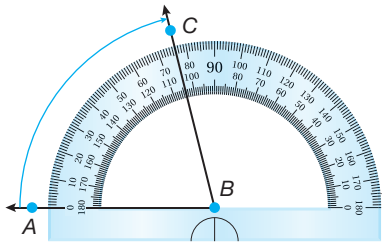
Skills Bank Lesson 13

Angles are measured in degrees (°). Use a **protractor** to find the measure of an angle.

Example Using a Protractor to Measure an Angle

Use a protractor to find the measure of $\angle ABC$.

SOLUTION

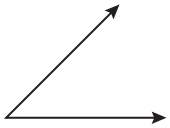


The measure of $\angle ABC$ is 76°. $m\angle ABC = 76°$.

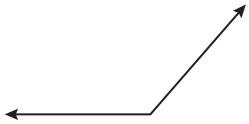
Skills Bank Practice

Use a protractor to measure each angle.

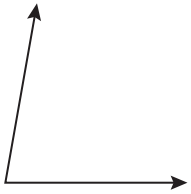
a.



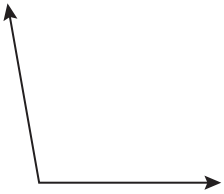
b.



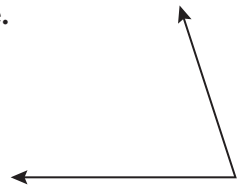
c.



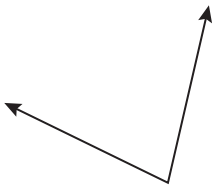
d.



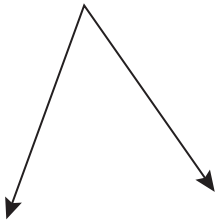
e.



f.

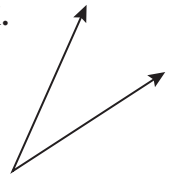


g.



h.

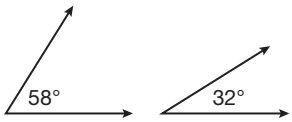
i.



Angle Relationships

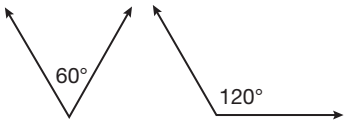
Skills Bank Lesson 14

Two angles are **complementary** if the sum of their measures is 90°.



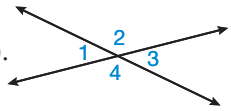
Since 58° + 32° = 90°, these angles are complementary.

Two angles are **supplementary** if the sum of their measures is 180°.



Since 60° + 120° = 180°, these angles are supplementary.

Vertical angles are formed when two lines intersect. They share a vertex. Vertical angles are congruent (they have the same measure).

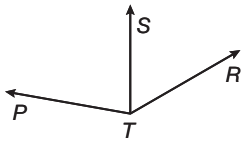


∠1 and ∠3 are vertical angles. m∠1 = m∠3

∠2 and ∠4 are vertical angles. m∠1 = m∠4

Adjacent angles share a vertex and a side, and do not overlap.

∠*PTS* and ∠*RTS* are adjacent angles. They share vertex *T* and side $\overrightarrow{TS}$.



Skills Bank Practice

Find the measure of an angle that is complementary to the angle with the given measure.

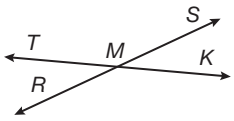
a. 30° **b.** 88° **c.** 17° **d.** 47°

Find the measure of an angle that is supplementary to the angle with the given measure.

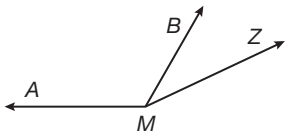
e. 19° **f.** 122° **g.** 163° **h.** 81°

Complete.

i. ∠*SMK* and ∠_____ are vertical angles.

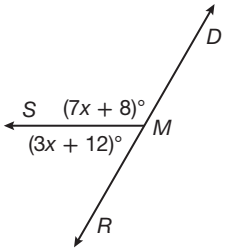


j. ∠_____ and ∠_____ are adjacent.

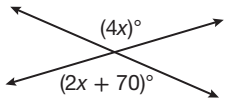


Solve for *x*.

k. ∠*RMS* and ∠*SMD* are supplementary.



l.



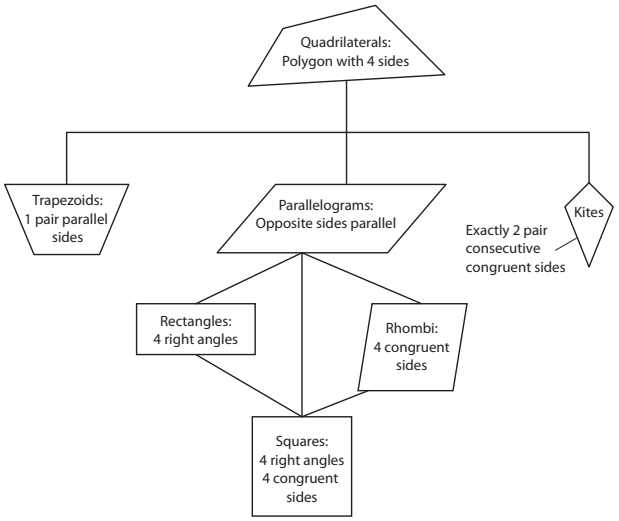
SKILLS BANK

Properties of Polygons

Skills Bank Lesson 15

A **polygon** is a closed plane figure that is made up of line segments called **sides.** Each side intersects exactly two others, at its endpoints. These endpoints are the **vertices** of the polygon. In an **equilateral** polygon, all sides are congruent. In an **equiangular** polygon, all interior angles are congruent. In a **regular** polygon, all sides are congruent and all interior angles are congruent.

Polygon	Number of Sides
Triangle	3
Quadrilateral	4
Pentagon	5
Hexagon	6
Heptagon	7
Octagon	8
Nonagon	9
Decagon	10
Hendecagon	11
Dodecagon	12
n-gon	n



The sum of the measures of each interior angle in an n-sided convex polygon is $180°(n - 2)$.

Example 1 Finding the Measure of Each Interior Angle of a Polygon

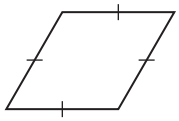
Find the measure of each interior angle of a regular hexagon.

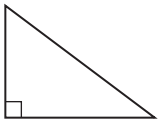
SOLUTION

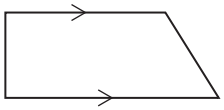
A hexagon has 6 sides, so $n = 6$. The sum of the angle measures is $180°(6 - 2)$, or $720°$. Each interior angle in a regular hexagon is congruent. So, each interior angle measures $720° \div 6$, or $120°$.

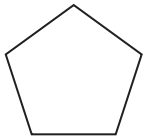
Skills Bank Practice

For a–d, give all the names that apply to each figure.

a. 

b. 

c. 

d. 

e. What is the sum of the interior angle measures in a pentagon?

f. An octagon has angles measuring 100°, 135°, 130°, 145°, 115°, and 105°. What is the measure of each of the other two angles if they have the same measure?

g. What is the measure of each interior angle in a regular decagon?

Geometric Formulas

Skills Bank Lesson 16

Use these formulas for area (A) and volume (V) using base area (B), perimeter (P), radius (r), length (l), width (w), and height (h). (Note: in the surface area of a regular pyramid P is the perimeter of the base, and l is the slant height of the pyramid.)

	Rectangular Prism	Cylinder	Sphere	Regular Pyramid
Volume	Bh	$\pi r^2 h$	$\frac{4}{3}\pi r^3$	$\frac{1}{3}Bh$
Surface Area	$2lw + 2wh + 2lh$	$2\pi r(r + h)$	$4\pi r^2$	$B + \frac{1}{2}Pl$

Example 1 Using a Volume Formula

Find the volume of a rectangular prism that has a length of 6 cm, width of 3 cm, and a height of 5 cm.

SOLUTION

$V = 6 \cdot 3 \cdot 5$

$= 90 \text{ cm}^3$

Example 2 Using a Surface Area Formula

Find the surface area of a rectangular prism that has a length of 14 cm, a height of 10 cm, and a base with an area of 168 cm^2.

SOLUTION

$SA = 2 \cdot 14 \cdot \frac{168}{14} + 2 \cdot \frac{168}{14} \cdot 10 + 2 \cdot 14 \cdot 10$

$= 336 + 240 + 280$

$= 856 \text{ cm}^3$

Skills Bank Practice

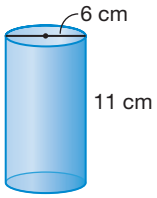
Find each measure. Round to the nearest hundredth.

a. Find the volume of a cylinder with a circumference of 14 m and a height of 7m.

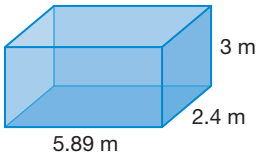
b. Find the radius of a sphere with a volume of 113.1 in^3.

c. Find the surface area of a regular pyramid with a slant height of 23 cm and a square base that has a length of 15 cm.

d. Find the surface area.



e. Find the volume.



f. Find the height a regular pyramid with a base area of 12 in.^2 and a volume of 32 in.^3.

g. Find the volume of a regular pyramid with an octagonal base that has an area of 27 ft^2 and a height of 7 ft.

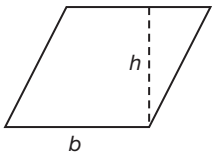
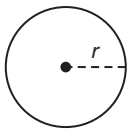
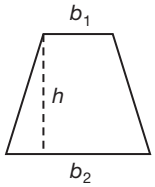
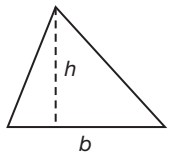
h. Find the radius of a cylinder with a height of 8 cm and a surface area of $96\pi \text{ cm}^2$.

i. Find the volume of a cylinder with radius and height of 9 cm and a sphere with a radius of 9 cm. Which is larger? By how much?

Area of Polygons, Circles, and Composite Figures

Skills Bank Lesson 17

Use formulas to find areas of figures. For a figure that is made up of one or more other shapes (a **composite** figure), break the figure down into basic shapes, then find the area of each one. Use the table of area formulas for some common polygons, to find the area of the composite figures.

Parallelogram	Circle	Trapezoid	Rectangle	Triangle
				
$A = bh$	$A = \pi r^2$	$A = \frac{1}{2}(b_1 + b_2)h$	$A = lw$	$A = \frac{1}{2}bh$

Example 1 Finding the Area of a Figure

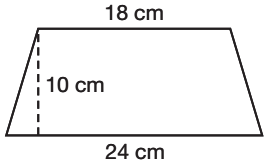
Find the area of each figure.

SOLUTION

$$A = \frac{1}{2}(b_1 + b_2)h$$

$$= \frac{1}{2}(18 + 24)10$$

$$= 210 \text{ cm}^2$$



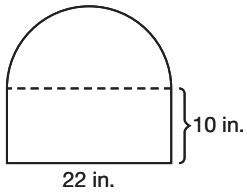
SOLUTION

$$A = lw + \frac{1}{2}\pi r^2$$

$$\approx 22 \cdot 10 + \frac{1}{2} \cdot 3.14 \cdot 11^2$$

$$\approx 220 + 189.97$$

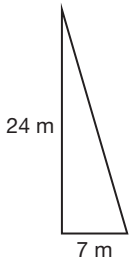
$$\approx 409.97 \text{ in}^2$$



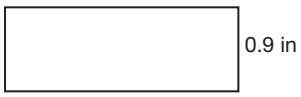
Skills Bank Practice

Find the area of each figure. (Use 3.14 for π.)

a.



b.



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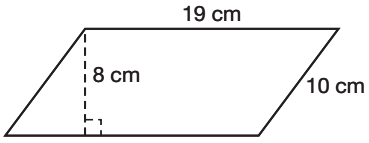

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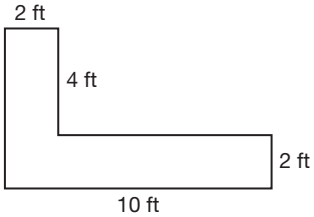

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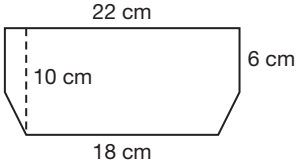

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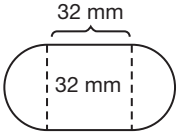
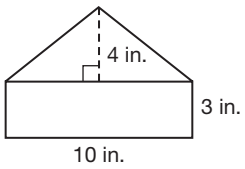


g.



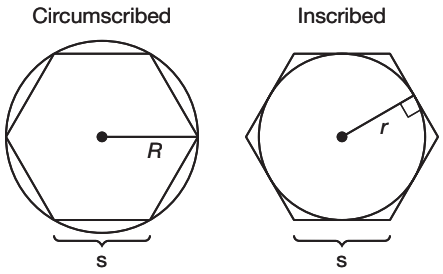
Angle Relationships in Circles and Polygons

Skills Bank Lesson 18

A **regular** polygon has congruent interior angles and congruent sides. The measure of each interior angle of a regular polygon with n sides is $\left(\frac{n-2}{n}\right) \cdot 180$.

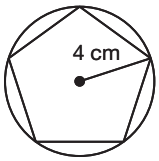
These formulas apply to a circle of radius R circumscribed and a circle of radius r inscribed in a regular n-gon.

$$s = 2R\sin\left(\frac{180°}{n}\right) \qquad r = R\cos\left(\frac{180°}{n}\right)$$



Example 1 Finding Side Length Using Circumscribed Circles

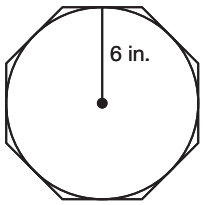
A circle with radius 4 cm is circumscribed around a regular pentagon. Find the length of each side of the pentagon. Round to the nearest hundredth.



SOLUTION $s = 2R\sin\left(\frac{180°}{n}\right) = 2 \cdot 4 \cdot \sin\left(\frac{180°}{5}\right) = 8 \cdot \sin(36°) \approx 4.70$ cm

Example 2 Finding Side Length Using Inscribed Circles

A circle with radius 6 inches is inscribed in a regular octagon. Find the length of each side of the octagon. Round to the nearest hundredth.



SOLUTION

Step 1: Find the value of R.

$$6 = R\cos\left(\frac{180°}{8}\right)$$

$$R = \frac{6}{\cos\left(\frac{180°}{8}\right)}$$

$$R \approx 6.49$$

Step 2: Find the value of s.

$$s = 2R\sin\left(\frac{180°}{n}\right)$$

$$s = 2 \cdot 6.49 \cdot \sin\left(\frac{180°}{8}\right)$$

$$s = 12.98 \cdot \sin(22.5°)$$

$$s \approx 4.97 \text{ in.}$$

The length of each side of the octagon is approximately 4.97 inches.

Skills Bank Practice

Find the measure of each interior angle of the given regular polygon. Round to the nearest hundreth.

a. A circle with radius 7 m is inscribed in a regular hexagon. Find the length of each side of the hexagon.

b. A circle with radius 10 ft is circumscribed around a regular triangle. Find the radius of the circle that can be inscribed in the same regular triangle.

c. A circle with radius 5 cm is inscribed in a square. Find the length of each side of the square.

d. A circle with radius 30 inches is circumscribed around a regular decagon. Find the radius of the circle that can be inscribed in the same regular decagon.

e. A circle is inscribed in a regular heptagon with side length 4 meters. Find the length of the radius of the circle.

Views of Solid Figures

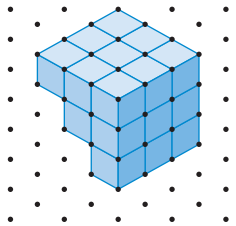
Skills Bank Lesson 19

An **isometric** drawing is a way of drawing a three dimensional figure using isometric dot paper, which has equally spaced dots in a repeating triangular pattern.

Example 1 Drawing an Isometric View of a Figure

Draw an isometric view of the figure shown.

SOLUTION

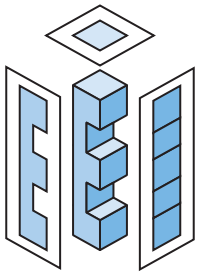


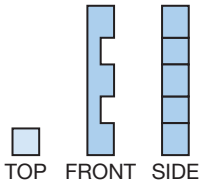
An **orthographic view** is a two-dimensional view of a three-dimensional figure, taken from a position directly in front of, above, or to the side of the figure.

Example 2 Finding the Orthographic View of a Figure

Show top, front, and side views of the figure.

SOLUTION

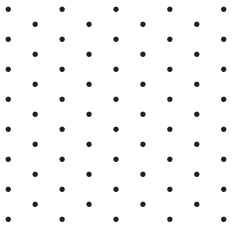




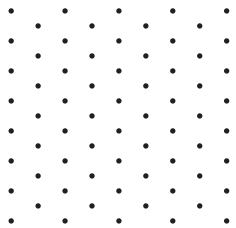
Skills Bank Practice

Draw the isometric view of each figure in the dot grids provided.

a.

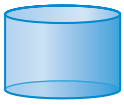


b.



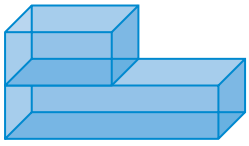
Draw the top, front, and side orthographic views of each figure.

c.



d.

e.



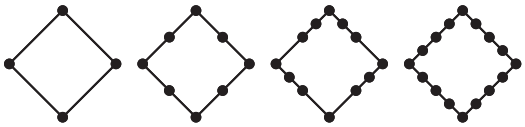
Geometric Patterns and Tessellations

Skills Bank Lesson 20

Patterns of figures can often be described numerically.

Example 1 Finding the Next Stage of a Geometric Pattern

The first four stages of a pattern are shown. Write a sequence for the number of dots in each stage. Explain the pattern and find the next term.



SOLUTION A table can help you see a pattern in the sequence of numbers: In each stage, the number of dots increases by 4. There will be 20 dots in the 5th stage.

Stage	1	2	3	4
Dots	4	8	12	16

A **tessellation** is a repeating pattern of plane shapes that completely covers an area without any gaps or overlaps. One simple example of a tessellation is square tiles completely covering a floor. No tiles overlap, and there are no gaps between tiles.

Example 2 Finding a Numeric Pattern in a Tessellation

In a tessellation of squares, squares are added to create the stages that are numbered inside the squares. How many squares will be added to create the 3rd stage?

2	2	2
2	1	2
2	2	2

SOLUTION

Add squares around the outside of the figure.

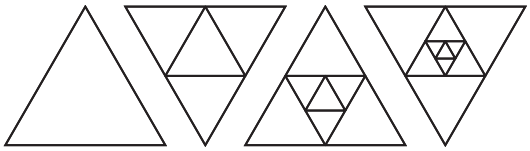
You will use 16 squares.

3	3	3	3	3
3	2	2	2	3
3	2	1	2	3
3	2	2	2	3
3	3	3	3	3

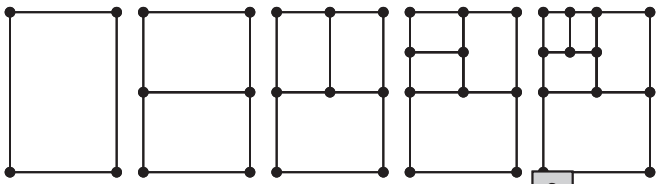
Skills Bank Practice

Find and describe the pattern in the described sequence. Give the next number in the sequence.

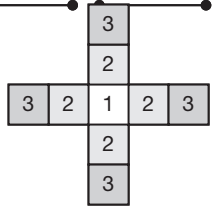
a. The number of non-overlapping triangles in each stage.



b. The number of dots in each stage.



c. The total number of squares used in each stage.



Stem-and-Leaf Plots

Skills Bank Lesson 21

A **stem-and-leaf plot** is a way to arrange the numbers in a data set according to place value.

Example 1 Making a Stem-and-Leaf Plot

Make a stem-and-leaf plot of the data.

18, 22, 15, 22, 31, 35, 27, 19, 29, 9, 25, 20, 58, 12, 56, 25, 15, 23

SOLUTION

Write the data in order from least to greatest:

9, 12, 15, 15, 18, 19, 20, 22, 22, 23, 25, 25, 27, 29, 31, 35, 56, 58

In the stem-and-leaf plot, the tens place digit of each number is a stem.

The ones digit of each number is a leaf.

Stems	Leaves
0	9
1	2, 5, 5, 8, 9
2	0, 2, 2, 3, 5, 5, 7, 9
3	1, 5
4	
5	6, 8

Skills Bank Practice

For a and b, make a stem-and-leaf plot for the set of data.

a. Number of Minutes Students Practiced Piano

9, 12, 15, 15, 18, 19, 20, 22, 22, 23, 25, 25, 27, 29, 31, 35, 56, 58

b. Chapter 1 Test Scores

70, 83, 88, 90, 68, 82, 99, 79, 79, 81, 100, 90, 86, 90, 70, 79, 90, 92, 94,75, 62, 73, 83, 96, 64, 98

c. The stem-and-leaf plot shows the number of boxes of popcorn sold at basketball games last season. Use the stem-and-leaf plot to answer the questions.

Stem	Leaves
8	3, 5, 5, 8, 9
9	0, 3, 6
10	0, 2, 8, 4, 6, 6, 6
11	3, 6
12	0, 1

i. For how many games was data collected?

ii. What was the least number of boxes sold?

iii. What was the greatest number of boxes sold?

iv. What was the median number of boxes sold?

v. What was the mode number of boxes sold?

Statistical Graphs

Skills Bank Lesson 22

You can display data using a **statistical graph.**

Use a **bar graph** to compare amounts. Use a **circle graph** to compare parts of a whole.

Example 1 Making a Bar Graph and Circle Graph

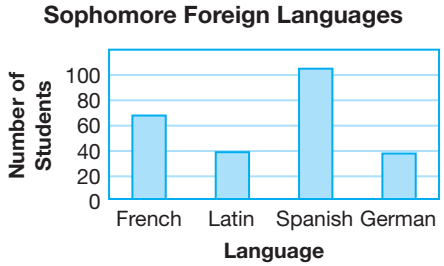
Make a bar graph and circle graph for the data.

Sophomore Class Foreign Languages

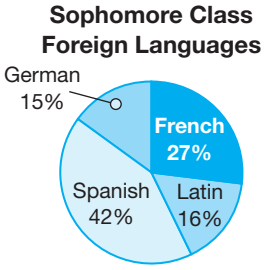
French	Latin	Spanish	German
68	39	105	38

SOLUTION

Place the languages along the horizontal axis and numbers along the vertical axis. Label each axis and give the graph a title.



Find the percent of the total for each language. Multiply the percent by 360° to find the measure of the central angle for each language.



Skills Bank Practice

a. Draw a circle graph of the data.

School Population by Grade			
9th	10th	11th	12th
254	233	261	203

b. Draw a bar graph of the data.

Number of Songs Downloaded						
Mon	Tue	Wed	Thu	Fri	Sat	Sun
28	32	25	16	30	18	32

SKILLS BANK

Proofs

Skills Bank Lesson 23

A **proof** is a logical argument that shows a conclusion to be true or false. In a mathematical proof, each step must be justified by a property, theorem, definition, or other accepted rule.

Example 1 Proving Expressions Equal

Prove that $(x + 1)^2 - 9 = (x + 4)(x - 2)$

SOLUTION Simplify each side and determine whether the expressions are equivalent.

$(x + 1)^2 - 9 \stackrel{?}{=} (x + 4)(x - 2)$ Given

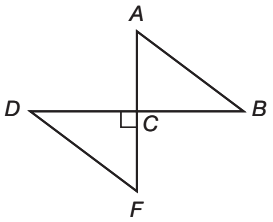
$x^2 + 2x + 1 - 9 \stackrel{?}{=} x^2 - 2x + 4x - 8$ Distributive Property

$x^2 + 2x - 8 = x^2 + 2x - 8$ ✓ Add.

The sides are equal. So, $(x + 1)^2 - 9 = (x + 4)(x - 2)$.

Example 2 Proofs in Geometry

Prove that ΔABC is a right triangle.



SOLUTION Make justified conclusions.

Conclusion	Justification
$\angle DCF$ is a right angle.	Given
$\angle DCF$ and $\angle ACB$ are vertical angles.	Definition of vertical angles
$\angle ACB$ is a right angle	Vertical angles are congruent.
ΔABC is a right triangle.	Definition of a right triangle.

Skills Bank Practice

Prove whether each statement is true or false.

a. $(x + 5)^2 = x^2 + 25$

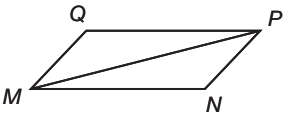
b. $x(x - 3) + 5x - 9 = (x + 1)^2 - 10$

c. $(x + 3)(x + 2)(x + 1) = x^3 + 6x^2 + 11x + 6$

d. $(x + 7)(x - 4) = (x + 2)(x + 1)$

e. Prove that $\Delta MQP \cong \Delta PNM$

Given: $MQPN$ is a parallelogram



Venn Diagrams

Skills Bank Lesson 24

A **Venn Diagram** shows relationships among the elements of two or more sets.

Example Making a Venn Diagram

Make a Venn Diagram using the positive integers from 1 to 20.

A: the set of even numbers

B: the set of factors of 12

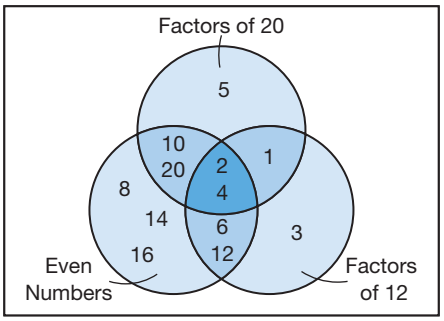
C: the set of factors of 20.

SOLUTION

Draw three overlapping circles.

Label one circle for each set.

Place the elements in the appropriate region of the diagram.



Skills Bank Practice

Draw a Venn Diagram of the sets.

a. *A*: factors of 40
B: factors of 15

b. *A*: positive integers less than 21
B: factors of 10
C: the first 5 multiples of 2

c. *A*: the primary colors
B: the colors in the American Flag

d. *A*: weekdays (business days)
B: weekend days

Draw a Venn Diagram to represent the situations described.

e. There are 12 students in a Math class and 13 students in a Science class. Five of the students are in both the Math and the Science class.

Properties and Formulas

Properties

Addition and Subtraction Properties for Inequalities
(10)

For real numbers a, b, and c, if $a < b$, then $a + c < b + c$ and $a - c < b - c$.

The relationship also holds true for $>$, $\leq$, and $\geq$.

Addition Property of Equality
(7)

If $a = b$, then $a + c = b + c$.

Arithmetic Sequence
(92)

The nth term of an arithmetic sequence is given by $a_n = a_1 + (n - 1)d$.

Associative Property of Addition
(1)

Let a, b, and c be real numbers, then $(a + b) + c = a + (b + c)$.

Associative Property of Multiplication
(1)

Let a, b, and c be real numbers, then $(ab)c = a(bc)$.

Binomial Theorem
(49)

If n is a nonnegative integer, then

$$(a + b)^n = ({}_nC_0)a^nb^0 + ({}_nC_1)a^{n-1}b^1 + \ldots + ({}_nC_{n-1})a^1b^{n-1} + ({}_nC_n)a^0b^n = \sum_{r=0}^{n}({}_nC_r)a^{n-r}b^r$$

where ${}_nC_r = \frac{n!}{r!(n-r)!}$.

Cartesian to Polar
(96)

$\tan\theta = \frac{y}{x}$

$r^2 = x^2 + y^2$

Change of Base Formula
(72)

For $a > 0$ and $a \neq 1$ and any base b such that $b > 0$, and $b \neq 1$, $\log_b x = \frac{\log_a x}{\log_a b}$.

Closure Property of Addition
(1)

If a and b are real numbers, then $a + b$ is a real number.

Closure Property of Multiplication
(1)

If a and b are real numbers, then ab is a real number.

Commutative Property of Addition
(1)

Let a and b be real numbers, then $a + b = b + a$.

Commutative Property of Multiplication
(1)

Let a and b be real numbers, then $ab = ba$.

Converse of Pythagorean Theorem
(41)

If the sum of the squares of the lengths of the two shorter sides of a triangle equals the square of the length of the longest side, then the triangle is a right triangle.

Cramer's Rule
(16)

The solutions of the linear system $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ are $x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D}$ and $y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D}$, where D is the determinant of the coefficient matrix.

Difference of Squares
(78)

$$a^2 - b^2 = (a + b)(a - b)$$

Difference of Two Cubes
(61)

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Discriminant
(74)

The discriminant of a quadratic equation $ax^2 + bx + c = 0$, is $b^2 - 4ac$.

If $b^2 - 4ac > 0$, there are two real solutions.

If $b^2 - 4ac = 0$, there is one real solution.

If $b^2 - 4ac < 0$, there are no real solutions.

Distance Formula
(41)

The distance d between any two points with coordinates (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Distributive Property
(1)

Let a, b, and c be real numbers, $a(b + c) = ab + ac$.

Division Property of Equality
(7)

If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.

Dot product
(99)

$A \cdot B = |A||B| \cos \theta$

Factor Theorem
(61)

For polynomial $P(x)$, $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.

Fundamental Theorem of Algebra
(106)

Every polynomial function of degree $n \geq 1$ has at least one zero in the set of complex numbers.

Geometric Sequence
(97)

The nth term of a geometric sequence is given by $a_n = a_1 r^{n-1}$, where r is the common ratio.

Heron's Formula
(77)

$A = \sqrt{s(s - a)(s - b)(s - c)}$, where $s = \frac{1}{2}(a + b + c)$.

Identity Property of Addition
(1)

$$a + 0 = a,\ 0 + a = a$$

Identity Property of Multiplication
(1)

$$a \cdot 1 = a,\ 1 \cdot a = a$$

Inverse of a 2 × 2 Matrix
(32)

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - cb \neq 0$, then the inverse of A is: $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Inverse Property of Addition
(1)

$a + (-a) = 0$

Inverse Property of Multiplication
(1)

$a \cdot \frac{1}{a} = 1,\ a \neq 0$

Irrational Root Theorem
(66)

If a polynomial $P(x)$ has rational coefficients, and $a + b\sqrt{c}$ is a root of $P(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $P(x) = 0$.

Law of Cosines
(77)

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

Law of Sines
(71)

For ΔABC with side a opposite angle A, side b opposite angle B, and side c opposite angle C, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Matrix Addition and Subtraction
(5)

For matrices $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$,

$A + B = \begin{bmatrix} a_1 \pm b_1 & a_2 \pm b_2 \\ a_3 \pm b_3 & a_4 \pm b_4 \end{bmatrix}$.

Matrix Determinant
(14)

For a square 2 × 2 matrix such as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant equals $ad - cb$.

Matrix Multiplicative Identity
(9)

The product of any matrix A and the multiplicative identity matrix I is matrix A.

$AI = IA = A$

Midpoint Formula
(91)

For a segment whose endpoints are at (x_1, y_1) and (x_2, y_2), $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Multiplication and Division Properties for Inequalities
(10)

For real numbers *a, b,* and *c,*

If $c < 0$ and $a < b$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

If $c > 0$ and $a < b$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

Also holds true for $>$, $\leq$, and $\geq$.

Multiplication Property of Equality
(7)

If $a = b$ and $c \neq 0$, then $ac = bc$.

Negative Exponent Property
(3, 59)

If n is any real number and x is any real number that is not zero, $x^{-n} = \frac{1}{x^n}$.

Percent of Change
(6)

$$\text{percent of change} = \frac{\text{amount of increase or decrease}}{\text{original amount}}$$

Polar Coordinates to Cartesian
(96)

$x = r \cos \theta$

$y = r \sin \theta$

Power of a Power Property
(59)

For any nonzero real number a and rational numbers m and n, $(a^m)^n = a^{m \cdot n}$.

Power of a Product Property
(59)

For all nonzero real numbers a and b and rational number m, $(ab)^m = a^m \cdot b^m$.

Power of a Quotient Property
(59)

For all nonzero real numbers a and b and rational number m, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

Power Property of Logarithms
(72)

For any real number p and positive numbers a and $b (b = 1)$, $\log_b a^p = p \log_b a$.

Power Property of Natural Logarithms
(81)

For any real number p and positive number a, $\ln a^p = p \ln a$.

Power Property for Exponents
(3)

If *m, n,* and *x* are real numbers, $(x^m)^n = x^{m \cdot n}$.

Product of Powers Property
(59)

For any nonzero real number a and rational numbers m and n, $a^m \cdot a^n = a^{m+n}$.

Product Property of Natural Logarithms
(81)

For any positive numbers a and b, $\ln ab = \ln a + \ln b$.

Product Property of Logarithms
(72)

For any positive numbers, *m, n,* and b $(b \neq 1)$, $\log_b mn = \log_b m + \log_b n$.

Product Property of *n*th Roots
(59)

For $a > 0$ and $b > 0$, $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Product Property for Exponents
(3)

If *m, n,* and *x* are real numbers and $x \neq 0$, $x^m \cdot x^n = x^{m+n}$.

Product Rule for Radicals
(40)

Given that a and b are real numbers and n is an integer greater than 1, $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.

Pythagorean Theorem
(41)

If a triangle is a right triangle, the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse.

Quotient of Powers Property
(28)

For $a \neq 0$, and integers m and n, $\frac{a^m}{a^n} = a^{m-n}$.

Quotient Property of Logarithms
(72)

For any positive numbers, m, n, and b $(b \neq 1)$, $\log_b \frac{m}{n} = \log_b m - \log_b n$.

Quotient Property of Natural Logarithms
(81)

For any positive numbers a and b, $\ln \frac{a}{b} = \ln a - \ln b$.

Quotient Property of *n*th Roots
(59)

For $a > 0$ and $b > 0$, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Rational Exponents Property
(59)

For m and n integers and $n \neq 0$:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

Rational Root Theorem
(66)

If a polynomial $P(x)$ has integer coefficients, then every rational root of $P(x) = 0$ can be written in the form $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient of $P(x)$.

Remainder Theorem
(51)

If a polynomial $f(x)$ is divided by $x - k$, the remainder is $r = f(k)$.

Remainder Theorem
(95)

For polynomials $P(x)$ and $(x - a)$, $P(x) = (x - a)Q(x) + r(x)$. The term $Q(x)$ is the quotient and the term $r(x)$ is the remainder. In particular, when $x = a$, $P(a) = r(a)$.

Scalar Multiplication
(5)

For matrix $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and any real number k,

$$k \cdot A = k\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix}.$$

Scientific Notation
(3)

A number written as the product of two factors in the form $a \times 10^n$, where $1 \leq a < 10$, and n is an integer.

Square Root Property
(58)

If $x^2 = a$, then $x = \pm\sqrt{a}$ for any $a > 0$.

Subtraction Property of Equality
(7)

If $a = b$, then $a - c = b - c$.

Sum of a Finite Geometric Series
(113)

The sum of the first n terms of a geometric series, S_n, is $a_1\left(\frac{1 - r^n}{1 - r}\right)$.

Sum of an Arithmetic Series
(105)

The sum of the first n terms of an arithmetic series, S_n, is $n\left(\frac{a_1 + a_n}{2}\right)$.

Sum of an Infinite Geometric Series
(113)

The sum of an infinite geometric series, S, is $\frac{a_1}{1 - r}$, where r is the common ratio and $0 < |r| < 1$.

Sum of Two Cubes
(61)

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Transitive Property for Inequalities
(10)

For real numbers a, b, and c, If $a < b$ and $b < c$, then $a < c$.
Also holds true for $>$, $\leq$, and $\geq$.

Trigonometric Ratios
(46)

$$\text{cosecant of } \angle A = \frac{1}{\text{sine of } \angle A} = \frac{\text{length of hypotenuse}}{\text{length of leg opposite to } \angle A}$$

$$\text{secant of } \angle A = \frac{1}{\text{cosine of } \angle A} = \frac{\text{length of hypotenuse}}{\text{length of leg adjacent to } \angle A}$$

$$\text{cotangent of } \angle A = \frac{1}{\text{tangent of } \angle A} = \frac{\text{length of leg adjacent to } \angle A}{\text{length of leg opposite } \angle A}$$

Zero Exponent Property
(59)

For any nonzero real number a, $a^0 = 1$.

Zero Product Property
(23, 35)

Let a and b be real numbers. If $ab = 0$, then $a = 0$ or $b = 0$.

Formulas

Perimeter

Rectangle $P = 2l + 2w$ or $P = 2(l + w)$

Square $P = 4s$

Circumference

Circle $C = \pi d$ or $C = 2\pi r$

Area

Rectangle $A = lw$

Triangle $A = \frac{1}{2}bh$

Trapezoid $A = \frac{1}{2}(b_1 + b_2)h$

Circle $A = \pi r^2$

Surface Area

Cube $S = 6s^2$

Cylinder $S = 2\pi r^2 + 2\pi rh$

Cone $S = \pi r^2 + \pi rl$

Volume

Where B is the area of the base of a solid figure,

Prism or cylinder $V = Bh$

Pyramid or cone $V = \frac{1}{3}Bh$

Linear Equations

Slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$

Slope-intercept form $y = mx + b$

Point-slope form $y - y_1 = m(x - x_1)$

Standard form $Ax + By = C$

Quadratic Equations

Standard form $ax^2 + bx + c = 0$

Axis of symmetry $x = -\frac{b}{2a}$

Discriminant $b^2 - 4ac$

Quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Sequences

*n*th term of an arithmetic sequence

$$a_n = a_1 + (n - 1)d$$

*n*th term of an geometric sequence

$$a_n = a_1 \cdot r^{n-1}$$

Trigonometric Ratios

$$\text{sine of } \angle A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}}$$

$$\text{cosine of } \angle A = \frac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}}$$

$$\text{tangent of } \angle A = \frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A}$$

Percents

$$\text{Percent of change} = \frac{\text{amount of increase or decrease}}{\text{original amount}}$$

Permutations and Combinations

$P(n, r)$ permutation of n objects taken r at a time

$${}_nP_r = \frac{n!}{(n - r)!}$$

$C(n, r)$ combination of n objects taken r at a time

$${}_nC_r = \frac{n!}{r!(n - r)!}$$

$n!$ $n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 1$

Probability

$P(\text{event}) =$ $\frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$

$P(A)$ probability of event A

Probability of Complement

$$P(\text{not event}) = 1 - P(\text{event})$$

Probability of Independent events

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Probability of Dependent events

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ after } A)$$

Probability of Mutually Exclusive Events

$$P(A \text{ or } B) = P(A) + P(B)$$

Probability of Inclusive Events

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Additional Formulas

Direct variation $y = kx$

Inverse variation $y = \frac{k}{x}; x \neq 0$

Distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Distance traveled $d = rt$

Exponential decay $y = kb^x; k > 0, 0 < b < 1$

Exponential growth $y = kb^x; k > 0, b > 1$

Midpoint of a segment $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Symbols

Comparison Symbols

$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\neq$	not equal to
$\approx$	approximately equal to

Geometry

$\cong$	is congruent to
$\sim$	is similar to
$^\circ$	degree(s)
$\angle ABC$	angle ABC
$m\angle ABC$	the measure of angle ABC
$\triangle ABC$	triangle ABC
$\overleftrightarrow{AB}$	line AB
$\overline{AB}$	segment AB
$\overrightarrow{AB}$	ray AB
AB	length of $\overline{AB}$
$\llcorner$	right angle
$\perp$	is perpendicular to
$\parallel$	is parallel to

Real Numbers

$\mathbb{R}$	the set of real numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Z}$	the set of integers
$\mathbb{W}$	the set of whole numbers
$\mathbb{N}$	the set of natural numbers

Additional Symbols

$\pm$	plus or minus
$a \cdot b$, ab, or $a(b)$	a times b
$\lvert -5 \rvert$	the absolute value of -5
%	percent
π	pi, $\pi \approx 3.14$, or $\pi \approx \frac{22}{7}$
$f(x)$	function notation: f of x
a^n	a to nth power
a_n	nth term of a sequence
(x, y)	ordered pair
$x:y$	ratio of x to y, or $\frac{x}{y}$
{ }	set braces
$\sqrt{x}$	nonnegative square root of x

Table of Metric Measures

Length

1 kilometer (km) = 1000 meters (m)

1 meter = 100 centimeters (cm)

1 centimeter = 10 millimeters (mm)

Capacity and Volume

1 liter (L) = 1000 milliliters (mL)

Mass

1 kilogram (kg) = 1000 grams (g)

1 gram = 1000 milligrams (mg)

Table of Customary Measures

Length

1 mile (mi) = 5280 feet (ft)

1 mile = 1760 yards (yd)

1 yard = 3 feet

1 yard = 36 inches (in.)

1 foot = 12 inches

Capacity and Volume

1 gallon (gal) = 4 quarts (qt)

1 quart = 2 pints (pt)

1 pint = 2 cups (c)

1 cup = 8 fluid ounces (fl oz)

Weight

1 ton = 2000 pounds (lb)

1 pound = 16 ounces (oz)

Customary and Metric Measures

1 inch = 2.54 centimeters

1 yard ≈ 0.9 meters

1 mile ≈ 1.6 kilometers

Time

1 year = 365 days

1 year = 12 months

1 month ≈ 4 weeks

1 year = 52 weeks

1 week = 7 days

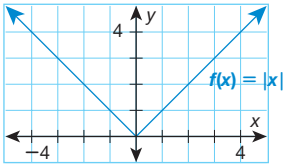
1 day = 24 hours

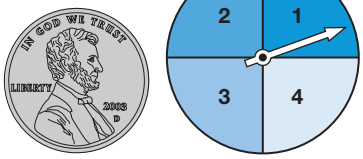
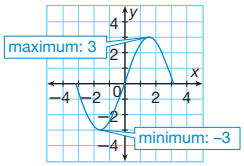
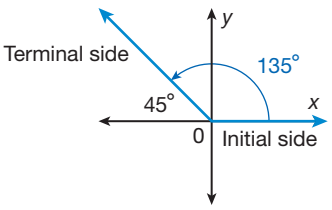
1 hour (hr) = 60 minutes (min)

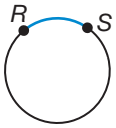
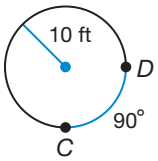
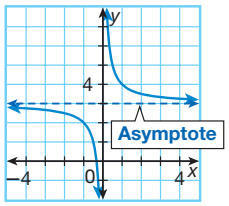
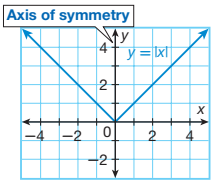
1 minute = 60 seconds (s)

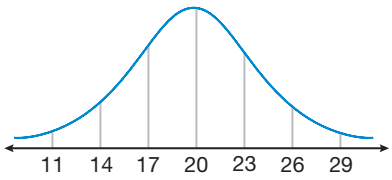
English/Spanish Glossary

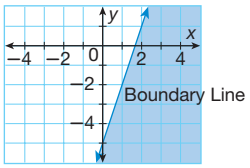
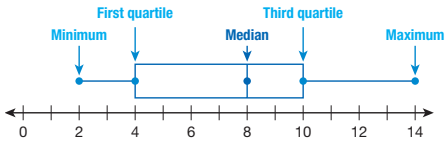
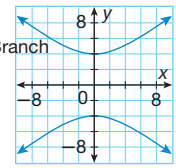
A

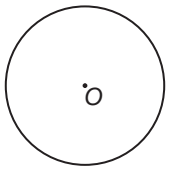
English	Example	Spanish
absolute value of a complex number *(69)* The absolute value of $a + bi$ is the distance from the origin to the point (a, b) in the complex plane and is denoted $\lvert a + bi \rvert = \sqrt{a^2 + b^2}$.	$\lvert 2 + 3i \rvert = \sqrt{2^2 + 3^2} = \sqrt{13}$	**valor absoluto de un número complejo** *(69)* El valor absoluto de $a + bi$ es la distancia desde el origen hasta el punto (a, b) en el plano complejo y se expresa $\lvert a + bi \rvert = \sqrt{a^2 + b^2}$.
absolute value of a real number *(17)* The absolute value of x is the distance from zero to x on a number line, denoted $\lvert x \rvert$. $\lvert x \rvert = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$	$\lvert 3 \rvert = 3$ $\lvert -3 \rvert = 3$	**valor absoluto de un número real** *(17)* El valor absoluto de x es la distancia desde cero hasta x en una recta numérica y se expresa $\lvert x \rvert$. $\lvert x \rvert = \begin{cases} x & \text{si } x \geq 0 \\ -x & \text{si } x < 0 \end{cases}$
absolute value function *(17)* A function whose rule contains absolute-value expressions.		**función de valor absoluto** *(17)* Función cuya regla contiene expresiones de valor absoluto.
abstract equation *(88)* An equation with two or more variables; also called a literal equation.	$V = \frac{1}{3}bh$	**ecuación abstracta** *(88)* Una ecuación con dos o más variables; también llamada una ecuación literal.
accuracy *(18)* An indication of how close a measurement corresponds to the actual value being measured.	For an object with a length of 2.35 cm, a measurement of 2.37 cm is more accurate than a measurement of 2.31.	**exactitud** *(18)* Un indicador de qué tan cercana está una medida con respecto al valor real medido.

English	Example	Spanish
A		
addition counting principle *(33)* If one outcome can occur in n_1 ways and a second outcome can occur in n_2 ways, then there are $n_1 + n_2$ total outcomes. For k different categories of outcomes, the total number of outcomes is $n_1 + n_2 + \ldots + n_k$.	There are 6 total outcomes in the event *flip a coin or spin the spinner.*  2 outcomes 4 outcomes	**principio de conteo** *(33)* Si un resultado puede ocurrir en n_1 maneras y un segundo resultado puede ocurrir en n_2 maneras, entonces hay un total de $n_1 + n_2$ de resultados. Para k diferentes categorías de resultados, el número total de resultados es $n_1 + n_2 + \ldots + n_k$.
additive inverse of a matrix *(5)* A matrix where each entry is the opposite of each entry in another matrix. Two matrices are additive inverses if their sum is the zero matrix.	$\begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ 0 & -4 \end{bmatrix}$ are additive inverses.	**inverso aditivo de una matriz** *(5)* Matriz en la cual cada entrada es el opuesto de cada entrada en otra matriz. Dos matrices son inversos aditivos si su suma es la matriz cero.
address *(5)* The location of an entry in a matrix, given by the row and column in which the entry appears. In matrix A, the address of the entry in row i and column j is a_{ij}.	In the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, the address of the entry 2 is a_{11}, the address of the entry 3 is a_{12}.	**dirección** *(5)* Ubicación de una entrada en una matriz, indicada por la fila y la columna en las que aparece la entrada. En la matriz A, la dirección de la entrada de la fila i y la columna j es a_{ij}.
algebraic expression *(2)* An expression that contains at least one variable.	$2x + 3y$	**expresión algebraica** *(2)* Expresión que contiene por lo menos una variable.
amplitude *(82)* The amplitude of a periodic function is half the difference of the maximum and minimum values (always positive).	 amplitude $= \frac{1}{2}\left[3 - (-3)\right] = 3$	**amplitud** *(82)* La amplitud (siempre positiva) de una función periódica es la mitad de la diferencia entre los valores máximo y mínimo.
angle of rotation *(56)* An angle formed by a rotating ray, called the terminal side, and a stationary reference ray, called the initial side.	 	**ángulo de rotación** *(56)* Ángulo formado por un rayo en rotación, denominado lado terminal, y un rayo de referencia estático, denominado lado inicial.

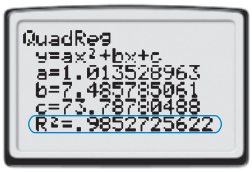
English	Example	Spanish
A		
arc *(63)*		**arco** *(63)*
An unbroken part of a circle consisting of two points on the circle, called the endpoints, and all the points on the circle between them.		Parte continua de un círculo formada por dos puntos del círculo denominados extremos y todos los puntos del círculo comprendidos entre éstos.
arc length *(63)*	 $m\overset{\frown}{CD} = 5\pi$ ft	**longitud de arco** *(63)*
The distance along an arc measured in linear units.		Distancia a lo largo de un arco medida en unidades lineales.
arithmetic sequence *(92)*	4, 7, 10, 13, 16, ... +3 +3 +3 +3 $d = 3$	**sucesión aritmética** *(92)*
A sequence whose successive terms differ by the same nonzero number d, called the *common difference*.		Sucesión cuyos términos sucesivos difieren en el mismo número distinto de cero d, denominado *diferencia común*.
arithmetic series *(105)*	$4 + 7 + 10 + 13 + 16 + ...$	**serie aritmética** *(105)*
The indicated sum of the terms of an arithmetic sequence.		Suma indicada de los términos de una sucesión aritmética.
asymptote *(47)*		**asíntota** *(47)*
A line that a graph approaches as the value of a variable becomes extremely large or small.		Línea recta a la cual se aproxima una gráfica a medida que el valor de una variable se hace sumamente grande o pequeño.
axis of symmetry *(27)*		**eje de simetría** *(27)*
A line that divides a plane figure or a graph into two congruent reflected halves.		Línea que divide una figura plana o una gráfica en dos mitades reflejadas congruentes.
B		
base of a power *(SB 3)*	$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$ base	**base de una potencia** *(SB 3)*
The number in a power that is used as a factor.		Número de una potencia que se utiliza como factor.

English	Example	Spanish
B		
base of an exponential function *(maintained)* The value of b in a function of the form $f(x) = ab^x$, where a and b are real numbers with $a \neq 0$, $b > 0$, and $b \neq 1$.	$f(x) = 5(2)^x$ (base)	**base de una función exponencial** *(repaso)* Valor de b en una función del tipo $f(x) = ab^x$, donde a y b son números reales con $a \neq 0$, $b > 0$, y $b \neq 1$.
bell curve *(80)* A single-peaked symmetric curve formed by drawing a line through the tops of the bars of a histogram, representing the normal distribution of a data set.		**curva de bell** *(80)* Una curva simétrica con sólo una cresta que se forma dibujando una línea a través de las partes superiores de las barras de un histograma que representa la distribución normal de un conjunto de datos.
bias *(73)* Systematic error that creates a sample that is not representative of its population, favoring some groups more than others.	A sample by which a surveyor questions every person on a certain street corner is biased against those that do not walk that route.	**sesgo** *(73)* Error sitemático que crea una muestra que no es representativa de su población, favoreciendo a algunos grupos más que a otros.
binomial *(11)* A polynomial with two terms.	$x + y$ $2a^2 + 3$ $4m^3n^2 + 6mn^4$	**binomio** *(11)* Polinomio con dos términos.
binomial experiment *(49)* A probability experiment consists of n identical and independent trials whose outcomes are either successes or failures, with a constant probability of success p and a constant probability of failure q, where $q = 1 - p$ or $p + q = 1$.	A multiple-choice quiz has 10 questions with 4 answer choices. The number of trials is 10. If each question is answered randomly, the probability of success for each trial is $\frac{1}{4} = 0.25$ and the probability of failure is $\frac{3}{4} = 0.75$.	**experimento binomial** *(49)* Experimento de probabilidades que comprende n pruebas idénticas e independientes cuyos resultados son éxitos o fracasos, con una probabilidad constante de éxito p y una probabilidad constante de fracaso q, donde $q = 1 - p$ ó $p + q = 1$.

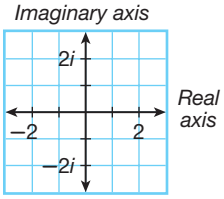
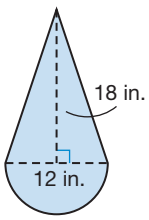
English	Example	Spanish
B		
binomial probability *(49)* In a binomial experiment, the probability of r successes out of n trials is $P(r) = {}_nC_r \cdot p^r q^{n-r}$.	In the binomial experiment earlier, the probability of randomly guessing 6 problems correctly is $P = {}_{10}C_6(0.25)^6(0.75)^4 \approx 0.016$.	**probabilidad binomial** *(49)* En un experimento binomial, la probabilidad de r éxitos de un total de n intentos es $P(r) = {}_nC_r \cdot p^r q^{n-r}$.
Binomial Theorem *(49)* For any positive integer n, $(x+y)^n = {}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \cdots + {}_nC_{n-1}x^1y^{n-1} + {}_nC_nx^0y^n$	$(x+2)^4 = {}_4C_0x^42^0 + {}_4C_1x^32^1 + {}_4C_2x^22^2 + {}_4C_1x^12^3 + {}_4C_4x^02^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$	**Teorema de los binomios** *(49)* Dado un entero positivo n, $(x+y)^n = {}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \cdots + {}_nC_{n-1}x^1y^{n-1} + {}_nC_nx^0y^n$
boundary line *(39)* A line that divides a coordinate plane into two half-planes.		**línea de límite** *(39)* Línea que divide un plano coordenado en dos semiplanos.
box-and-whisker plot *(25)* A method of showing how data is distributed by using the median, quartiles, and minimum and maximum values; also called a *box plot*.		**gráfica de mediana y rango** *(25)* Método para demostrar la distribución de datos utilizando la mediana, los cuartiles y los valores mínimos y máximos; también llamado *gráfica de caja.*
branch of a hyperbola *(109)* One of the two symmetrical parts of the hyperbola.		**rama de una hipérbola** *(109)* Una de las dos partes simétricas de la hipérbola.
C		
capture-recapture method *(73)* A sampling method used to estimate the size of a population. It requires two separate visits, where subjects are captured during both visits.	Visit 1: capture and mark 30 animals Visit 2: capture 50 animals, 20 are marked There are about n animals in the population, where $\frac{20}{30} = \frac{50}{n}$, so $n = 75$.	**método de captura-recaptura** *(73)* Un método de muestreo utilizado para estimar el tamaño de una población. Requiere de dos visitas separadas donde los sujetos son capturados durante ambas visitas.

English	Example	Spanish
C		
change of base formula *(72)* For $a > 0$ and $a \neq 1$ and any base b such that $b > 0$ and $b \neq 1$, $\log_b x = \frac{\log_a x}{\log_a b}$.	$\log_4 8 = \frac{\log_2 8}{\log_2 4}$	**fórmula para cambiar de base** *(72)* Para $a > 0$ y $a \neq 1$ y cualquier base b tal que $b > 0$ y $b \neq 1$, $\log_b x = \frac{\log_a x}{\log_a b}$.
circle *(91)* The set of points in a plane that are a fixed distance from a given point called the center of the circle.		**círculo** *(91)* Conjunto de puntos en un plano que se encuentran a una distancia fija de un punto determinado denominado centro del círculo.
closure *(maintained)* A set of numbers is said to be closed, or to have closure, under a given operation if the result of the operation on any two numbers in the set is also in the set.	The natural numbers are closed under addition because the sum of two natural numbers is always a natural number.	**cerradura** *(repaso)* Se dice que un conjunto de números es cerrado, o tiene cerradura, respecto de una operación determinada, si el resultado de la operación entre dos números cualesquiera del conjunto también está en el conjunto.
cluster sampling *(Inv 7)* Sampling in which the population is first divided into groups, called clusters, and then a number of clusters are randomly selected.	Blocks of houses are randomly selected from all the blocks in a given neighborhood. Every house in a chosen block belongs in the sample.	**muestreo por cúmulos** *(Inv 7)* Muestreo en el cual la población primero se divide en grupos, llamados cúmulos, y después un número de cúmulos son seleccionados al azar.
coefficient *(maintained)* A number multiplied by a variable.	In the expression $2x + 3y$, 2 is the coefficient of x and 3 is the coefficient of y.	**coeficiente** *(repaso)* Número multiplicado por una variable.
coefficient matrix *(16)* The matrix of the coefficients of the variables in a linear system of equations.	System of equations $2x + 3y = 11$, $5x - 4y = 16$ Coefficient matrix $\begin{bmatrix} 2 & 3 \\ 5 & -4 \end{bmatrix}$	**matriz de coeficientes** *(16)* Matriz de los coeficientes de las variables en un sistema lineal de ecuaciones.

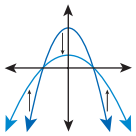
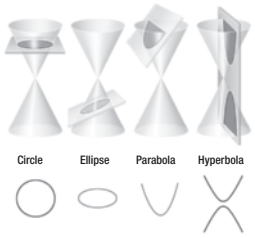
GLOSSARY/ GLOSARIO

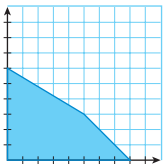
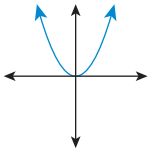
English	Example	Spanish
C		
coefficient of determination *(116)* The number R^2, with $0 \le R^2 \le 1$, that shows the fraction of the data that are close to the curve of best fit and, thus, how well the curve fits the data.		**coeficiente de determinación** *(116)* El número R^2, con $0 \le R^2 \le 1$, que muestra la fracción de los datos cercanos a la línea de mejor ajuste y, por lo tanto, cuánto se ajusta la línea de mejor ajuste a los datos.
combination *(42)* A selection of a group of objects in which order is *not* important. The number of combinations of r objects chosen from a group of n objects is denoted ${}_nC_r$.	For 4 objects *A, B, C,* and *D*, there are ${}_4C_2 = 6$ different combinations of 2 objects: *AB, AC, AD, BC, BD, CD.*	**combinación** *(42)* Selección de un grupo de objetos en la cual el orden *no* es importante. El número de combinaciones de r objetos elegidos de un grupo de n objetos se expresa así: ${}_nC_r$.
common difference *(92)* In an arithmetic sequence, the nonzero constant difference of any term and the previous term.	In the arithmetic sequence 3, 5, 7, 9, 11, ..., the common difference is 2.	**diferencia común** *(92)* En una sucesión aritmética, diferencia constante distinta de cero entre cualquier término y el término anterior.
common logarithm *(64)* A logarithm whose base is 10, denoted $\log_{10}$ or just log.	$\log 100 = \log_{10} 100 = 2$, since $10^2 = 100$.	**logaritmo común** *(64)* Logaritmo de base 10, que se expresa $\log_{10}$ o simplemente log.
common ratio *(97)* In a geometric sequence, the constant ratio of any term and the previous term.	In the geometric sequence 32, 16, 18, 4, 2 ..., the common ratio is $\frac{1}{2}$.	**razón común** *(97)* En una sucesión geométrica, la razón constante r entre cualquier término y el término anterior.
completing the square *(58)* A process used to form a perfect-square trinomial. To complete the square of $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.	$x^2 + 6x + \blacksquare$ Add $\left(\frac{6}{2}\right)^2 = 9$. $x^2 + 6x + 9$ $(x + 3)^2$ is a perfect square.	**completar el cuadrado** *(58)* Proceso utilizado para formar un trinomio cuadrado perfecto. Para completar el cuadrado de $x^2 + bx$, hay que sumar $\left(\frac{b}{2}\right)^2$.

C

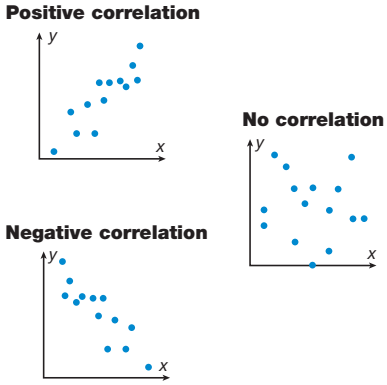
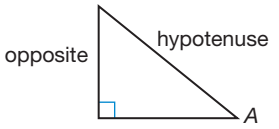
English	Example	Spanish
complex conjugate *(69)* The complex conjugate of any complex number $a + bi$, denoted $\overline{a + bi}$, is $a - bi$.	$\overline{4 + 3i} = 4 - 3i$ $\overline{4 - 3i} = 4 + 3i$	**conjugado complejo** *(69)* El conjugado complejo de cualquier número complejo $a + bi$, expresado como $\overline{a + bi}$, es $a - bi$.
complex fraction *(48)* A fraction that contains one or more fractions in the numerator, the denominator, or both.	$\dfrac{\frac{1}{2}}{1 + \frac{2}{3}}$	**fracción compleja** *(48)* Fracción que contiene una o más fracciones en el numerador, en el denominador, o en ambos.
complex number *(62)* Any number that can be written as $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.	$4 + 2i$ $5 + 0i = 5$ $0 - 7i = -7i$	**número complejo** *(62)* Todo número que se puede expresar como $a + bi$, donde a y b son números reales e $i = \sqrt{-1}$.
complex plane *(69)* A set of coordinate axes in which the horizontal axis is the real axis and the vertical axis is the imaginary axis; used to graph complex numbers.		**plano complejo** *(69)* Conjunto de ejes coordenado en el cual el eje horizontal es el eje real y el eje vertical es el eje imaginario; se utiliza para representar gráficamente números complejos.
composite figure *(SB 17)* A plane figure made up of triangles, rectangles, trapezoids, circles, and other simple shapes, or a three-dimensional figure made up of prisms, cones, pyramids, cylinders, and other simple three-dimensional figures.		**figura compuesta** *(SB 17)* Figura plana compuesta por triángulos, rectángulos, trapecios, círculos y otras formas simples, o figura tridimensional compuesta por prismas, conos, pirámides, cilindros y otras figuras tridimensionales simples.
composite function *(53)* A function of the form $(f \circ g)(x)$, or $f(g(x))$, where the input for function f is the output of function g.	If $f(x) = x + 2$ and $g(x) = x^2$, then $f(g(x)) = x^2 + 2$.	**función compuesta** *(53)* Una función de la forma $(f \circ g)(x)$, o $f(g(x))$, donde la entrada para la función f es la salida de la función g.

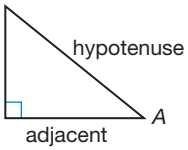
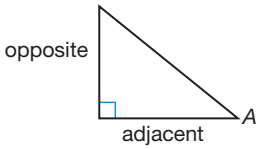
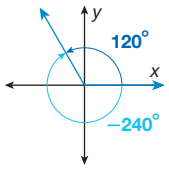
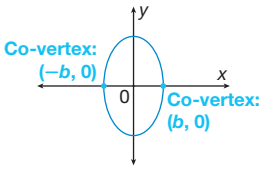
GLOSSARY/ GLOSARIO

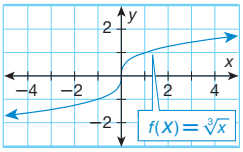
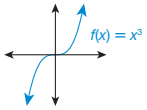
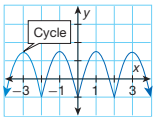
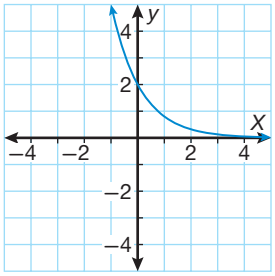
English	Example	Spanish
C		
composition of functions *(53)*		**composición de funciones** *(53)*
The composition of functions f and g, written as $(f \circ g)(x)$ and defined as $f(g(x))$ uses the output of $g(x)$ as the input for $f(x)$.	If $f(x) = x^2$ and $g(x) = x + 1$, the composite function $(f \circ g)(x) = (x + 1)^2$.	La composición de las funciones f y g, expresada como $(f \circ g)(x)$ y definida como $f(g(x))$ utiliza la salida de $g(x)$ como la entrada para $f(x)$.
compound event *(60)*	In the experiment of tossing a coin and rolling a number cube, the event of the coin landing heads and the number cube landing on 3.	**suceso compuesto** *(60)*
An event made up of two or more simple events.		Suceso formado por dos o más sucesos simples.
compression *(30)*		**compresión** *(30)*
A transformation that pushes the points of a graph horizontally toward the y-axis or vertically toward the x-axis.		Transformación que desplaza los puntos de una gráfica horizontalmente hacia el eje y o verticalmente hacia el eje x.
conditional probability *(55)*		**probabilidad condicional** *(55)*
The probability of event B, given that event A has already occurred or is certain to occur, denoted $P(B \mid A)$; used to find probability of dependent events.		Probabilidad del suceso B, dado que el suceso A ya ha ocurrido o es seguro que ocurrirá, expresada como $P(B \mid A)$; se utiliza para calcular la probabilidad de sucesos dependientes.
conic section *(91)*		**sección cónica** *(91)*
A plane figure formed by the intersection of a double right cone and a plane. Examples include circles, ellipses, hyperbolas, and parabolas.		Figura plana formada por la intersección de un cono regular doble y un plano. Algunos ejemplos son círculos, elipses, hipérbolas y parábolas.
conjugates *(44)*		**conjugados** *(44)*
Binomials of the form $a + b$ and $a - b$. The terms are the same; the signs before b are opposites.	$3 + \sqrt{2}$ and $3 - \sqrt{2}$ are radical conjugates.	Binomios de la forma $a + b$ y $a - b$. Los términos son los mismos; los signos antes de b son opuestos.

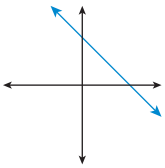
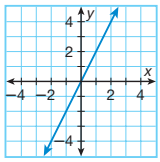
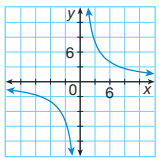
English	Example	Spanish
C		
conjunction *(Inv 1)* A compound statement that uses the word *and*.	3 is less than 5 AND greater than 0.	**conjunción** *(Inv 1)* Enunciado compuesto que contiene la palabra *y*.
consistent system *(15)* A system of equations or inequalities that has at least one solution.	$\begin{cases} x + y = 6 \\ x - y = 4 \end{cases}$ solution: $(5, 1)$	**sistema consistente** *(15)* Sistema de ecuaciones o desigualdades que tiene por lo menos una solución.
constant of variation *(8)* The constant k in direct, inverse, joint, and combined variation equations.	$y = 5x$ ↑ constant of variation	**constante de variación** *(8)* La constante k en ecuaciones de variación directa, inversa, conjunta y combinada.
constant term *(2)* A term in a function or expression that does not contain variables.	$f(x) = 3x + 5$ ↑ Constant term	**término constante** *(2)* Término de una función o expresión que no contiene variables.
constraint *(54)* One of the inequalities that define the feasible region in a linear-programming problem.	**Constraints:** $x > 0$ $y > 0$ $x + y \leq 8$ $3x + 5y \leq 30$ **Feasible region** 	**restricción** *(54)* Una de las desigualdades que definen la región factible en un problema de programación lineal.
continuous function *(22)* A function whose graph is an unbroken line or curve with no gaps or breaks.		**función continua** *(22)* Función cuya gráfica es una línea recta o curva continua, sin espacios ni interrupciones.
convenience sampling *(73)* Creating a sample by using who or what is most convenient or available.	A student surveys the same people she talks to everyday: her own friends, family, and teachers.	**muestrear convenientemente** *(73)* Crear una muestra utilizando a personas o cosas que sean las más convenientes y que estén disponibles.

C

English	Example	Spanish
converge *(113)* An infinite series converges when the common ratio $\lvert r \rvert < 1$ and the partial sums approach a fixed number.	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ converges to 1.	**convergir** *(113)* Una sucesión o serie infinita converge cuando la razón común $\lvert r \rvert < 1$ y las sumas parciales se aproximan a un número fijo.
conversion factor *(18)* The ratio of two equal quantities, each measured in different units.	$\frac{12 \text{ inches}}{1 \text{ foot}}$	**factor de conversión** *(18)* Razón entre dos cantidades iguales, cada una medida en unidades diferentes.
correlation *(45)* A measure of the strength and direction of the relationship between two variables or data sets.		**correlación** *(45)* Medida de la fuerza y dirección de la relación entre dos variables o conjuntos de datos.
correlation coefficient *(45)* A number r, where $-1 \le r \le 1$, that describes how closely the points in a scatter plot cluster around the least-squares line.	An r-value close to 1 describes a strong positive correlation. An r-value close to 0 describes a weak correlation or no correlation. An r-value close to -1 describes a strong negative correlation.	**coeficiente de correlación** *(45)* Número r, donde $-1 \le r \le 1$, que describe a qué distancia de la recta de mínimos cuadrados se agrupan los puntos de un diagrama de dispersión.
cosecant *(46)* In a right triangle, the cosecant of angle A is the ratio of the length of the hypotenuse to the length of the side opposite A. It is the reciprocal of the sine function.	 $\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin A}$	**cosecante** *(46)* En un triángulo rectángulo, la cosecante del ángulo A es la razón entre la longitud de la hipotenusa y la longitud del cateto opuesto a A. Es la inversa de la función seno.

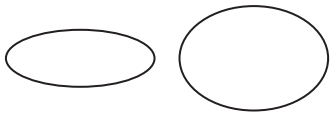
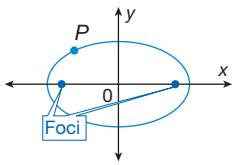
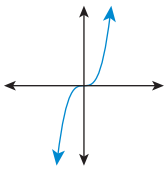
English	Example	Spanish
C		
cosine *(46)* In a right triangle, the cosine of angle A is the ratio of the length of the side adjacent to angle A to the length of the hypotenuse. It is the reciprocal of the secant function.	 $\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sec A}$	**coseno** *(46)* En un triángulo rectángulo, el coseno del ángulo A es la razón entre la longitud del cateto adyacente al ángulo A y la longitud de la hipotenusa. Es la inversa de la función secante.
cotangent *(46)* In a right triangle, the cotangent of angle A is the ratio of the length of the side adjacent to A to the length of the side opposite A. It is the reciprocal of the tangent function.	 $\cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\tan A}$	**cotangente** *(46)* En un triángulo rectángulo, la cotangente del ángulo A es la razón entre la longitud del cateto adyacente a A y la longitud del cateto opuesto a A. Es la inversa de la función tangente.
coterminal angles *(56)* Two angles in standard position with the same terminal side.		**ángulos coterminales** *(56)* Dos ángulos en posición estándar con el mismo lado terminal.
co-vertices of an ellipse *(98)* The endpoints of the minor axis.		**co-vértices de una elipse** *(98)* Extremos del eje menor.
Cramer's rule *(16)* A method of solving systems of linear equations by using determinants.	For the system $\begin{cases} x - y = 3 \\ 2x - y = -1 \end{cases}$, $D = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1(-1) - 2(-1) = 1$ $x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix}}{1} = \frac{-3-1}{1} = -4$ $y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}}{1} = \frac{-1-6}{1} = -7$	**regla de Cramer** *(16)* Método para resolver sistemas de ecuaciones lineales utilizando determinantes.

English	Example	Spanish
C		
cross products *(SB 8)* In the statement $\frac{a}{b} = \frac{c}{d}$, bc and ad are the cross products.	$\frac{1}{2} = \frac{3}{6}$ Cross products: $2 \cdot 3 = 6$ *and* $1 \cdot 6 = 6$	**productos cruzados** *(SB 8)* En el enunciado $\frac{a}{b} = \frac{c}{d}$, bc y ad son los productos cruzados.
cube root function *(75)* The function $f(x) = \sqrt[3]{x}$.		**función de raíz cúbica** *(75)* La función $f(x) = \sqrt[3]{x}$.
cubic function *(101)* A polynomial function of degree 3.		**función cúbica** *(101)* Función polinomial de grado 3.
cycle of a periodic function *(82)* The shortest repeating part of a periodic graph or function.		**ciclo de una función periódica** *(82)* La parte repetida más corta de una gráfica o función periódica.
D		
decreasing function *(57)* A function whose output value decreases as its input value increases.		**función decreciente** *(57)* Una función cuyo valor de salida decrece conforme su valor de entrada aumenta.
degree of a monomial *(11)* The sum of the exponents of the variables in the monomial.	$4x^2y^5z^3$ Degree: $2 + 5 + 3 = 10$ 5 Degree: 0 $(5 = 5x^0)$	**grado de un monomio** *(11)* Suma de los exponentes de las variables del monomio.
degree of a polynomial *(11)* The degree of the term of the polynomial with the greatest degree.	$3x^2y^2 + 4xy^5 - 12x^3y^2$ Degree 6 Degree 4 Degree 6 Degree 5	**grado de un polinomio** *(11)* Grado del término del polinomio con el grado máximo.
dependent events *(33)* Events for which the occurrence or nonoccurrence of one event affects the probability of the other event.	From a bag containing 3 red marbles and 2 blue marbles, drawing a red marble, and then drawing a blue marble without replacing the first marble.	**sucesos dependientes** *(33)* Dos sucesos son dependientes si el hecho de que uno de ellos se cumpla o no afecta la probabilidad del otro.

English	Example	Spanish
D		
dependent system *(15)* A system of equations that has infinitely many solutions.	 $\begin{cases} x + y = 3 \\ 2x + 2y = 6 \end{cases}$	**sistema dependiente** *(15)* Sistema de ecuaciones que tiene infinitamente muchas soluciones.
dependent variable *(4)* The output of a function; a variable whose value depends on the value of the input, or independent variable.	$y = 2x + 1$ dependent variable.	**variable dependiente** *(4)* Salida de una función; variable cuyo valor depende del valor de la entrada, o variable independiente.
determinant *(14)* A real number associated with a square matrix. The determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\lvert A \rvert = ad - bc$.	$\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 2(4) - (-1)(3) = 11$	**determinante** *(14)* Número real asociado con una matriz cuadrada. La determinante de $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ es $\lvert A \rvert = ad - bc$.
difference of two squares *(23)* A polynomial of the form $a^2 - b^2$, which may be written as the product $(a + b)(a - b)$.	$x^2 - 4 = (x + 2)(x - 2)$	**diferencia de dos cuadrados** *(23)* Polinomio del tipo $a^2 - b^2$, que se puede expresar como el producto $(a + b)(a - b)$.
dimensions of a matrix *(5)* A matrix with m rows and n columns has dimensions $m \times n$, read "m by n."	$\begin{bmatrix} -3 & 2 & 1 & -1 \\ 4 & 0 & -5 & 2 \end{bmatrix}$ Dimensions 2×4	**dimensiones de una matriz** *(5)* Una matriz con m filas y n columnas tiene dimensiones $m \times n$, expresadas "m por n".
direct variation *(8)* A linear relationship between two variables, x and y, that can be written in the form $y = kx$, where k is a nonzero constant.	 $y = 2x$	**variación directa** *(8)* Relación lineal entre dos variables, x e y, que puede expresarse en la forma $y = kx$, donde k es una constante distinta de cero.
discontinuous function *(22)* A function whose graph has one or more jumps, breaks, or holes.		**función discontinua** *(22)* Función cuya gráfica tiene uno o más saltos, interrupciones u hoyos.

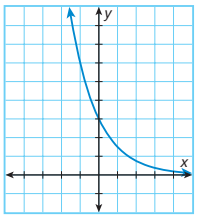
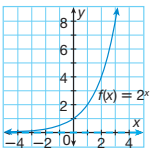
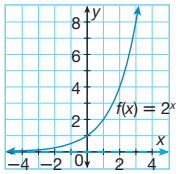
GLOSSARY/ GLOSARIO

English	Example	Spanish
D		
discriminant *(74)*	The discriminant of $2x^2 - 5x - 3$ is $(-5)^2 - 4(2)(-3) = 25 + 24 = 49$.	**discriminante** *(74)*
The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$.		El discriminante de la ecuación cuadrática $ax^2 + bx + c = 0$ es $b^2 - 4ac$.
disjoint events *(60)*	When rolling a number cube, rolling an even number and rolling a 3 are disjoint events.	**sucesos excluyentes** *(60)*
Events that have no outcomes in common.		Sucesos que no tienen resultados posibles en común.
disjunction *(Inv 1)*	John will walk to work OR he will stay home.	**disyunción** *(Inv 1)*
A compound statement that uses the word *or*.		Enunciado compuesto que contiene la palabra *o*.
distance formula *(41)*	The distance from $(2, 1)$ to $(6, 4)$ is $d = \sqrt{(6-2)^2 + (4-1)^2}$ $= \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = 5$.	**fórmula de distancia** *(41)*
In a coordinate plane, the distance from (x_1, y_1) to (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.		En un plano coordenado, la distancia desde (x_1, y_1) hasta (x_2, y_2) es $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
diverge *(113)*	$1 + 2 + 4 + 8 + 16 + \ldots$ diverges.	**divergir** *(113)*
An infinite series diverges when the common ratio $\lvert r \rvert \geq 1$ and the partial sums do not approach a fixed number.		Una serie infinita diverge cuando la razón común $\lvert r \rvert \geq 1$ y las sumas parciales no se aproximan a un número fijo.
domain *(4)*	The domain of the function $f(x) = \sqrt{x}$ is $\{x \mid x \geq 0\}$.	**dominio** *(4)*
The set of all possible input values of a relation or function.		Conjunto de todos los posibles valores de entrada de una función o relación.
dot product *(99)*	$A = (3, 7)$ and $B = (-2, -5)$ $A \cdot B = (3)(-2) + (7)(-5)$ $= -6 - 35$ $= -41$	**producto punto** *(99)*
The sum of the products of the x- and y-coordinates of the endpoints of two vectors that begin at the origin.		La suma de los prductos de las coordenadas x e y de los extremos de dos vectores que comienzan en el origen.
double roots *(35)*	$x^2 - 4x + 4 = 0$ has a double root of 2. $x^2 - 4x + 4 = 0$ $(x - 2)(x - 2) = 0$ $x - 2 = 0$ or $x - 2 = 0$ $x = 2$ or $x = 2$	**raíces dobles** *(35)*
Two identical roots, or solutions, of an equation.		Dos raíces idénticas, o soluciones, de una ecuación.

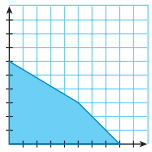
English	Example	Spanish
E		
eccentricity *(98)* A number that denotes how close or how far an ellipse is from being a circle. The eccentricity of a circle is 0. The eccentricity of an ellipse is greater than 0 and less than 1.	The eccentricity of the ellipse on the left is greater than that of the ellipse on the right. 	**excentricidad** *(98)* Un número que denota qué tan cerca o tan lejos está una elipse de ser un círculo. La excentricidad de una elipse es mayor que 0 y es menor que 1.
elimination method *(24)* A method used to solve systems of equations in which one variable is eliminated by adding or subtracting two equations of the system.		**método de eliminación** *(24)* Método utilizado para resolver sistemas de ecuaciones por el cual se elimina una variable sumando o restando dos ecuaciones del sistema.
ellipse *(98)* The set of all points P in a plane such that the sum of the distances from P to two fixed points F_1 and F_2, called the foci, is constant.		**elipse** *(98)* Conjunto de todos los puntos P de un plano tal que la suma de las distancias desde P hasta los dos puntos fijos F_1 y F_2, denominados focos, es constante.
element *(5)* Each value in a matrix; also called an entry.	3 is the element in the first row and second column of $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, denoted a_{12}.	**elemento** *(5)* Cada valor de una matriz; también denominado entrada.
end behavior *(101)* The trends in the y-values of a function as the x-values approach positive and negative infinity.	 End behavior: $f(x) \to \infty$ as $x \to \infty$ $f(x) \to -\infty$ as $x \to -\infty$	**comportamiento extremo** *(101)* Tendencia de los valores de y de una función a medida que los valores de x se aproximan al infinito positivo y negativo.
equally likely outcomes *(55)* Outcomes are equally likely if they have the same probability of occurring. If an experiment has n equally likely outcomes, then the probability of each outcome is $\frac{1}{n}$.	If a coin is tossed, and heads and tails are equally likely, then $P(\text{heads}) = P(\text{tails}) = \frac{1}{2}$.	**resultados igualmente probables** *(55)* Los resultados son igualmente probables si tienen la misma probabilidad de ocurrir. Si un experimento tiene n resultados igualmente probables, entonces la probabilidad de cada resultado es $\frac{1}{n}$.

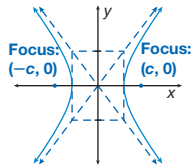
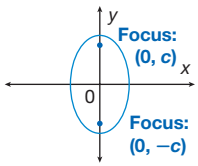
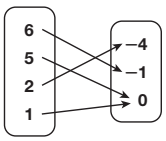
E

English	Example	Spanish
equation *(7)* A mathematical statement that two expressions are equivalent.	$x + 4 = 7$ $2 + 3 = 6 - 1$ $(x - 1)^2 + (y + 2)^2 = 4$	**ecuación** *(7)* Enunciado matemático que indica que dos expresiones son equivalentes.
evaluate *(2)* To find the value of an algebraic expression by substituting a number for each variable and simplifying by using the order of operations.	Evaluate $2x + 7$ for $x = 3$. $2x + 7$ $2(3) + 7$ $6 + 7$ 13	**evaluar** *(2)* Calcular el valor de una expresión algebraica sustituyendo cada variable por un número y simplificando mediante el orden de las operaciones.
event *(33)* An outcome or set of outcomes in a probability experiment.	In the experiment of rolling a number cube, the event "an odd number" consists of the outcomes 1, 3, and 5.	**suceso** *(33)* Resultado o conjunto de resultados en un experimento de probabilidad.
expected value *(maintained)* The weighted average of the numerical outcomes of a probability experiment.	The table shows the probability of getting a given score by guessing on a three-question quiz. **Score** 0 1 2 3 **Probability** 0.42 0.42 0.14 0.02 The expected value is a score of $0(0.42) + 1(0.42) + 2(0.14) + 3(0.02) = 0.76$.	**valor esperado** *(repaso)* Promedio ponderado de los resultados numéricos de un experimento de probabilidad.
experiment *(33)* An operation, process, or activity in which outcomes can be used to estimate probability.	Tossing a coin 10 times and noting the number of heads.	**experimento** *(33)* Una operación, proceso o actividad cuyo resultado se puede usar para estimar la probabilidad.
experimental probability *(55)* The ratio of the number of times an event occurs to the number of trials, or times, that an activity is performed.	Kendra made 6 of 10 free throws. The experimental probability that she will make her next free throw is $P(\text{free throw}) = \frac{\text{number made}}{\text{number attempted}} = \frac{6}{10}$.	**probabilidad experimental** *(55)* Razón entre la cantidad de veces que ocurre un suceso y la cantidad de pruebas, o veces, que se realiza una actividad.
explicit formula *(92)* A formula that defines the nth term a_n, or general term, of a sequence as a function of n.	Sequence: 4, 7, 10, 13, 16, 19, ... Explicit formula: $a_n = 1 + 3n$	**fórmula explícita** *(92)* Fórmula que define el enésimo término a_n, o término general, de una sucesión como una función de n.

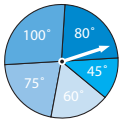
English	Example	Spanish
E		
exponent *(3)* The number that indicates how many times the base in a power is used as a factor.	$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$ exponent	**exponente** *(3)* Número que indica la cantidad de veces que la base de una potencia se utiliza como factor.
exponential decay *(57)* An exponential function of the form $f(x) = ab^x$ in which $a > 0$ and $0 < b < 1$. If r is the rate of decay, then the function can be written $y = a(1 - r)^t$, where a is the initial amount and t is the time.	 $y = 3\left(\frac{1}{2}\right)^x$	**decremento exponencial** *(57)* Función exponencial del tipo $f(x) = ab^x$ en la cual $0 < b < 1$. Si r es la tasa decremental, entonces la función se puede expresar como $y = a(1 - r)^t$, donde a es la cantidad inicial y t es el tiempo.
exponential equation *(93)* An equation that contains one or more exponential expressions.	$2^{x+1} = 8$	**ecuación exponencial** *(93)* Ecuación que contiene una o más expresiones exponenciales.
exponential function *(47)* A function of the form $f(x) = ab^x$, where a and b are real numbers with $a \neq 0$, $b > 0$, and $b \neq 1$.		**función exponencial** *(47)* Función del tipo $f(x) = ab^x$, donde a y b son números reales con $a \neq 0$, $b > 0$ y $b \neq 1$.
exponential growth *(57)* An exponential function of the form $f(x) = ab^x$ in which $a > 0$ and $b > 1$. If r is the rate of growth, then the function can be written $y = a(1 + r)^t$, where a is the initial amount and t is the time.		**crecimiento exponencial** *(57)* Función exponencial del tipo $f(x) = ab^x$ en la que $b > 1$. Si r es la tasa de crecimiento, entonces la función se puede expresar como $y = a(1 + r)^t$, donde a es la cantidad inicial y t es el tiempo.
exponential regression *(116)* A statistical method used to fit an exponential model to a given data set.		**regresión exponencial** *(116)* Método estadístico utilizado para ajustar un modelo exponencial a un conjunto de datos determinado.

GLOSSARY/ GLOSARIO

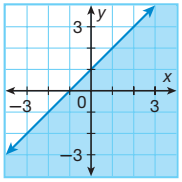
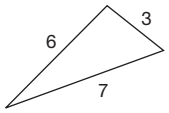
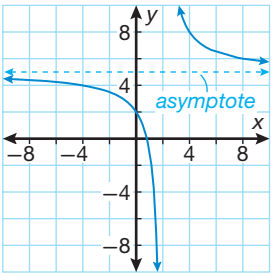
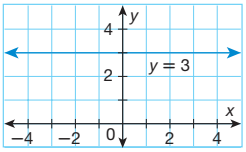
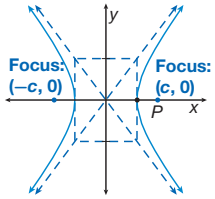
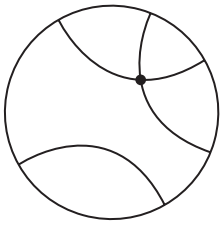
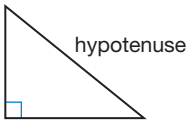
English	Example	Spanish
E		
extraneous solution *(17)* A solution of a derived equation that is not a solution of the original equation.	To solve $\sqrt{x} = -2$, square both sides; $x = 4$. Check $\sqrt{4} = -2$ is false; so 4 is an extraneous solution.	**solución extraña** *(17)* Solución de una ecuación derivada que no es una solución de la ecuación original.
F		
Factor Theorem *(95)* For any polynomial $P(x)$, $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.	$(x - 1)$ is a factor of $P(x) = x^2 - 1$ because $P(1) = 1^2 - 1 = 0$.	**Teorema del factor** *(95)* Dado el polinomio $P(x)$, $(x - a)$ es un factor de $P(x)$ si y sólo si $P(a) = 0$.
factorial *(42)* If n is a positive integer, then n factorial, written $n!$, is $n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 2 \cdot 1$. The factorial of 0 is defined to be 1.	$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ $0! = 1$	**factorial** *(42)* Si n es un entero positivo, entonces el factorial de n, expresado como $n!$, es $n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 2 \cdot 1$. Por definición, el factorial de 0 es 1.
factoring *(23)* The process of writing a number or algebraic expression as a product.	$x^2 - 4x - 21 = (x - 7)(x + 3)$	**factoreo** *(23)* Proceso por el que se expresa un número o expresión algebraica como un producto.
favorable outcome *(55)* The occurrence of one of several possible outcomes of a specified event or probability experiment.	In the experiment of rolling an odd number on a number cube, the favorable outcomes are 1, 3, and 5.	**resultado favorable** *(55)* Cuando se produce uno de varios resultados posibles de un suceso específico o experimento de probabilidades.
feasible region *(54)* The set of points that satisfy the constraints in a linear-programming problem.	Constraints: $x > 0$ $y > 0$ $x + y \leq 8$ $3x + 5y \leq 30$ Feasible region 	**región factible** *(54)* Conjunto de puntos que cumplen con las restricciones de un problema de programación lineal.

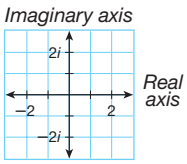
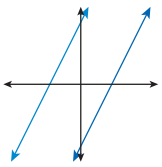
English	Example	Spanish
F		
finite sequence *(92)* A sequence with a finite number of terms.	1, 2, 3, 4, 5	**sucesión finita** *(92)* Sucesión con un número finito de términos.
first differences *(maintained)* The differences between y-values of a function for evenly spaced x-values.	x: 0, 1, 2, 3 y: 3, 7, 11, 15 first differences +4 +4 +4	**primeras diferencias** *(repaso)* Diferencias entre los valores de y de una función para valores de x espaciados uniformemente.
first quartile *(25)* The median of the lower half of a data set, denoted Q_1. Also called *lower quartile.*	Lower half | Upper half 18, 23, 28, | 49, 36, 42 First quartile	**primer cuartil** *(25)* Mediana de la mitad inferior de un conjunto de datos, expresada como Q_1. También se llama *cuartil inferior.*
focus (pl. foci) of a hyperbola *(109)* One of two fixed points F_1 and F_2 that are used to define a hyperbola. For every point P on the hyperbola, $PF_1 - PF_2$ is constant.		**foco de una hipérbola** *(109)* Uno de los dos puntos fijos F_1 y F_2 utilizados para definir una hipérbola, Para cada punto P de la hipérbola, $PF_1 - PF_2$ es constante.
focus (pl. foci) of an ellipse *(98)* One of two fixed points F_1 and F_2 that are used to define an ellipse. For every point P on the ellipse, $PF_1 + PF_2$ is constant.		**foco de una elipse** *(98)* Uno de los dos puntos fijos F_1 y F_2 utilizados para definir una elipse. Para cada punto P de la elipse, $PF_1 + PF_2$ es constante.
function *(4)* A relation in which every input is paired with exactly one output.		**función** *(4)* Una relación en la que cada entrada corresponde exactamente a una salida.
function notation *(4)* If x is the independent variable and y is the dependent variable, then the function notation for y is $f(x)$, read "f of x," where f names the function.	equation: $y = 2x$ function notation: $f(x) = 2x$	**notación de función** *(4)* Si x es la variable independiente e y es la variable dependiente, entonces la notación de función para y es $f(x)$, que se lee "f de x", donde f nombra la función.

GLOSSARY/ GLOSARIO

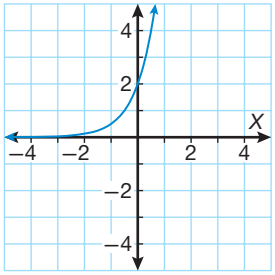
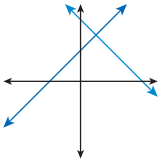
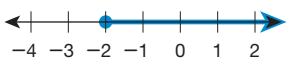
English	Example	Spanish
F		
Fundamental Counting Principle *(33)*		**Principio fundamental de conteo** *(33)*
For n items, if there are m_1 ways to choose a first item, m_2 ways to choose a second item after the first item has been chosen, and so on, then there are $m_1 \cdot m_2 \cdot \ldots \cdot m_n$ ways to choose n items.	If there are 4 colors of shirts, 3 colors of pants, and 2 colors of shoes, then there are $4 \cdot 3 \cdot 2 = 24$ possible outfits.	Dados n elementos, si existen m_1 formas de elegir un primer elemento, m_2 formas de elegir un segundo elemento después de haber elegido el primero, y así sucesivamente, entonces existen $m_1 \cdot m_2 \cdot \ldots \cdot m_n$ formas de elegir n elementos.
Fundamental Theorem of Algebra *(106)*		**Teorema fundamental del álgebra** *(106)*
Every polynomial function of degree $n \geq 1$ has at least one zero, where a zero may be a complex number.	$y = x^3 - x^2 - 5x + 9$ has at least one zero.	Cada función polinomial de grado $n \geq 1$ tiene por lo menos un cero, donde un cero puede ser un número complejo.
G		
general form of a conic section *(114)*		**forma general de una sección cónica** *(114)*
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A and B are not both 0.	A circle with a vertex at $(1, 2)$ and radius 3 has the general form $x^2 + y^2 - 2x - 4y - 4 = 0$.	$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, donde A y B no son los dos 0.
geometric probability *(55)*	 The probability of the pointer landing on red is $\frac{5}{24}$.	**probabilidad geométrica** *(55)*
A form of theoretical probability determined by a ratio of geometric measures such as lengths, areas, or volumes.		Método para calcular probabilidades basado en una medida geométrica como la longitud, el área o el volumen.
geometric sequence *(97)*		**sucesión geométrica** *(97)*
A sequence in which the ratio of successive terms is a constant r, called the common ratio, where $r \neq 0$ and $r \neq 1$.	$1, \underset{\cdot 2}{2}, \underset{\cdot 2}{4}, \underset{\cdot 2}{8}, \underset{\cdot 2}{16}, \ldots$ $r = 2$	Sucesión en la que la razón de los términos sucesivos es una constante r, denominada razón común, donde $r \neq 0$ y $r \neq 1$.
geometric series *(113)*		**serie geométrica** *(113)*
The indicated sum of the terms of a geometric sequence.	$1 + 2 + 4 + 8 + 16 + \ldots$	Suma indicada de los términos de una sucesión geométrica.

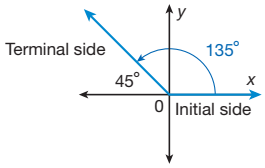
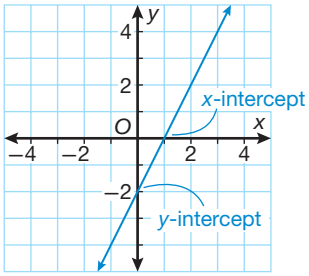
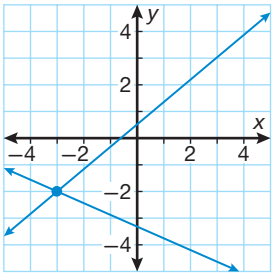
English	Example	Spanish
G		
greatest common factor (GCF) *(maintained)* The product of the greatest integer and the greatest power of each variable that divides evenly into each term.	The GCF of $4x^3y$ and $6x^2y$ is $2x^2y$. The GCF of 27 and 45 is 9.	**máximo común divisor (MCD)** *(repaso)* Producto del entero mayor y la potencia mayor de cada variable que divide exactamente cada término.
greatest integer function *(Inv 9)* A function denoted by $f(x) = [x]$ or $f(x) = \lfloor x \rfloor$ in which the number x is rounded down to the greatest integer that is less than or equal to x.	$\lfloor 4.98 \rfloor = 4$ $\lfloor -2.1 \rfloor = -3$	**función de entero mayor** *(Inv 9)* Función expresada como $f(x) = [x]$ ó $f(x) = \lfloor x \rfloor$ en la cual el número x se redondea hacia abajo hasta el entero mayor que sea menor que o igual a x.
greatest common monomial factor *(23)* A monomial that divides evenly into every term of a polynomial; its coefficient is the greatest common factor of the coefficients in the polynomial; its variables must occur in every term of the polynomial, the highest degree of each of its variables being the lowest degree of that variable in the polynomial.	The greatest monomial factor of $12x^3y^2 - 18x^4y + 6x^2y^2 + 6x^2yz$ is $6x^2y$.	**máximo monomio común** *(23)* Un monomio que divide sin residuo a cada término de un polinomio; su coeficiente es el máximo factor común de los coeficientes en el polinomio; sus variables deben de estar presentes en cada término del polinomio, el grado más alto de cada una de sus variables debe ser el grado más bajo de esa variable en el polinomio.
growth factor *(57)* The base $1 + r$ in an exponential expression.	$12{,}000(1 + 0.14)^t$ ↑ growth factor	**factor de crecimiento** *(57)* La base $1 + r$ en una expresión exponencial.
H		
half-life *(57)* The half-life of a substance is the time it takes for one-half of the substance to decay.	Carbon-14 has a half-life of 5730 years, so 5 g of an initial amount of 10 g will remain after 5730 years.	**vida media** *(57)* La vida media de una sustancia es el tiempo que tarda la mitad de la sustancia en desintegrarse y transformarse en otra sustancia.

English	Example	Spanish
H		
half-plane *(39)* The part of the coordinate plane on one side of a line, which may include the line.		**semiplano** *(39)* Parte del plano coordenado de un lado de una línea, que puede incluir la línea.
Heron's Formula *(77)* A triangle with side lengths a, b, and c has area $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is one-half the perimeter, or $s = \frac{1}{2}(a+b+c)$.	 $s = \frac{1}{2}(3+6+7) = 8$ $A = \sqrt{8(8-3)(8-6)(8-7)}$ $= \sqrt{80} = 4\sqrt{5}$ square units	**fórmula de Herón** *(77)* Un triángulo con longitudes de lado a, b y c tiene un área $= \sqrt{s(s-a)(s-b)(s-c)}$, donde s es la mitad del perímetro ó $s = \frac{1}{2}(a+b+c)$.
horizontal asymptote *(47)* A horizontal line that a graphed function approaches.		**azíntota horizontal** *(47)* Una línea horizontal a la que se aproxima la gráfica de una función.
horizontal line *(34)* A line described by the equation $y = b$, where b is the y-intercept.		**recta horizontal** *(34)* Línea descrita por la ecuación $y = b$, donde b es la intersección con el eje y.
hyperbola *(109)* The set of all points P in a plane such that the difference of the distances from P to two fixed points F_1 and F_2, called the foci, is a constant $d = \lvert PF_1 - PF_2 \rvert$.		**hipérbola** *(109)* Conjunto de todos los puntos P en un plano tal que la diferencia de las distancias de P a dos puntos fijos F_1 y F_2, llamados focos, es una constante $d = \lvert PF_1 - PF_2 \rvert$.
hyperbolic geometry *(109)* A non-Euclidean geometry; in this geometry, through a point not on a line there are at least two lines parallel to the given line.		**geometría hiperbólica** *(109)* Una geometría no-Euclideana; en esta geometría, a través de un punto que no está en una línea, hay por lo menos dos líneas paralelas a dicha línea.
hypotenuse *(41)* The side opposite the right angle in a right triangle.		**hipotenusa** *(41)* Lado opuesto al ángulo recto de un triángulo rectángulo.

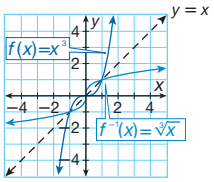
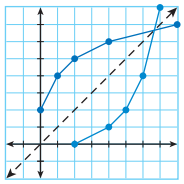
English	Example	Spanish
I		
imaginary axis *(69)* The vertical axis in the complex plane, it graphically represents the purely imaginary part of complex numbers.		**eje imaginario** *(69)* Eje vertical de un plano complejo. Representa gráficamente la parte puramente imaginaria de los números complejos.
imaginary number *(62)* The square root of a negative number, written in the form bi, where b is a real number and i is the imaginary unit, $\sqrt{-1}$. Also called a *pure imaginary number*.	$\sqrt{-16} = \sqrt{16} \cdot \sqrt{-1} = 4i$	**número imaginario** *(62)* Raíz cuadrada de un número negativo, expresado como bi, donde b es un número real e i es la unidad imaginaria, $\sqrt{-1}$. También se denomina *número imaginario puro.*
imaginary part of a complex number *(62)* For a complex number of the form $a + bi$, the real number b is called the imaginary part, represented graphically as b units on the imaginary axis of a complex plane.	$5 + 6i$ real part — imaginary part	**parte imaginaria de un número complejo** *(62)* Dado un número complejo del tipo $a + bi$, el número real b se denomina parte imaginaria y se representa gráficamente como b unidades en el eje imaginario de un plano complejo.
imaginary unit *(62)* The unit in the imaginary number system, $\sqrt{-1}$.	$\sqrt{-1} = i$	**unidad imaginaria** *(62)* Unidad del sistema de números imaginarios, $\sqrt{-1}$.
inclusive events *(60)* Events that have one or more outcomes in common.	In the experiment of rolling a number cube, rolling an even number and rolling a number less than 3 are inclusive events because the outcome 2 is both even and less than 3.	**sucesos inclusivos** *(60)* Sucesos que tienen uno o más resultados en común.
inconsistent system *(15)* A system of equations or inequalities that has no solution.	 $\begin{cases} y = 2.5x + 5 \\ y = 2.5x - 5 \end{cases}$ is inconsistent.	**sistema inconsistente** *(15)* Sistema de ecuaciones o desigualdades que no tiene solución.

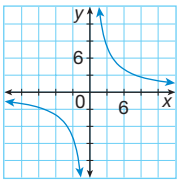
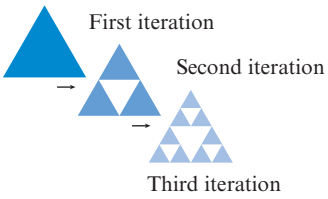
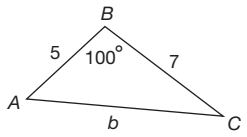
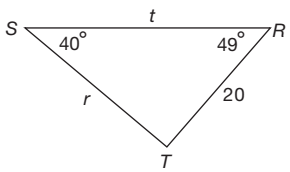
GLOSSARY/ GLOSARIO

English	Example	Spanish
I		
increasing function *(57)* A function whose output value increases as its input value increases.		**función creciente** *(57)* Una función cuyo valor de salida aumenta conforme su valor de entrada aumenta.
independent events *(33)* Events for which the occurrence or non-occurrence of one event does not affect the probability of the other event.	From a bag containing 3 red marbles and 2 blue marbles, drawing a red marble, replacing it, and then drawing a blue marble.	**sucesos independientes** *(33)* Dos sucesos son independientes si el hecho de que se produzca o no uno de ellos no afecta la probabilidad del otro suceso.
independent system *(15)* A system of equations that has exactly one solution.	 $\begin{cases} y = -x + 4 \\ y = x + 2 \end{cases}$ Solution: (1, 3)	**sistema independiente** *(15)* Sistema de ecuaciones que tiene exactamente una solución.
independent variable *(4)* The input of a function; a variable whose value determines the value of the output, or dependent variable.	$y = 2x + 1$ independent variable	**variable independiente** *(4)* Entrada de una función; variable cuyo valor determina el valor de la salida, o variable dependiente.
index *(40)* In the radical $\sqrt[n]{x}$, which represents the nth root of x, n is the index. In the radical $\sqrt{x}$, the index is understood to be 2.	The radical $\sqrt[3]{8}$ has an index of 3.	**índice** *(40)* En el radical $\sqrt[n]{x}$, que representa la enésima raíz de x, n es el índice. En el radical $\sqrt{x}$, se da por sentado que el índice es 2.
inequality *(10)* A statement that compares two expressions by using one of the following symbols: $<$, $>$, $\leq$, $\geq$, or $\neq$.	 $x \geq -2$	**desigualdad** *(10)* Enunciado que compara dos expresiones utilizando uno de los siguientes símbolos: $<$, $>$, $\leq$, $\geq$, ó $\neq$.
infinite geometric series *(113)* A geometric series with infinitely many terms.	$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10{,}000} + \ldots$	**serie geométrica infinita** *(113)* Serie geométrica con una cantidad infinita de términos.

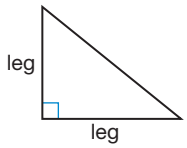
English	Example	Spanish
I		
infinite sequence *(92)* A sequence with infinitely many terms.	1, 3, 5, 7, 9, 11, …	**sucesión infinita** *(92)* Sucesión con un número infinito de términos.
initial side *(56)* The ray that lies on the positive x-axis when an angle is drawn in standard position.		**lado inicial** *(56)* El rayo que se encuentra en el eje positivo x cuando se traza un ángulo en la posición estándar.
integer *(1)* A member of the set of whole numbers and their opposites.	… −3, −2, −1, 0, 1, 2, 3 …	**entero** *(1)* Miembro del conjunto de números cabales y sus opuestos.
intercepts *(13)* On a coordinate plane, the points where a graph intersects the axes.		**intersecciones** *(13)* En un plano coordenado, los puntos donde una gráfica interseca a los ejes.
intersect *(15)* When two or more lines, or segments, cross or meet; meeting point is called point of intersection.	The lines intersect at (−3, −2). 	**intersecar** *(15)* Cuando dos o más rectas o segmentos de rectas se cruzan; el punto donde se cruzan es llamado punto de intersección.
inverse cosine function *(67)* If the domain of the cosine function is restricted to $[0, \pi]$, then the function $\text{Cos}\,\theta = a$ has an inverse function $\text{Cos}^{-1} a = \theta$, also called *arccosine*.	$\text{Cos}^{-1}\frac{1}{2} = \frac{\pi}{3}$	**función coseno inverso** *(67)* Si el dominio de la función coseno se restringe a $[0, \pi]$, entonces la función $\text{Cos}\,\theta = a$ tiene una función inversa $\text{Cos}^{-1} a = \theta$, también llamada *arco coseno*.

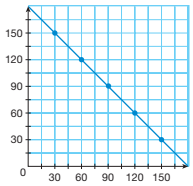
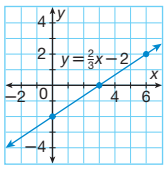
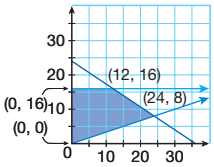
GLOSSARY/ GLOSARIO

English	Example	Spanish
I		
inverse function *(50)* The function that results from exchanging the input and output values of a one-to-one function. The inverse of $f(x)$ is denoted $f^{-1}(x)$.		**función inversa** *(50)* Función que resulta de intercambiar los valores de entrada y salida de una función uno a uno. La función inversa de $f(x)$ se expresa $f^{-1}(x)$.
inverse matrix *(32)* A square matrix such that the product of it and another matrix forms an identity matrix.	The inverse matrix of $\begin{bmatrix}5 & 2\\ 8 & 4\end{bmatrix}$ is $\begin{bmatrix}1 & -\frac{1}{2}\\ -2 & 1\frac{1}{4}\end{bmatrix}$ because $\begin{bmatrix}5 & 2\\ 8 & 4\end{bmatrix} \cdot \begin{bmatrix}1 & -\frac{1}{2}\\ -2 & 1\frac{1}{4}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.	**matriz inversa** *(32)* Una matriz cuadrada tal que el producto de ella por otra matriz resulta en la matriz identidad.
inverse relation *(50)* The inverse of the relation consisting of all ordered pairs (x, y) is the set of all ordered pairs (y, x). The graph of an inverse relation is the reflection of the graph of the relation across the line $y = x$.		**relación inversa** *(50)* La inversa de la relación que consta de todos los pares ordenados (x, y) es el conjunto de todos los pares ordenados (y, x). La gráfica de una relación inversa es el reflejo de la gráfica de la relación sobre la línea $y = x$.
inverse sine function *(67)* If the domain of the sine function is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then the function $\text{Sin}\,\theta = a$ has an inverse function, $\text{Sin}^{-1} a = \theta$, also called *arcsine*.	$\text{Sin}^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$	**función seno inverso** *(67)* Si el dominio de la función seno se restringe a $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, entonces la función $\text{Sin}\,\theta = a$ tiene una función inversa, $\text{Sin}^{-1} a = \theta$, también llamada *arco seno*.
inverse tangent function *(67)* If the domain of the tangent function is restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the function $\text{Tan}\,\theta = a$ has an inverse function, $\text{Tan}^{-1} a = \theta$, also called *arctangent*.	$\text{Tan}^{-1}\sqrt{3} = \frac{\pi}{3}$	**función tangente inversa** *(67)* Si el dominio de la función tangente se restringe a $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, entonces la función $\text{Tan}\,\theta = a$ tiene una función inversa, $\text{Tan}^{-1} a = \theta$, también llamada *arco tangente*.

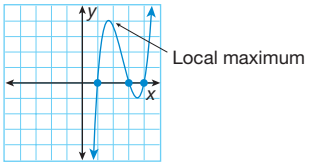
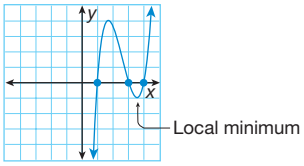
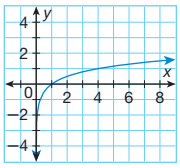
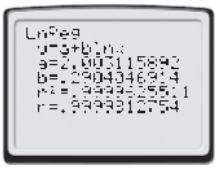
English	Example	Spanish
I		
inverse variation *(12)* A relationship between two variables, x and y, that can be written in the form $y = \frac{k}{x}$, where k is a nonzero constant and $x \neq 0$.	 $y = \frac{24}{x}$	**variación inversa** *(12)* Relación entre dos variables, x e y, que puede expresarse en la forma $y = \frac{k}{x}$, donde k es una constante distinta de cero y $x \neq 0$.
irrational number *(1)* A real number that cannot be expressed as the ratio of two integers.	$\sqrt{2}$, π, e	**número irracional** *(1)* Número real que no se puede expresar como una razón de enteros.
iteration *(maintained)* The repetitive application of the same rule.		**iteración** *(repaso)* Aplicación repetitiva de la misma regla.
J		
joint variation *(12)* A relationship among three variables that can be written in the form $y = kxz$, where k is a nonzero constant.	$y = 3xz$	**variación conjunta** *(12)* Relación entre tres variables que se puede expresar de la forma $y = kxz$, donde k es una constante distinta de cero.
L		
Law of Cosines *(77)* For $\triangle ABC$ with side lengths a, b, and c, $a^2 = b^2 + c^2 - 2bc \cos A$ $b^2 = a^2 + c^2 - 2ac \cos B$ $c^2 = a^2 + b^2 - 2ab \cos C$.	 $b^2 = 7^2 + 5^2 - 2(7)(5) \cos 100°$ $b^2 \approx 86.2$ $b \approx 9.3$	**Ley de cosenos** *(77)* Dado $\triangle ABC$ con longitudes de lado a, b y c, $a^2 = b^2 + c^2 - 2bc \cos A$ $b^2 = a^2 + c^2 - 2ac \cos B$ $c^2 = a^2 + b^2 - 2ab \cos C$.
Law of Sines *(71)* For $\triangle ABC$ with side lengths a, b, and c, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.	 $\frac{\sin 49°}{r} = \frac{\sin 40°}{20}$ $r = \frac{20 \sin 49°}{\sin 40°} \approx 23.5$	**Ley de senos** *(71)* Dado $\triangle ABC$ con longitudes de lado a, b y c, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

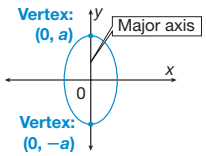
GLOSSARY/ GLOSARIO

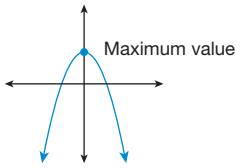
English	Example	Spanish
L		
leading coefficient *(11)* The coefficient of the first term of a polynomial in standard form.	$3x^2 + 7x - 2$ Leading coefficient	**coeficiente principal** *(11)* Coeficiente del primer término de un polinomio en forma estándar.
least common denominator (LCD) *(maintained)* The least common multiple of two or more given denominators.	The LCD of $\frac{3}{4}$ and $\frac{5}{6}$ is 12.	**mínimo común denominador (mcd)** *(repaso)* Mínimo común múltiplo de dos o más denominadores dados.
least common multiple (LCM) *(maintained)* The smallest positive integer (or polynomial) that is a multiple of two numbers (or polynomials.)	The LCM of 10 and 18 is 90. The LCM of $2x^2$ *and* $5x^3$ *is* $10x^3$.	**mínimo común múltiplo (mcm)** *(repaso)* El número (o polinomio) positivo más pequeño que es un múltiplo de dos números (o polinomios).
leg of a right triangle *(41)* One of the two sides of the right triangle that form the right angle.		**cateto de un triángulo rectángulo** *(41)* Uno de los dos lados de un triángulo rectángulo que forman el ángulo recto.
likelihood *(55)* a measure of the chance of something happening.	The likelihood of snow in Miami in June is very low.	**posibilidad** *(55)* Una medida de la probabilidad de que algo ocurra.
like radical terms *(40)* Radical terms having the same radicand and index.	$3\sqrt{2x}$ and $\sqrt{2x}$ Like radicals $\sqrt{3x}$ and $\sqrt{2x}$ Unlike radicals	**radicales semejantes** *(40)* Términos radicales que tienen el mismo radicando e índice.
like terms *(2)* Terms with the same variables raised to the same exponents.	$3a^3b^2$ and $7a^3b^2$ Like terms $4xy^2$ and $6x^2y$ Unlike terms	**términos semejantes** *(2)* Términos con las mismas variables elevadas a los mismos exponentes.
limit *(113)* A number (or infinity) that the terms of an infinite sequence or series approach as the term number increases.	The series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ has a limit of 1.	**límite** *(113)* Número (o infinito) al que se aproximan los términos de una sucesión o serie infinita a medida que aumenta el número de términos.

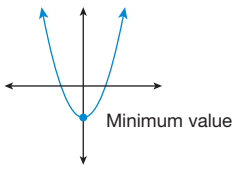
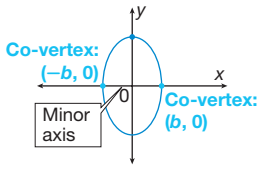
English	Example	Spanish
L		
line of best fit *(45)* The line that comes closest to all of the points in a data set.		**línea de mejor ajuste** *(45)* Línea que más se acerca a todos los puntos de un conjunto de datos.
linear equation in one variable *(maintained)* An equation that can be written in the form $ax = b$, where a and b are constants and $a \neq 0$.	$x + 1 = 7$	**ecuación lineal en una variable** *(repaso)* Ecuación que puede expresarse en la forma $ax = b$, donde a y b son constantes y $a \neq 0$.
linear function *(34)* A function that can be written in the form $f(x) = mx + b$, where x is the independent variable and m and b are real numbers. Its graph is a line.		**función lineal** *(34)* Función que puede expresarse en la forma $f(x) = mx + b$, donde x es la variable independiente y m y b son números reales. Su gráfica es una línea.
linear inequality in two variables *(39)* An inequality that can be written in one of the following forms: $y < mx + b$, $y > mx + b$, $y \leq mx + b$, $y \geq mx + b$, or $y \neq mx + b$, where m and b are real numbers.	$2x + 3y \leq 6$ $y > \frac{1}{2}x - 7$	**desigualdad lineal en dos variables** *(39)* Desiqualdad que puede expresarse de una de las siguientes formas: $y < mx + b$, $y > mx + b$, $y \leq mx + b$, $y \geq mx + b$, o $y \neq mx + b$, donde m y b son números reales.
linear programming *(54)* A method of finding a maximum or minimum value of a linear function, called the *objective function*, that satisfies a given set of conditions, called *constraints*.	Constraints $\begin{cases} x \geq 0 \\ 40x + 60y \leq 1440 \\ y \geq \frac{1}{3}x \\ y \leq 16 \end{cases}$ Feasible Region  For the given constraints, the objective function $P = 18x + 25y$ is maximized at $(24, 8)$.	**programación lineal** *(54)* Método para calcular un valor máximo o mínimo de una función lineal, denominada *función objetiva*, que cumple con una serie dada de condiciones, denominadas *restricciones*.
linear regression *(45)* A statistical method used to fit a linear model to a given data set.		**regresión lineal** *(45)* Método estadístico utilizado para ajustar un modelo lineal a un conjunto de datos determinado.

GLOSSARY/ GLOSARIO

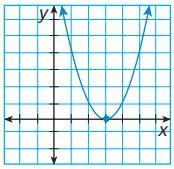
English	Example	Spanish
L		
linear system *(15)* A system of equations containing only linear equations.	$\begin{cases} y = 2x + 1 \\ x + y = 8 \end{cases}$	**sistema lineal** *(15)* Sistema de ecuaciones que contiene sólo ecuaciones lineales.
local maximum *(101)* For a function f, $f(a)$ is a local maximum if there is an interval around a such that $f(x) < f(a)$ for every x-value in the interval except a.		**máximo local** *(101)* Dada una función f, $f(a)$ es el máximo local si hay un intervalo en a tal que $f(x) < f(a)$ para cada valor de x en el intervalo excepto a.
local minimum *(101)* For a function f, $f(a)$ is a local minimum if there is an interval around a such that $f(x) > f(a)$ for every x-value in the interval except a.		**mínimo local** *(101)* Dada una función f, $f(a)$ es el mínimo local si hay un intervalo en a tal que $f(x) > f(a)$ para cada valor de x en el intervalo excepto a.
logarithm *(64)* The exponent that a specified base must be raised to in order to get a certain value.	$\log_2 8 = 3$, because 3 is the power that 2 is raised to in order to get 8; or $2^3 = 8$.	**logaritmo** *(64)* Exponente al cual debe elevarse una base determinada a fin de obtener cierto valor.
logarithmic equation *(102)* An equation that contains a logarithm of a variable.	$\log x + 3 = 7$	**ecuación logarítmica** *(102)* Ecuación que contiene un logaritmo de una variable.
logarithmic function *(110)* A function of the form $f(x) = \log_b x$, where $b \neq 1$ and $b > 0$, which is the inverse of the exponential function $f(x) = b^x$.	 $f(x) = \log_4 x$	**función logarítmica** *(110)* Función del tipo $f(x) = \log_b x$, donde $b \neq 1$ y $b > 0$, que es la inversa de la función exponencial $f(x) = b^x$.
logarithmic inequality *(102)* An inequality that contains a logarithm of a variable.	$\log x - \log 2 \leq \log 75$	**desigualdad logarítmica** *(102)* Una desigualdad que contiene un logaritmo de una variable.
logarithmic regression *(116)* A statistical method used to fit a logarithmic model to a given data set.		**regresión logarítmica** *(116)* Método estadístico utilizado para ajustar un modelo logarítmico a un conjunto de datos determinado.

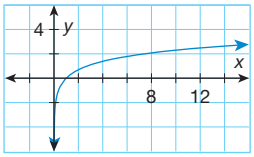
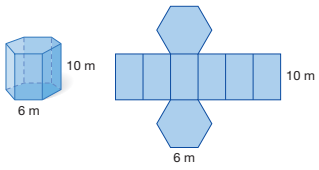
English	Example	Spanish
M		
main diagonal (of a matrix) *(9)* The diagonal from the upper left corner to the lower right corner of a matrix.	$\begin{bmatrix} 3 & 1 & 2 \\ 5 & 0 & 1 \\ 2 & 7 & 6 \end{bmatrix}$	**diagonal principal (de una matriz)** *(9)* Diagonal que se extiende desde la esquina superior izquierda hasta la esquina inferior derecha de una matriz.
major axis *(98)* The longer axis of an ellipse. The foci of the ellipse are located on the major axis, and its endpoints are the *vertices of the ellipse*.		**eje mayor** *(98)* El eje más largo de una elipse. Los focos de la elipse se encuentran sobre el eje mayor y sus extremos son los *vértices de la elipse*.
mathematical induction *(Inv 12)* A type of mathematical proof. To prove that a statement is true for all natural numbers n, first show that the statement is true for $n = 1$; then assume it is true for some number k and prove that it is true for $k + 1$. It follows that the statement is true for all values of n.		**inducción matemática** *(Inv 12)* Tipo de demostración matemática. Para demostrar que un enunciado se cumple para todos los números naturales n, primero se demuestra que el enunciado se cumple para $n = 1$; luego se supone que se cumple para un número k y se demuestra que se cumple para $k + 1$. Por lo tanto, el enunciado se cumplirá para todos los valores de n.
matrix *(5)* A rectangular array of numbers.	$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & -5 \\ 7 & -6 & 3 \end{bmatrix}$	**matriz** *(5)* Arreglo rectangular de números.
matrix addition *(5)* Adding each element in one matrix to the element that is in the same location in a second matrix.	$\begin{bmatrix} 1 & 0 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 9 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} -4 & 9 \\ 13 & 7 \end{bmatrix}$	**adición de matrices** *(5)* Sumar cada elemento en una matriz al elemento que está en el mismo lugar en una segunda matriz.
matrix equation *(5)* An equation of the form $AX = B$, where A is the coefficient matrix, X is the variable matrix, and B is the constant matrix of a system of equations.	$\begin{bmatrix} 3 & 7 \\ 9 & 1 \end{bmatrix} + X = \begin{bmatrix} 5 & 9 \\ 2 & -6 \end{bmatrix}$	**ecuación matricial** *(5)* Ecuación del tipo $AX = B$, donde A es la matriz de coeficientes, X es la matriz de variables y B es la matriz de constantes de un sistema de ecuaciones.

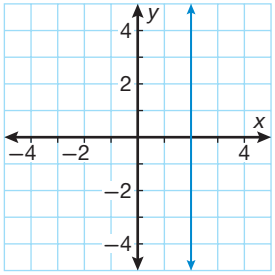
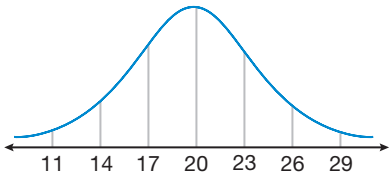
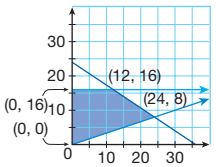
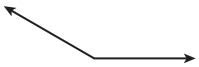
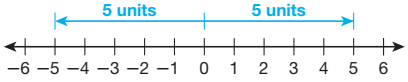
English	Example	Spanish
M		
matrix of constants *(32)* A matrix consisting of the constants used in a system of equations.	For $\begin{cases} 2x + 4y = 8 \\ 3x - y = -2 \end{cases}$, it is $\begin{bmatrix} 8 \\ -2 \end{bmatrix}$.	**matriz de constantes** *(32)* Una matriz que consiste en de las constantes utilizadas en un sistema de ecuaciones.
matrix of variables *(32)* A matrix consisting of the variables used in a system of equations.	For $\begin{cases} 2x + 4y + z = 8 \\ 3x - y - 7z = -2 \\ x + 6y - 2z = 1 \end{cases}$, it is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.	**matriz de variables** *(32)* Una matriz que consiste en las variables utilizadas en un sistema de ecuaciones.
matrix subtraction *(maintained)* Subtracting each element in one matrix from the element that is in the same location in a second matrix.	$\begin{bmatrix} 1 & 0 \\ 7 & 4 \end{bmatrix} - \begin{bmatrix} -5 & 9 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 1 & 1 \end{bmatrix}$	**resta de matrices** *(repaso)* Restar cada elemento en una matriz del elemento que está en el mismo lugar en una segunda matriz.
maximum value of a function *(30)* The y-value of the highest point on the graph of the function.		**máximo de una función** *(30)* Valor de y del punto más alto en la gráfica de la función.
mean *(25)* The sum of all the values in a data set divided by the number of data values; also called the *average*.	Data set: 4, 6, 7, 8, 10 Mean: $\frac{4 + 6 + 7 + 8 + 10}{5} = \frac{35}{5} = 7$	**media** *(25)* Suma de todos los valores de un conjunto de datos dividido entre el número de valores de datos; también llamada *promedio.*
measure of central tendency *(25)* A measure that describes the center of a data set.	the mean, median, or mode	**medida de tendencia dominante** *(25)* Medida que describe el centro de un conjunto de datos.
measure of dispersion *(25)* A statistic that indicates how spread out, or dispersed, the data values are; common measures are range and standard deviation.	For data set: 5, 8, 12, 14, 16 range = 11, standard deviation ≈ 4.47 For data set: 1, 9, 41, 60, 95 range = 94, standard deviation ≈ 38.41	**medida de dispersión** *(25)* Una estadística que indica que tan alejados o tan dispersados están los valores de datos; medidas comunes son rango y desviación estándar.

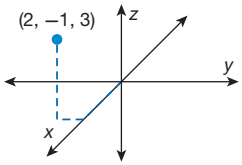
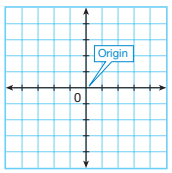
English	Example	Spanish
M		
measure of variation *(maintained)* A measure that describes the spread of a data set.	the range, variance, standard deviation, or interquartile range	**medida de variación** *(repaso)* Medida que describe la dispersión de un conjunto de datos.
median *(25)* For an ordered data set with an odd number of values, the median is the middle value. For an ordered data set with an even number of values, the median is the average of the two middle values.	8, 9, 9, 12, 15 Median: 9 4, 6, 7, 10, 10, 12 Median: $\frac{7+10}{2} = 8.5$	**mediana de un conjunto de datos** *(25)* Dado un conjunto de datos ordenados con un número impar de valores, la mediana es el valor del medio. Dado un conjunto de datos ordenados con un número par de valores, la mediana es el promedio de los dos valores del medio.
minimum value of a function *(30)* The y-value of the lowest point on the graph of the function.		**mínimo de una función** *(30)* Valor y del punto más bajo en la gráfica de la función.
minor *(14)* The minor of an element in a matrix is the determinant of the terms that remain when the row and column for that element are deleted.	For $\begin{bmatrix} 2 & -1 & 6 \\ 5 & 7 & 4 \\ 1 & 0 & 3 \end{bmatrix}$, the minor of 2 is $\begin{vmatrix} 7 & 4 \\ 0 & 3 \end{vmatrix}$.	**menor** *(14)* El menor de un elemento en una matriz es el determinante de los términos que quedan cuando la fila y la columa para ese elemento son eliminadas.
minor axis *(98)* The shorter axis of an ellipse. Its endpoints are the co-vertices of the ellipse.		**eje menor** *(98)* El eje más corto de una elipse. Sus extremos son los co-vértices de la elipse.
mode *(25)* The value or values that occur most frequently in a data set; if all values occur with the same frequency, the data set is said to have no mode.	Data set: 3, 6, 8, 8, 10 Mode: 8 Data set: 2, 5, 5, 7, 7 Modes: 5 and 7 Data set: 2, 3, 6, 9, 11 No mode	**moda** *(25)* El valor o los valores que se presentan con mayor frecuencia en un conjunto de datos. Si todos los valores se presentan con la misma frecuencia, se dice que el conjunto de datos no tiene moda.

M

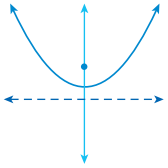
English	Example	Spanish
monomial *(11)*	$8x$, 9, $3x^2y^4$	**monomio** *(11)*
A number or a product of numbers and variables with whole-number exponents, or a polynomial with one term.		Número o producto de números y variables con exponentes de números cabales, o polinomio con un término.
monomial factor *(23)*	$14x^2 + 20x^3$ has a monomial factor of $2x^2$.	**factor monomial** *(23)*
A common factor of a polynomial that is a number, variable, or product of numbers and variables.		Un factor común de un polinomio que es un número, variable, o producto de números y variables.
multiple root *(maintained)*	 3 is a multiple root of $P(x) = (x - 3)^2$.	**raíz múltiple** *(repaso)*
A root r is a multiple root when the factor $(x - r)$ appears in the equation more than once.		Una raíz r es una raíz múltiple cuando el factor $(x - r)$ aparece en la ecuación más de una vez.
multiplicative identity matrix *(9)*	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	**matriz de identidad multiplicativa** *(9)*
A square matrix with 1 in every entry of the main diagonal and 0 in every other entry.		Una matriz cuadrada que contiene 1 en cada entrada de la diagonal principal y 0 en las demás entradas.
multiplicative inverse of a square matrix *(32)*	The multiplicative inverse of $A = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -3 & -5 \\ -1 & -2 \end{bmatrix}$, because $AA^{-1} = A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.	**inverso multiplicativo de una matriz cuadrada** *(32)*
The multiplicative inverse of square matrix A, if it exists, is notated A^{-1}, where the product of A and A^{-1} is the identity matrix.		El inverso multiplicativo de una matriz cuadrada A, si existe, se escribe A^{-1}, donde el producto de A y A^{-1} es la matriz de identidad.
multiplicity *(66)*	For $P(x) = (x - 3)^2$, the root 3 has a multiplicity of 2.	**multiplicidad** *(66)*
If a polynomial $P(x)$ has a multiple root at r, the multiplicity of r is the number of times $(x - r)$ appears as a factor in $P(x)$.		Si un polinomio $P(x)$ tiene una raíz múltiple en r, la multiplicidad de r es la cantidad de veces que $(x - r)$ aparece como factor en $P(x)$.

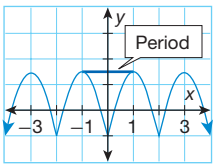
English	Example	Spanish
M		
mutually exclusive events *(33)* Two events are mutually exclusive if they cannot both occur in the same trial of an experiment.	In the experiment of rolling a number cube, rolling a 3 and rolling an even number are mutually exclusive events.	**sucesos mutuamente excluyentes** *(33)* Dos sucesos son mutuamente excluyentes si ambos no pueden ocurrir en la misma prueba de un experimento.
N		
natural logarithm *(81)* A logarithm with base e, written as ln.	$\ln 5 = \log_e 5 \approx 1.6$	**logaritmo natural** *(81)* Logaritmo con base e, que se escribe ln.
natural logarithmic function *(110)* The function $f(x) = \ln x$, which is the inverse of the natural exponential function $f(x) = e^x$. Domain is $\{x \mid x > 0\}$; range is all real numbers.		**función logarítmica natural** *(110)* Función $f(x) = \ln x$, que es la inversa de la función exponencial natural $f(x) = e^x$. El dominio es $\{x \mid x > 0\}$; el rango es todos los números reales.
negative exponent *(3)* A base raised to a negative exponent is equal to the reciprocal of that base raised to the opposite exponent: $b^{-n} = \frac{1}{b^n}$.	$5^{-3} = \frac{1}{5^3} = \frac{1}{125}$	**exponente negativo** *(3)* Una base elevada a un exponente negativo es igual al recíproco de dicha base elevado al exponente opuesto: $b^{-n} = \frac{1}{b^n}$.
net *(maintained)* A diagram of the faces of a three-dimensional figure arranged in such a way that the diagram can be folded to form the three-dimensional figure.		**plantilla** *(repaso)* Diagrama de las caras y superficies de una figura tridimensional que se puede plegar para formar la figura tridimensional.
nonlinear system of equations *(117)* A system in which at least one of the equations is not linear.	$y = 2x^2$ $y = -3x^2 + 5$	**sistema no lineal de ecuaciones** *(117)* Sistema en el cual por lo menos una de las ecuaciones no es lineal.

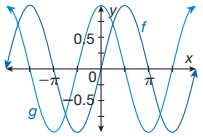
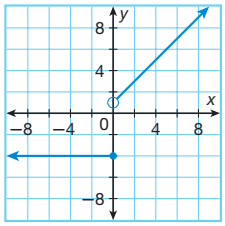
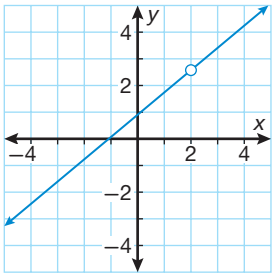
English	Example	Spanish
N		
no slope *(maintained)* The slope of a vertical line; the run equals 0.		**sin pendiente** *(repaso)* La pendiente de una línea vertical; la distancia horizontal es igual a 0.
normal distribution *(80)* A distribution of data that is bell-shaped and symmetric about the mean.	mean: 20, standard deviation: 3 	**distribución normal** *(80)* Una distribución de datos que tiene la forma de una campana y que es simétrica con respecto a la media.
***n*th root** *(59)* The nth root of a number a, written as $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$, is a number that is equal to a when it is raised to the nth power.	$\sqrt[5]{32} = 2$, because $2^5 = 32$.	**enésima raíz** *(59)* La enésima raíz de un número a, que se escribe como $\sqrt[n]{a}$ o $a^{\frac{1}{n}}$, es un número igual a a cuando se eleva a la enésima potencia.
O		
objective function *(54)* The function to be maximized or minimized in a linear programming problem.	 The objective function $P = 18x + 25y$ is maximized at $(24, 8)$.	**función objetiva** *(54)* Función que se debe maximizar o minimizar en un problema de programación lineal.
obtuse angle *(maintained)* An angle that measures greater than 90° and less than 180°.		**ángulo obtuso** *(repaso)* Ángulo que mide más de 90° y menos de 180°.
opposite *(SB 7)* The opposite of a number a, denoted $-a$, is the number that is the same distance from zero as a, on the opposite side of the number line. The sum of opposites is 0.	 5 and -5 are opposites.	**opuesto** *(SB 7)* El opuesto de un número a, expresado $-a$, es el número que se encuentra a la misma distancia de cero que a, del lado opuesto de la recta numérica. La suma de los opuestos es 0.

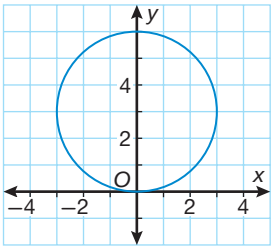
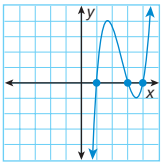
English	Example	Spanish
O		
order of operations *(maintained)* A process for evaluating expressions: First, perform operations in parentheses or other grouping symbols. Second, evaluate powers and roots. Third, perform all multiplication and division from left to right. Fourth, perform all addition and subtraction from left to right.	$2 + 3^2 - (7 + 5) \div 4 \cdot 3$ $2 + 3^2 - 12 \div 4 \cdot 3$ Add inside parentheses. $2 + 9 - 12 \div 4 \cdot 3$ Evaluate the power. $2 + 9 - 3 \cdot 3$ Divide. $2 + 9 - 9$ Multiply. $11 - 9$ Add. 2 Subtract.	**orden de las operaciones** *(repaso)* Proceso para evaluar las expresiones: Primero, realizar las operaciones entre paréntesis u otros símbolos de agrupación. Segundo, evaluar las potencias y las raíces. Tercero, realizar todas las multiplicaciones y divisiones de izquierda a derecha. Cuarto, realizar todas las sumas y restas de izquierda a derecha.
ordered triple *(29)* A set of three numbers that can be used to locate a point (x, y, z) in a three-dimensional coordinate system.		**tripleta ordenada** *(29)* Conjunto de tres números que se pueden utilizar para ubicar un punto (x, y, z) en un sistema de coordenadas tridimensional.
origin *(SB 10)* The intersection of the x- and y-axes in a coordinate plane. The coordinates of the origin are $(0, 0)$.		**origen** *(SB 10)* Intersección de los ejes x e y en un plano coordenado. Las coordenadas del origen son $(0, 0)$.
outcome *(33)* A possible result of a probability experiment.	In the experiment of rolling a number cube, the possible outcomes are 1, 2, 3, 4, 5, and 6.	**resultado** *(33)* Resultado posible en un experimento de probabilidades.
outlier *(25)* A data value that is far removed from the rest of the data. A value less than $Q_1 - 1.5(\text{IQR})$ or greater than $Q_3 + 1.5(\text{IQR})$ is considered to be an outlier.		**valor extremo** *(25)* Valor de datos que está muy alejado del resto de los datos. Un valor menor que $Q_1 - 1.5(\text{IQR})$ o mayor que $Q_3 + 1.5(\text{IQR})$ se considera un valor extremo.

GLOSSARY/ GLOSARIO

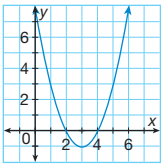
English	Example	Spanish
O		
overlapping events *(60)*	When rolling a number cube, rolling an even number and rolling a 2 are overlapping events.	**sucesos coincidentes** *(60)*
Events that have at least one outcome in common.		Sucesos que tienen por lo menos un resultado posible en común.
P		
parabola *(27)*		**parábola** *(27)*
The shape of the graph of a quadratic function. All parabolas have a symmetric *u*-shape.		Forma de la gráfica de una función cuadrática. Todas las parábolas tienen una forma de *u* simétrica.
parameter *(Inv 2)*	$y = (x - h)^2 + k$ parameters	**parámetro** *(Inv 2)*
One of the constants in a function or equation that may be changed. Also the third variable in a set of parametric equations.		Una de las constantes en una función o ecuación que se puede cambiar. También es la tercera variable en un conjunto de ecuaciones paramétricas.
parametric equations *(Inv 2)*	$x(t) = t + 1$ $y(t) = -2t$	**ecuaciones paramétricas** *(Inv 2)*
A pair of equations that define the *x*- and *y*-coordinates of a point in terms of a third variable called a parameter.		Par de ecuaciones que definen las coordenadas *x* e *y* de un punto en función de una tercera variable denominada parámetro.
parent function *(17)*	$f(x) = x^2$ is the parent function for $g(x) = x^2 + 4$ and $h(x) = 5(x + 2)^2 - 3$.	**función madre** *(17)*
The simplest function with the defining characteristics of the family. Functions in the same family are transformations of their parent function.		La función más básica con las características de la familia. Las funciones de la misma familia son transformaciones de su función madre.
partial sum *(105)*	For the sequence $a_n = n^2$, the fourth partial sum of the infinite series $\sum_{k=1}^{\infty} k^2$ is $\sum_{k=1}^{4} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$.	**suma parcial** *(105)*
Indicated by $S_n = \sum_{i=1}^{n} a_i$, the sum of a specified number of terms n of a sequence whose total number of terms is greater than n.		Expresada por $S_n = \sum_{i=1}^{n} a_i$, la suma de un número específico n de términos de una sucesión cuyo número total de términos es mayor que n.

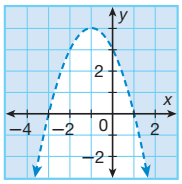
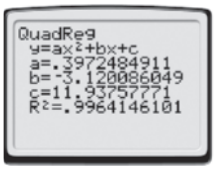
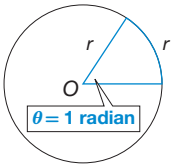
English	Example	Spanish
P		
Pascal's triangle *(42)*	1 1 1 1 2 1 1 3 3 1 1 4 6 4 1	**triángulo de Pascal** *(42)*
A triangular arrangement of numbers in which every row starts and ends with 1 and each other number is the sum of the two numbers above it.		Arreglo triangular de números en el cual cada fila comienza y termina con 1 y cada uno de los demás números es la suma de los dos números que están encima de él.
percent of change *(6)*		**porcentaje de cambio** *(6)*
An increase or decrease given as a percent of the original amount. Percent increase describes an amount that has grown. Percent decrease describes an amount that has been reduced.		Incremento o disminución dada como un porcentaje de la cantidad original. El porcentaje de incremento describe una cantidad que ha aumentado. El porcentaje de disminución describe una cantidad que se ha reducido.
perfect square *(maintained)*	36 is a perfect square because $\sqrt{36} = 6$.	**cuadrado perfecto** *(repaso)*
A number whose positive square root is a whole number.		Número cuya raíz cuadrada positiva es un número cabal.
perfect-square trinomial *(23)*	$x^2 + 6x + 9$ is a perfect-square trinomial, because $x^2 + 6x + 9 = (x + 3)^2$.	**trinomio cuadrado perfecto** *(23)*
A trinomial whose factored form is the square of a binomial. A perfect-square trinomial has the form $a^2 - 2ab + b^2 = (a - b)^2$ or $a^2 + 2ab + b^2 = (a + b)^2$.		Trinomio cuya forma factorizada es el cuadrado de un binomio. Un trinomio cuadrado perfecto tiene la forma $a^2 - 2ab + b^2 = (a - b)^2$ ó $a^2 + 2ab + b^2 = (a + b)^2$.
period of a periodic function *(82)*		**periodo de una función periódica** *(82)*
The length of a cycle measured in units of the independent variable (usually time in seconds).		Longitud de un ciclo medido en unidades de la variable independiente (generalmente el tiempo en segundos).
periodic function *(82)*	$\sin(x)$, $\cos(x)$, and $\tan(x)$ are all periodic functions.	**función periódica** *(82)*
A function that repeats exactly in regular intervals, called *periods*.		Función que se repite exactamente a intervalos regulares denominados *períodos.*

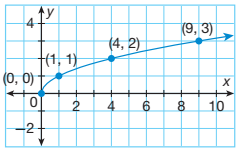
English	Example	Spanish
P		
permutation *(42)* An arrangement of a group of objects in which order is important. The number of permutations of r objects from a group of n objects is denoted ${}_nP_r$.	For 4 objects A, B, C, and D, there are ${}_4P_2 = 12$ different permutations of 2 objects: AB, AC, AD, BC, BD, CD, BA, CA, DA, CB, DB, and DC.	**permutación** *(42)* Arreglo de un grupo de objetos en el cual el orden es importante. El número de permutaciones de r objetos de un grupo de n objetos se expresa ${}_nP_r$.
phase shift *(86)* A horizontal translation of a periodic function.	 g is a phase shift of $f\frac{\pi}{2}$ units left.	**cambio de fase** *(86)* Traslación horizontal de una función periódica.
piecewise function *(79)* A function that is a combination of one or more functions.	 $f(x) = \begin{cases} -4 & \text{if } x \le 0 \\ x + 1 & \text{if } x > 0 \end{cases}$	**función a trozos** *(79)* Función que es una combinación de una o más funciones.
point of discontinuity *(22)* A point on a graph that is not connected; appears as a hole in the graph.		**punto de discontinuidad** *(22)* Un punto en una gráfica que no está conectado; aparece como un agujero en la gráfica.
point-slope form *(26)* The point-slope form of a linear equation is $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line.	The equation of the line through $(2, 1)$ with slope 3 is $y - 1 = 3(x - 2)$.	**forma de punto y pendiente** *(26)* La forma de punto y pendiente de una ecuación lineal es $y - y_1 = m(x - x_1)$, donde m es la pendiente y (x_1, y_1) es un punto en la línea.

English	Example	Spanish
P		
polar coordinate *(96)* An ordered pair (r, θ) where r is the directed distance from O to P and θ is the directed angle measure counterclockwise from the polar axis to $\overline{OP}$.		**coordenada polar** *(96)* Un par ordenado (r, θ) donde r es la distancia dirigida desdo O hasta P y θ es la medida del ángulo dirigido en el sentido contrario al de las manecillas de reloj, desde el eje polar hasta $\overline{OP}$.
polar equation *(96)* An equation involving r and θ, where r determines the radius from the origin and [theta] indicates the angle formed with the positive x-axis.	$r(\theta) = 6 \sin \theta$ 	**ecuación polar** *(96)* Una ecuación que involucra r y θ, donde r determina el radio desde el origen y [theta] indica el ángulo formado con el eje x positivo.
polynomial *(11)* A monomial or a sum or difference of monomials.	$2x^2 + 3x - 7$	**polinomio** *(11)* Monomio o suma o diferencia de monomios.
polynomial factor *(76)* A factor of a polynomial that is a polynomial.	$x - 2$ and $x + 5$ are binomial factors of $x^2 + 3x - 10$ because $x^2 + 3x - 10 = (x - 2)(x + 5)$.	**factor polinomial** *(76)* Un factor de un polinomio que es un polinomio.
polynomial function *(11)* A function whose rule is a polynomial.	 $f(x) = x^3 - 8x^2 + 19x - 12$	**función polinomial** *(11)* Función cuya regla es un polinomio.
polynomial roots *(76)* The solutions of a polynomial equation; the zeros of the related polynomial function.	$x^3 - 13x - 12 = 0$ $(x + 1)(x + 3)(x - 4) = 0$ $x = -1$ or $x = -3$ or $x = 4$ The roots of $x^3 - 13x - 12 = 0$ are -1, -3, and 4.	**raíces polinomiales** *(76)* Las soluciones de una ecuación polinomial; los ceros de la función polinomial relacionada.
population *(73)* A group of individuals about which information is desired.	A mayor wanting to know how many people will vote for him will survey citizens registered to vote in his city; the population is every citizen in his city that is registered to vote.	**población** *(73)* Un grupo de individuos de los cuales se desea información.

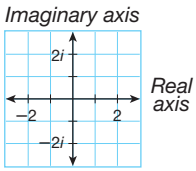
English	Example	Spanish
P		
power *(SB 3)* An expression written with a base and an exponent or the value of such an expression.	$2^3 = 8$, so 8 is the third power of 2.	**potencia** *(SB 3)* Expresión escrita con una base y un exponente o el valor de dicha expresión.
precision *(18)* The number of significant digits in a measurement.	A measurement of 4.3 cm is more precise than a measure of 4 cm.	**precisión** *(18)* El número de dígitos significativos en una medición.
prime polynomial *(23)* A polynomial that cannot be factored	$x + 5$ $x^2 - x + 14$	**polinomio primo** *(23)* Un polinomio que no puede ser factorizado.
principal root *(40)* The positive root of a number, indicated by the radical sign.	$\sqrt{36} = 6$	**raíz principal** *(40)* Raíz cuadrada positiva de un número, expresada por el signo de radical.
probability *(55)* A number from 0 to 1 (or 0% to 100%) that is the measure of how likely an event is to occur.	A bag contains 3 red marbles and 4 blue marbles. The probability of choosing a red marble is $\frac{3}{7}$.	**probabilidad** *(55)* Número entre 0 y 1 (o entre 0% y 100%) que describe cuán probable es que ocurra un suceso.
probability distribution for an experiment *(maintained)* The function that pairs each outcome with its probability.	A number cube is rolled 10 times. The results are shown in the table. **Outcome** | 1 | 2 | 3 | 4 | 5 | 6 **Probability** | $\frac{1}{10}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | 0 | $\frac{3}{10}$ | $\frac{1}{5}$	**distribución de probabilidad para un experimento** *(repaso)* Función que asigna a cada resultado su probabilidad.
probability experiment *(55)* An occurrence whose outcome is uncertain.	**Probability Experiments** spinning a spinner, flipping a coin, choosing a name from a hat without looking	**experimento de probabilidad** *(55)* Un suceso cuyo resultado no está definido.
probability sampling *(73)* Sampling in which every individual in the population has a known probability of being selected and this probability is greater than 0.	Choosing names from a hat: knowing the number of names in the hat results in knowing the probability of each name being selected.	**muestreo de probabilidad** *(73)* Muestreo en el cual cada individuo en la población tiene una probabilidad conocida de ser seleccionado y esta probabilidad es mayor que 0.

English	Example	Spanish
P		
proportion *(SB 8)* A statement that two ratios are equal; $\frac{a}{b} = \frac{c}{d}$.	$\frac{2}{3} = \frac{4}{6}$	**proporción** *(SB 8)* Enunciado que establece que dos razones son iguales; $\frac{a}{b} = \frac{c}{d}$.
Q		
quadratic equation *(27)* An equation that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.	$x^2 + 3x - 4 = 0$ $x^2 - 9 = 0$	**ecuación cuadrática** *(27)* Ecuación que se puede expresar como $ax^2 + bx + c = 0$, donde a, b y c son números reales y $a \neq 0$.
quadratic formula *(Inv 6)* The formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which gives solutions, or roots, of equations in the form $ax^2 + bx + c = 0$, where $a \neq 0$.	The solutions of $2x^2 - 5x - 3 = 0$ are given by $x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)}$ $= \frac{5 \pm \sqrt{25 + 24}}{4} = \frac{5 \pm 7}{4}$; $x = 3$ or $x = -\frac{1}{2}$.	**fórmula cuadrática** *(Inv 6)* La fórmula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, que da soluciones, o raíces, para las ecuaciones del tipo $ax^2 + bx + c = 0$, donde $a \neq 0$.
quadratic function *(27)* A function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$, or in the form $f(x) = a(x - h)^2 + k$, where a, h, and k are real numbers and $a \neq 0$.	 $f(x) = x^2 - 6x + 8$	**función cuadrática** *(27)* Función que se puede expresar como $f(x) = ax^2 + bx + c$, donde a, b y c son números reales y $a \neq 0$, o como $f(x) = a(x - h)^2 + k$, donde a, h y k son números reales y $a \neq 0$.
quadratic inequality in one variable *(89)* An inequality that can be written in the form $ax^2 + bx = c < d$, where a, b, c, and d are real numbers and $a \neq 0$. The symbol $<$ can be replaced with $>$, $\leq$, or $\geq$.	$x^2 + 8x + 24 > 10$	**desigualdad cuadrática de una variable** *(89)* Una desigualdad que puede ser escrita en la forma $ax^2 + bx + c < d$, donde a, b, c, y d son números reales y $a \neq 0$. El símbolo $<$ puede ser reemplazdo por $>$, $\leq$, o $\geq$

English	Example	Spanish
Q		
quadratic inequality in two variables *(89)* An inequality that can be written in one of the following forms: $y < ax^2 + bx + c$, $y > ax^2 + bx + c$, $y \leq ax^2 + bx + c$, $y \geq ax^2 + bx + c$, or $y \neq ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.	 $y > -x^2 - 2x + 3$	**desigualdad cuadrática en dos variables** *(89)* Desigualdad que puede expresarse de una de las siguientes formas: $y < ax^2 + bx + c$, $y > ax^2 + bx + c$, $y \leq ax^2 + bx + c$, $y \geq ax^2 + bx + c$, o $y \neq ax^2 + bx + c$, donde a, b y c son números reales y $a \neq 0$.
quadratic model *(116)* A quadratic function used to represent a set of data.	x: 4, 6, 8, 10 $f(x)$: 27, 52, 89, 130 A quadratic model for the data is $f(x) = x^2 + 3.3x - 2.6$.	**modelo cuadrático** *(116)* Función cuadrática que se utiliza para representar un conjunto de datos.
quadratic regression *(116)* A statistical method used to fit a quadratic model to a given data set.		**regresión cuadrática** *(116)* Método estadístico utilizado para ajustar un modelo cuadrático a un conjunto de datos determinado.
quartic function *(11)* A polynomial function of degree 4.	$f(x) = x^4 + 2x^3 - x^2 - 1$	**función de cuarto grado** *(11)* Una función polinomial de cuarta potencia.
R		
radian *(63)* A unit of angle measure based on arc length. In a circle of radius r, if a central angle has a measure of 1 radian, then the length of the intercepted arc is r units.	 2π radians $= 360°$ 1 radian $\approx 57°$	**radián** *(63)* Unidad de medida de un ángulo basada en la longitud del arco. En un círculo de radio r, si un ángulo central mide 1 radián, entonces la longitud del arco abarcado es r unidades. 2π radianes $= 360°$ 1 radián $\approx 57°$

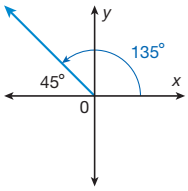
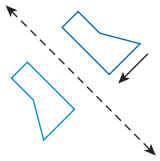
English	Example	Spanish
R		
radical *(40)* An indicated root of a quantity.	$\sqrt{36} = 6, \sqrt[3]{27} = 3$	**radical** *(40)* Raíz indicada de una cantidad.
radical equation *(70)* An equation that contains a variable within a radical.	$\sqrt{x + 3} + 4 = 7$	**ecuación radical** *(70)* Ecuación que contiene una variable dentro de un radical.
radical function *(75)* A function whose rule contains a variable within a radical.	 $f(x) = \sqrt{x}$	**función radical** *(75)* Función cuya regla contiene una variable dentro de un radical.
radical symbol *(40)* The symbol $\sqrt{\ }$ used to denote a root. The symbol is used alone to indicate a square root or with an index, $\sqrt[n]{\ }$, to indicate the nth root.	$\sqrt{36} = 6, \sqrt[3]{27} = 3$	**símbolo de radical** *(40)* Símbolo $\sqrt{\ }$ que se utiliza para expresar una raíz. Puede utilizarse solo para indicar una raíz cuadrada, o con un índice, $\sqrt[n]{\ }$, para indicar la enésima raíz.
radicand *(40)* The expression under a radical sign.	$\sqrt{x + 3} - 2$ ↑ Radicand	**radicando** *(40)* Número o expresión debajo del signo de radical.
random *(55)* Occurring by chance.	A random number is a number chosen without using any system or pattern.	**aleatorio** *(55)* Algo que ocurre al azar.
random event *(55)* An event that occurs by chance.	Getting heads on the flip of a coin and rolling an even number on a number cube are random events.	**suceso aleatorio** *(55)* Un suceso que ocurre al azar.
random sample *(73)* A sample selected from a population so that each member of the population has an equal chance of being selected.	Mr. Hansen chose a random sample of the class by writing each student's name on a slip of paper, mixing up the slips, and drawing five slips without looking.	**muestra aleatoria** *(73)* Muestra seleccionada de una población tal que cada miembro de ésta tenga igual probabilidad de ser seleccionado.
range of a data set *(25)* The difference of the greatest and least values in the data set.	The data set {3, 3, 5, 7, 8, 10, 11, 11, 12} has a range of $12 - 3 = 9$.	**rango de un conjunto de datos** *(25)* La diferencia del mayor y menor valores en un conjunto de datos.

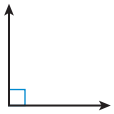
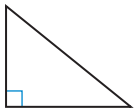
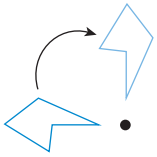
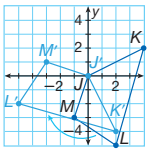
English	Example	Spanish
R		
range of a function or relation *(4)* The set of output values of a function or relation.	The range of $y = x^2$ is $\{y \mid y \geq 0\}$.	**rango de una función o relación** *(4)* Conjunto de los valores de salida de una función o relación.
rate *(maintained)* A ratio that compares two quantities measured in different units.	$\frac{55 \text{ miles}}{1 \text{ hour}} = 55 \text{ mi/h}$	**tasa** *(repaso)* Razón que compara dos cantidades medidas en diferentes unidades.
ratio *(SB 8)* A comparison of two quantities by division.	$\frac{1}{2}$ or 1:2	**razón** *(SB 8)* Comparación de dos números mediante una división.
rational equation *(84)* An equation that contains one or more rational expressions.	$\frac{x+2}{x^2+3x-1} = 6$	**ecuación racional** *(84)* Ecuación que contiene una o más expresiones racionales.
rational exponent *(59)* An exponent that can be expressed as $\frac{m}{n}$ such that if m and n are integers, then $b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$.	$4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$ $4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 2^3 = 8$	**exponente racional** *(59)* Exponente que se puede expresar como $\frac{m}{n}$ tal que, si m y n son números enteros, entonces $b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$.
rational expression *(28)* An algebraic expression whose numerator and denominator are polynomials and whose denominator has a degree ≥ 1.	$\frac{x+2}{x^2+3x-1}$	**expresión racional** *(28)* Expresión algebraica cuyo numerador y denominador son polinomios y cuyo denominador tiene un grado ≥ 1.
rational function *(84)* A function whose rule can be written as a rational expression.	$f(x) = \frac{x+2}{x^2+3x-1}$	**función racional** *(84)* Función cuya regla se puede expresar como una expresión racional.
rational inequality *(94)* An inequality that contains one or more rational expressions.	$\frac{x+2}{x^2+3x-1} \geq 6$	**desigualdad racional** *(94)* Desigualdad que contiene una o más expresiones racionales.

English	Example	Spanish
R		
rationalizing the denominator *(40)* A method of rewriting a fraction by multiplying by another fraction that is equivalent to 1 in order to remove radical terms from the denominator.	$\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)=\frac{\sqrt{2}}{2}$	**racionalizar el denominador** *(40)* Método que consiste en escribir nuevamente una fracción multiplicándola por otra fracción equivalente a 1 a fin de eliminar los términos radicales del denominador.
rational number *(1)* A number that can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.	$3,\ 1.75,\ 0.\overline{3},\ -\frac{2}{3},\ 0$	**número racional** *(1)* Número que se puede expresar como $\frac{a}{b}$, donde a y b son números enteros y $b \neq 0$.
Rational Root Theorem *(85)* If a polynomial $P(x)$ has integer coefficients, then every rational root of $P(x) = 0$ can be written in the form $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient of $P(x)$.	For $3x^2 + 4x^2 - x + 6 = 0$, 6 is a possible rational root because $6 = \frac{6}{1}$ and 6 is a factor of 6 and 1 is a factor of 3.	**Teorema de la raíz racional** *(85)* Si un polonomio $P(x)$ tiene coeficientes enteros, entonces cada raíz racional de $P(x) = 0$ puede ser escrita en la forma $\frac{p}{q}$, donde p es un factor del término constante y q es un factor del primer coeficiente de $P(x)$.
real axis *(69)* The horizontal axis in the complex plane; it graphically represents the real part of complex numbers.		**eje real** *(69)* Eje horizontal de un plano complejo. Representa gráficamente la parte real de los números complejos.
real number *(1)* A rational or irrational number. Every point on the number line represents a real number.	$-5,\ 0,\ \frac{2}{3},\ \sqrt{2},\ 3.1,\ \pi$	**número real** *(1)* Número racional o irracional. Cada punto de la recta numérica representa un número real.
real part of a complex number *(62)* For a complex number of the form $a + bi$, a is the real part.	$5 + 6i$ Real part Imaginary part	**parte real de un número complejo** *(62)* Dado un número complejo del tipo $a + bi$, a es la parte real.

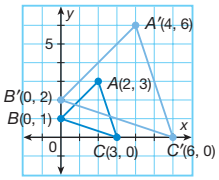
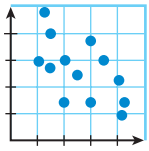
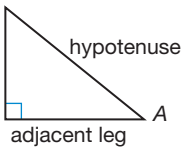
GLOSSARY/ GLOSARIO

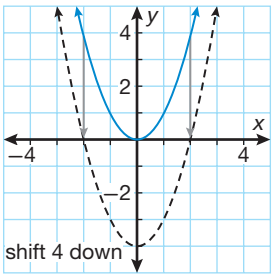
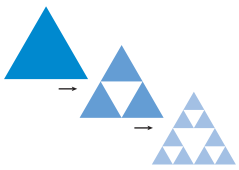
R

English	Example	Spanish
reciprocal *(31)* For a real number $a \neq 0$, the reciprocal of a is $\frac{1}{a}$. The product of reciprocals is 1.	$\frac{1}{2}$ is the reciprocal of 2. $\frac{5}{3}$ is the reciprocal of $\frac{3}{5}$.	**recíproco** *(31)* Dado el número real $a \neq 0$, el recíproco de a es $\frac{1}{a}$. El producto de los recíprocos es 1.
reference angle *(56)* For an angle in standard position, the reference angle is the positive acute angle formed by the terminal side of the angle and the x-axis.	 45° is the reference angle of 135° in standard position.	**ángulo de referencia** *(56)* Dado un ángulo en posición estándar, el ángulo de referencia es el ángulo agudo positivo formado por el lado terminal del ángulo y el eje x.
reflection *(27)* A transformation that reflects, or "flips," a graph or figure across a line, called the line of reflection, such that each reflected point is the same distance from the line of reflection but is on the opposite side of the line.		**reflexión** *(27)* Transformación que refleja, o invierte, una gráfica o figura sobre una línea, llamada la línea de reflexión, de manera tal que cada punto reflejado esté a la misma distancia de la línea de reflexión pero que se encuentre en el lado opuesto de la línea.
regression *(45)* The statistical study of the relationship between variables.		**regresión** *(45)* Estudio estadístico de la relación entre variables.
relation *(4)* A set of ordered pairs.	$\{(0, 5), (0, 4), (2, 3), (4, 0)\}$	**relación** *(4)* Conjunto de pares ordenados.
Remainder Theorem *(95)* If the polynomial function $P(x)$ is divided by $x - a$, then the remainder r is $P(a)$.	$P(x) = x^4 - 5x^3\,x - 2$ $P(x) \div (x - 3) = -P(3)$ $3\,\vert\,1 \quad -5 \quad -2 \quad 1 \quad -2$ $\quad\quad 3 \quad -6 \quad -24 \quad -69$ $1 \quad -2 \quad -8 \quad -23 \quad \boxed{-71}$ → remainder $P(3) = -71$	**Teorema del residuo** *(95)* Si la función polinomial $P(x)$ es dividia entre $x - a$, entonces el residuo r es $P(a)$.
replacement set *(maintained)* A set of numbers that can be substituted for a variable.	The solution set of $y = x + 3$ for the replacement set $\{1, 2, 3\}$ is $\{4, 5, 6\}$.	**conjunto de reemplazo** *(repaso)* Conjunto de números que pueden sustituir una variable.

English	Example	Spanish
R		
right angle *(maintained)* An angle that measures 90°.		**ángulo recto** *(repaso)* Ángulo que mide 90°.
right triangle *(41)* A triangle with one right angle.		**triángulo rectángulo** *(41)* Triángulo con un ángulo recto.
root of an equation *(35)* Any value of the variable that makes the equation true.	The roots of $(x - 2)(x + 1) = 0$ are 2 and -1.	**raíz de una ecuación** *(35)* Cualquier valor de la variable que transforme la ecuación en verdadera.
rotation *(56)* A transformation that rotates or turns a figure about a point called the center of rotation.		**rotación** *(56)* Transformación que hace rotar o girar una figura sobre un punto llamado centro de rotación.
rotation matrix *(112)* A matrix used to rotate a figure about the origin.	 Matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ was used to rotate the figure 90° clockwise.	**matriz de rotación** *(112)* Matriz utilizada para rotar una figura sobre el origen.
S		
sample *(73)* Part of a population.	A student wants to know what the teachers at a school think about the new salary plan. He chooses eight teachers to talk to. These eight teachers make up a sample.	**muestra** *(73)* Parte de una población.
sample size *(73)* The number of individuals in a sample.	A student interviewed 10 of the athletes on the basketball team. The sample size is 10.	**tamaño de una muestra** *(73)* El número de individuos en una muestra.

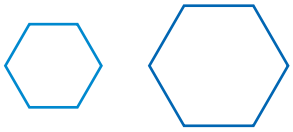
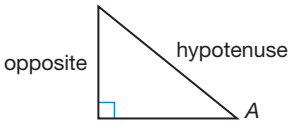
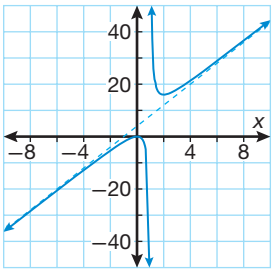
S

English	Example	Spanish
sample space *(33)* The set of all possible outcomes of a probability experiment.	In the experiment of rolling a number cube, the sample space is $\{1, 2, 3, 4, 5, 6\}$.	**espacio muestral** *(33)* Conjunto de todos los resultados posibles en un experimento de probabilidades.
sampling *(73)* The process of choosing a sample to represent a population.	To determine who to interview, a student obtained a class roster and chose every 15th name on the list.	**muestrear** *(73)* El proceso de escojer una muestra que representa a una población.
scalar *(5)* A number that is multiplied by a matrix.	$3\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix} = \begin{bmatrix}3 & -6\\ 6 & 9\end{bmatrix}$ scalar	**escalar** *(5)* Número que se multiplica por una matriz.
scale factor *(maintained)* The multiplier used on each dimension to change one figure into a similar figure.		**factor de escala** *(repaso)* El multiplicador utilizado en cada dimensión para transformar una figura en una figura semejante.
scatter plot *(maintained)* A graph with points plotted to show a possible relationship between two sets of data.		**diagrama de dispersión** *(repaso)* Gráfica con puntos dispersos para demostrar una relación posible entre dos conjuntos de datos.
scientific notation *(3)* A method of writing very large or very small numbers, by using powers of 10, in the form $m \times 10^n$, where $1 \le m < 10$ and n is an integer.	$1.256 \times 10^{13} = 12{,}560{,}000{,}000{,}000$ $7.5 \times 10^{-6} = 0.0000075$	**notación científica** *(3)* Método que consiste en escribir números muy grandes o muy pequeños utilizando potencias de 10 del tipo $m \times 10^n$, donde $1 \le m < 10$ y n es un número entero.
secant of an angle *(46)* In a right triangle, the ratio of the length of the hypotenuse to the length of the side adjacent to angle A. It is the reciprocal of the cosine function.	 $\sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos A}$	**secante de un ángulo** *(46)* En un triángulo rectángulo, la razón entre la longitud de la hipotenusa y la longitud del cateto adyacente al ángulo A. Es la inversa de la función coseno.

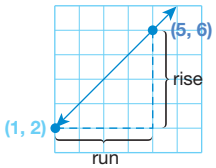
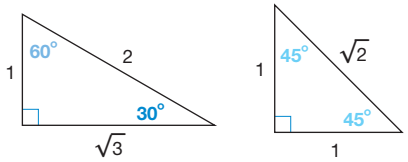
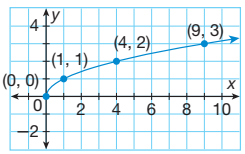
English	Example	Spanish
S		
second differences *(maintained)* Differences between first differences of a function.	x: 0, 1, 2, 3 y: 1, 4, 9, 16 first differences +3 +5 +7 second differences +2 +2	**segundas diferencias** *(repaso)* Diferencias entre las primeras diferencias de una función.
sequence *(92)* A list of numbers that often form a pattern.	1, 2, 4, 8, 16, …	**sucesión** *(92)* Lista de números que generalmente forman un patrón.
series *(105)* The indicated sum of the terms of a sequence.	$1 + 2 + 4 + 8 + 16 + \ldots$	**serie** *(105)* Suma indicada de los términos de una sucesión.
shift *(30)* A translation of a graph; the sliding of every point on the graph the same number of units in the same direction.		**desplazamiento** *(30)* Una traslación de una gráfica; el deslizamiento de cada punto en una gráfica en un número igual de unidades en la misma dirección.
Sierpinski triangle *(maintained)* A fractal formed from a triangle by removing triangles with vertices at the midpoints of the sides of each remaining triangle.		**triángulo de Sierpinski** *(repaso)* Fractal formado a partir de un triángulo al cual se le recortan triángulos cuyos vértices se encuentran en los puntos medios de los lados de cada triángulo restante.
sigma notation *(105)* A way of indicating the sum of a series; it uses the capital Greek letter, sigma.	$\sum_{k=1}^{5} 2k = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) = 30$	**notación sigma** *(105)* Una manera de indicar la suma de una serie; utiliza la letra griega mayúscula sigma.
significant digits *(18)* Any digit that is measured or estimated; includes all nonzero digits, zeros between nonzero digits, and zeros to the right of both the decimal point and the last nonzero digit; zeros used for placeholders are not significant.	**605**: 3 significant digits 0.000**2380**: 4 significant digits **720**: 2 significant digits	**dígitos significativos** *(18)* Cualquier dígito que es medido o estimado; incluye a todos los dígitos distintos de cero entre dígitos que no son cero y los ceros a la derecha del punto decimal y al último dígito que no es cero; los ceros que se usan para llenar lugares no son significativos.

GLOSSARY/ GLOSARIO

S

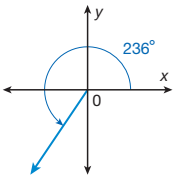
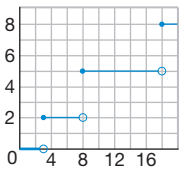
English	Example	Spanish
similar *(maintained)*		**semejantes** *(repaso)*
Two figures are similar if they have the same shape but not necessarily the same size.		Dos figuras son semejantes si tienen la misma forma pero no necesariamente el mismo tamaño.
simple random sample *(73)*	A teacher assigns every student a unique number and chooses numbers from a random number table.	**muestra simple al azar** *(73)*
A sample consisting of n individuals, where every individual has an equal chance of being chosen and every possible group of n individuals has an equal chance of being chosen.		Una muestra que consiste en n individuos, donde cada individuo tiene la misma posibilidad de ser escojido y cada grupo posible de n individuos tiene la misma oportunidad de ser escojido.
simplify *(maintained)*	$3(4) + 7$ $12 + 7$ 19	**simplificar** *(repaso)*
To perform all indicated operations.		Realizar todas las operaciones indicadas.
sine *(46)*	 $\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$.	**seno** *(46)*
In a right triangle, the ratio of the length of the side opposite $\angle A$ to the length of the hypotenuse.		En un triángulo rectángulo, razón entre la longitud del cateto opuesto a $\angle A$ y la longitud de la hipotenusa.
singular matrix *(32)*	$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ is a singular matrix because $\det A = 3(2) - (1)(6) = 0$.	**matriz singular** *(32)*
A matrix that does not have an inverse; its determinant is 0.		Una matriz que no tiene inversa; su determinante es cero.
slant asymptote *(107)*		**asíntota inclinada** *(107)*
An asymptote that is neither horizontal nor vertical, can be called an oblique asymptote.		Una asíntota que no es ni horizontal ni vertical; puede ser llamada una asíntota oblicua.

S

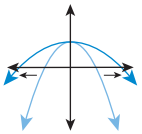
English	Example	Spanish
slope *(13)* A measure of the steepness of a line. If (x_1, y_1) and (x_2, y_2) are any two points on the line, the slope of the line, known as m, is represented by the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$.	 $m = \frac{6-2}{5-1} = \frac{4}{4} = 1$	**pendiente** *(13)* Medida de la inclinación de una línea. Dados dos puntos (x_1, y_1) y (x_2, y_2) en una línea, la pendiente de la línea, denominada m, se representa por la ecuación $m = \frac{y_2 - y_1}{x_2 - x_1}$.
slope-intercept form *(13)* The slope-intercept form of a linear equation is $y = mx + b$, where m is the slope and b is the y-intercept.	$y = -2x + 4$ slope, y-intercept	**forma de pendiente-intersección** *(13)* La forma de pendiente-intersección de una ecuación lineal es $y = mx + b$, donde m es la pendiente y b es la intersección y.
solution of an equation *(7)* The value or set of values that makes an equation true.	The solution set of $x^2 = 9$ is $\{-3, 3\}$.	**solución de una ecuación** *(7)* Un valor o conjunto de valores que hacen verdadero un enunciado.
special right triangle *(52)* A 45°-45°-90° triangle or a 30°-60°-90° triangle.		**triángulo rectángulo especial** *(52)* Triángulo de 45°-45°-90° ó triángulo de 30°-60°-90°.
square matrix *(9)* A matrix with the same number of rows as columns.	$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 1 & -3 & 1 \\ 2 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$	**matriz cuadrada** *(9)* Matriz con el mismo número de líneas y columnas.
square root *(40)* A number that is multiplied to itself to form a product is called a square root of that product.	-4 and 4 are square roots of 16 because $(-4)^2 = 16$ and $4^2 = 16$.	**raíz cuadrada** *(40)* El número que se multiplica por sí mismo para formar un producto se denomina la raíz cuadrada de ese producto.
square root function *(75)* A function whose rule contains a variable under a square root sign.	 $f(x) = \sqrt{x}$	**función de raíz cuadrada** *(75)* Función cuya regla contiene una variable bajo un signo de raíz cuadrada.

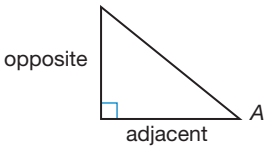
GLOSSARY/ GLOSARIO

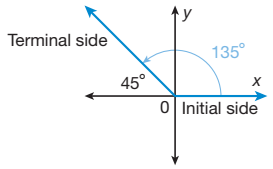
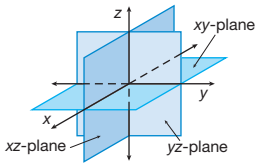
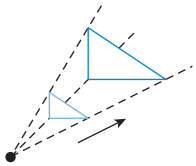
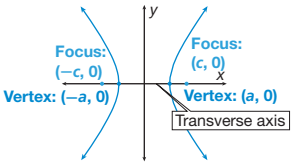
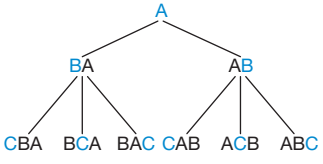
S

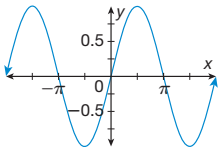
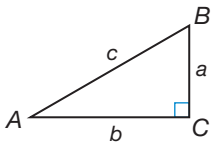
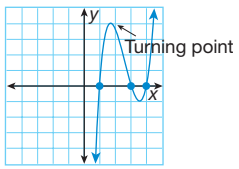
English	Example	Spanish
standard deviation *(25)* A measure of dispersion of a data set. The standard deviation σ is the square root of the variance.	Data set: $\{6, 7, 7, 9, 11\}$ Mean: $\frac{6+7+7+9+11}{5} = 8$ Variance: $\frac{1}{5}(4+1+1+1+9) = 3.2$ Standard deviation: $\sigma = \sqrt{3.2} \approx 1.8$	**desviación estándar** *(25)* Medida de dispersión de un conjunto de datos. La desviación estándar σ es la raíz cuadrada de la varianza.
standard form of a linear equation *(26)* $Ax + By = C$, where A, B, and C are real numbers.	$2x + 3y = 6$	**forma estándar de una ecuación lineal** *(26)* $Ax + By = C$, donde A, B y C son números reales.
standard form of a polynomial *(11)* A polynomial in one variable is written in standard form when the terms are in order from greatest degree to least degree.	$3x^3 - 5x^2 + 6x - 7$	**forma estándar de un polinomio** *(11)* Un polinomio de una variable se expresa en forma estándar cuando los términos se ordenan de mayor a menor grado.
standard form of a quadratic equation *(27)* $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.	$2x^2 + 3x - 1 = 0$	**forma estándar de una ecuación cuadrática** *(27)* $ax^2 + bx + c = 0$, donde a, b y c son números reales y $a \neq 0$.
standard position *(56)* An angle in standard position has its vertex at the origin and its initial side on the positive x-axis.		**posición estándar** *(56)* Ángulo cuyo vértice se encuentra en el origen y cuyo lado inicial se encuentra sobre el eje x.
statistics *(25)* The branch of mathematics that involves the collection, analysis, and comparison of sets of data.	A student was using statistics when she surveyed other students, made graphs of the data she collected, and calculated means and standard deviations of the data.	**estadística** *(25)* La rama de las matemáticas que involucra la recolección, análisis y comparación de conjuntos de datos.
step function *(79)* A piecewise function that is constant over each interval in its domain.		**función escalón** *(79)* Función a trozos que es constante en cada intervalo en su dominio.

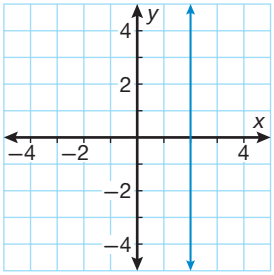
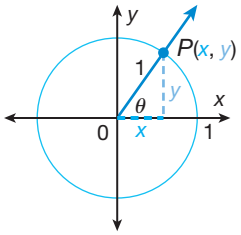
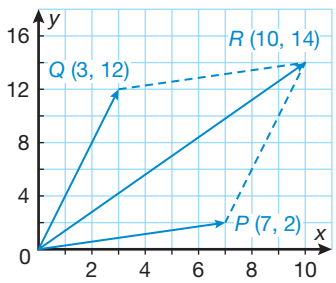
S

English	Example	Spanish
stratified sample *(73)*		**muestra estratificada** *(73)*
A sample chosen by dividing the population into mutually exclusive groups which have similar characteristics and performing a simple random sample on each subgroup.	Divide students into males and females and randomly choose students from each group.	Una muestra que se escoje dividiendo a la población en grupos mutuamente excluyentes, los cuales tienen características similares, y que se obtiene realizando un muestreo simple al azar de cada subgrupo.
stretch *(30)*		**estiramiento** *(30)*
A transformation that pulls the points of a graph horizontally away from the y-axis or vertically away from the x-axis.		Transformación que desplaza los puntos de una gráfica en forma horizontal alejándolos del eje y o en forma vertical alejándolos del eje x.
substitution method *(21)*	$\begin{cases} 2x + 3y = -1 \\ x - 3y = 4 \end{cases}$ Solve for x. $x = 4 + 3y$ Substitute into the first equation and solve. $2(4 + 3y) + 3y = -1$ $y = -1$ Then solve for x. $x = 4 + 3(-1) = 1$	**método de sustitución** *(21)*
A method used to solve systems of equations by solving an equation for one variable and substituting the resulting expression into the other equation(s).		Método utilizado para resolver sistemas de ecuaciones resolviendo una ecuación para una variable y sustituyendo la expresión resultante en las demás ecuaciones.
summation notation *(105)*		**notación de sumatoria** *(105)*
A method of notating the sum of a series using the Greek letter $\sum$ (capital *sigma*).	$\sum_{k=1}^{5} 3k = 3 + 6 + 9 + 12 + 15 = 45$	Método de notación de la suma de una serie que utiliza la letra griega $\sum$ (*sigma* mayúscula).
synthetic division *(51)*	$(x^3 - 7x + 6) \div (x - 2)$ $2 \rfloor\ 1 \quad 0 \quad -7 \quad 6$ $\qquad\quad 2 \quad 4 \quad 6$ $1 \quad 2 \quad -3 \quad \rfloor 0$ $(x^3 - 7x + 6) \div (x - 2)$ $= x^2 + 2x - 3$	**división sintética** *(51)*
A shorthand method of dividing by a linear binomial of the form $(x - a)$ by writing only the coefficients of the polynomials.		Método abreviado de división que consiste en dividir entre un binomio lineal del tipo $(x - a)$ escribiendo sólo los coeficientes de los polinomios.

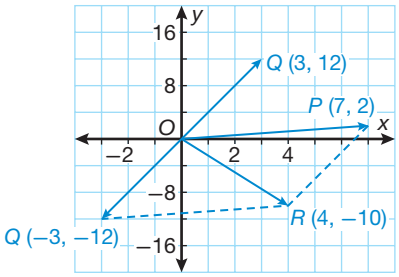
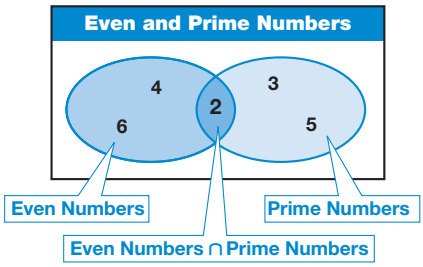
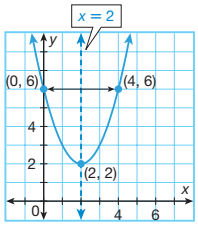
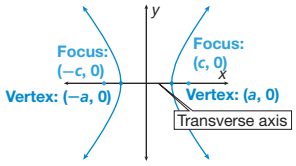
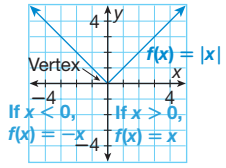
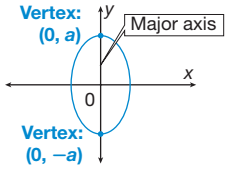
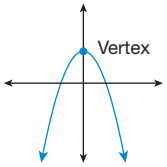
English	Example	Spanish
S		
synthetic substitution *(51)* The process of using synthetic division to evaluate a polynomial.	$P(x) = x^3 - 2x^2 + 4x + 3$ $5 \mid 1 \quad -2 \quad 4 \quad 3$ $\quad\quad\quad 5 \quad 15 \quad 95$ $1 \quad 3 \quad 19 \quad \boxed{98}$ $P(5) = 98$	**sustitución sintética** *(51)* El proceso de utilizar división sintética para evaluar a un polinomio.
systematic sampling *(73)* A method of sampling where the individuals in the population are listed and every nth individual is chosen.	Call every 10th phone number from a list of phone numbers.	**muestreo sistemático** *(73)* Un método de muestreo donde se hace una lista de los individuos en la población y cada n_{avo} individuo es escojido.
system of equations *(15)* A set of two or more equations that have two or more variables.	$\begin{cases} 2x + 3y = -1 \\ x^2 = 4 \end{cases}$	**sistema de ecuaciones** *(15)* Conjunto de dos o más ecuaciones que contienen dos o más variables.
system of linear equations *(15)* *See* linear system.		**sistema de ecuaciones lineales** *(15)* *Ver* sistema lineal.
system of linear inequalities *(43)* A system of inequalities in two or more variables in which all of the inequalities are linear.	$\begin{cases} 2x + 3y \geq -1 \\ x - 3y < 4 \end{cases}$	**sistema de desigualdades lineales** *(43)* Sistema de desigualdades en dos o más variables en el que todas las desigualdades son lineales.
T		
tangent of an angle *(46)* In a right triangle, the ratio of the length of the leg opposite $\angle A$ to the length of the leg adjacent to $\angle A$.	 $\tan A = \frac{\text{opposite}}{\text{adjacent}}$	**tangente de un ángulo** *(46)* En un triángulo rectángulo, razón entre la longitud del cateto opuesto a $\angle A$ y la longitud del cateto adyacente a $\angle A$.
term of an expression *(2)* The parts of the expression that are added or subtracted.	$3x^2 + 6x - 8$ Term Term Term	**término de una expresión** *(2)* Partes de la expresión que se suman o se restan.
term of a sequence *(92)* An element or number in the sequence.	5 is the third term in the sequence 1, 3, 5, 7, …	**término de una sucesión** *(92)* Elemento o número de una sucesión.

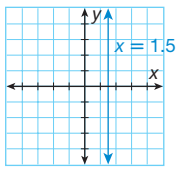
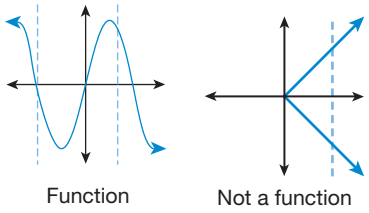
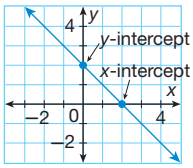
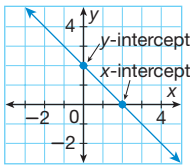
English	Example	Spanish
T		
terminal side *(56)* For an angle in standard position, the ray that is rotated relative to the positive x-axis.		**lado terminal** *(56)* Dado un ángulo en una posición estándar, el rayo que rota en relación con el eje positivo x.
theoretical probability *(55)* The ratio of the number of equally likely outcomes in an event to the total number of possible outcomes.	The theoretical probability of rolling an odd number on a number cube is $\frac{3}{6} = \frac{1}{2}$.	**probabilidad teórica** *(55)* Razón entre el número de resultados igualmente probables de un suceso y el número total de resultados posibles.
third quartile *(25)* The median of the upper half of a data set. Also called *upper quartile*.	Lower half Upper half 18, 23, 28, 29, 36, 42 Third quartile	**tercer cuartil** *(25)* La mediana de la mitad superior de un conjunto de datos. También se llama *cuartil superior*.
three-dimensional coordinate system *(Inv 3)* A space that is divided into eight regions by an x-axis, a y-axis, and a z-axis. The locations, or coordinates, of points are given by ordered triples.		**sistema de coordenadas tridimensional** *(Inv 3)* Espacio dividido en ocho regiones por un eje x, un eje y y un eje z. Las ubicaciones, o coordenadas, de los puntos son dadas por tripletas ordenadas.
transformation *(17)* A change in the position, size, or shape of a figure or graph.		**transformación** *(17)* Cambio en la posición, tamaño o forma de una figura o gráfica.
transverse axis *(109)* The axis of symmetry of a hyperbola that contains the vertices and foci.		**eje transversal** *(109)* Eje de simetría de una hipérbola que contiene los vértices y focos.
tree diagram *(33)* A branching diagram that shows all possible combinations or outcomes of an experiment.		**diagrama de árbol** *(33)* Diagrama con ramificaciones que muestra todas las combinaciones o resultados posibles de un experimento.

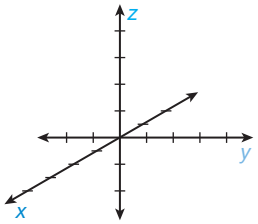
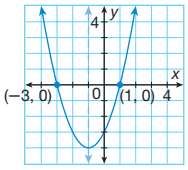
English	Example	Spanish
T		
trial *(33)* In probability, a single repetition or observation of an experiment.	In the experiment of rolling a number cube, each roll is one trial.	**prueba** *(33)* En repetición u observación de un experimento.
trigonometric function *(46)* A function whose rule is given by a trigonometric ratio.	 $f(x) = \sin x$	**función trigonométrica** *(46)* Función cuya regla es dada por una razón trigonométrica.
trigonometric identity *(108)* A trigonometric equation that is true for all values of the variable for which the statement is defined.	$\tan \theta = \frac{\sin \theta}{\cos \theta}$	**identidad trigonométrica** *(108)* Una ecuación trigonométrica que es verdadera para todos los valores de la variable para la cual se define el enunciado.
trigonometric ratio *(46)* Ratio of the lengths of two sides of a right triangle.	 $\sin A = \frac{a}{c}, \cos A = \frac{b}{c}, \tan A = \frac{a}{b}$	**razón trigonométrica** *(46)* Razón entre dos lados de un triángulo rectángulo.
trigonometry *(46)* The study of the measurement of triangles and of trigonometric functions and their applications.		**trigonometría** *(46)* Estudio de la medición de los triángulos y de las funciones trigonométricas y sus aplicaciones.
trinomial *(11)* A polynomial with three terms.	$4x^2 + 3xy - 5y^2$	**trinomio** *(11)* Polinomio con tres términos.
turning point *(101)* A point on the graph of a function that corresponds to a local maximum (or minimum) where the graph changes from increasing to decreasing (or vice versa).		**punto de inflexión** *(101)* Punto de la gráfica de una función que corresponde a un máximo (o mínimo) local donde la gráfica pasa de ser creciente a decreciente (o viceversa).

English	Example	Spanish
U		
undefined slope *(maintained)* The slope of a vertical line; the run equals 0; same as no slope.		**pendiente indefinida** *(repaso)* La pendiente de una línea vertical; la distancia horizontal es 0; lo mismo que sin pendiente.
unit circle *(63)* A circle with a radius of 1, centered at the origin.	 Unit circle	**círculo unitario** *(63)* Círculo con un radio de 1, centrado en el origen.
V		
variable *(2)* A symbol used to represent a quantity that can change.	$2x + 3$ ↑ variable	**variable** *(2)* Símbolo utilizado para representar una cantidad que puede cambiar.
variance *(25)* The average of squared differences from the mean. The square root of the variance is called the *standard deviation*.	Data set: is $\{6, 7, 7, 9, 11\}$ Mean: $\frac{6+7+7+9+11}{5} = 8$ Variance: $\frac{1}{5}(4+1+1+1+9) = 3.2$	**varianza** *(25)* Promedio de las diferencias cuadráticas en relación con la media. La raíz cuadrada de la varianza se denomina *desviación estándar.*
Vector *(99)* A quantity that has both a magnitude and a direction.		**Vector** *(99)* Una cantidad que tiene una magnitud y una dirección.
vector addition *(99)* The process of adding two or more vectors.	$P + Q = R$ 	**adición de vectores** *(99)* El proceso de sumar dos o más vectores.

GLOSSARY/ GLOSARIO

English	Example	Spanish
V		
vector subtraction *(99)* The process of subtracting one vector from another vector; equivalent to adding the opposite of a vector.	$P - Q = P + (-Q) = R$ 	**resta de vectores** *(99)* El proceso de restar un vector de otro vector; equivalente a sumar el opuesto de un vector.
Venn diagram *(SB 24)* A diagram used to show relationships between sets.		**diagrama de Venn** *(SB 24)* Diagrama utilizado para mostrar la relación entre conjuntos.
vertex form of a quadratic function *(30)* A quadratic function written in the form $f(x) = a(x - h)^2 + k$, where a, h, and k are constants and (h, k) is the vertex.	 $f(x) = (x - 2)^2 + 2$	**forma en vértice de una función cuadrática** *(30)* Una función cuadrática expresada en la forma $f(x) = a(x - h)^2 + k$, donde a, h y k son constantes y (h, k) es el vértice.
vertex of a hyperbola (vertices) *(109)* The endpoints of the transverse axis of the hyperbola.		**vértice de una hipérbola (vértices)** *(109)* Extremos del eje transversal de la hipérbola.
vertex of an absolute-value graph *(17)* The point on the axis of symmetry of the graph.		**vértice de una gráfica de valor absoluto** *(17)* Punto en el eje de simetría de la gráfica.
vertex of an ellipse (vertices) *(98)* The endpoints of the major axis of the ellipse.		**vértice de una elipse (vértices)** *(98)* Extremos del eje mayor de la elipse.
vertex of a parabola *(27)* The highest or lowest point on the parabola.		**vértice de una parábola** *(27)* Punto más alto o más bajo de una parábola.

English	Example	Spanish
V		
vertical line *(maintained)* A line whose equation is $x = a$, where a is the x-intercept. The slope of a vertical line is undefined.		**línea vertical** *(repaso)* Línea cuya ecuación es $x = a$, donde a es la intersección con el eje x. La pendiente de una línea vertical es indefinida.
vertical-line test *(4)* A test used to determine whether a relation is a function. If any vertical line crosses the graph of a relation more than once, the relation is not a function.		**prueba de la línea vertical** *(4)* Prueba utilizada para determinar si una relación es una función. Si una línea vertical corta la gráfica de una relación más de una vez, la relación no es una función.
voluntary response sampling *(73)* A sampling method in which the individuals in the sample choose themselves.	A news broadcast asks viewers to call in and answer *yes* or *no* to a poll question.	**muestreo de respuesta voluntaria** *(73)* Un método de muestreo en el cual los individuos en la muestra se escojen ellos mismos.
W		
whole number *(1)* The set of natural numbers and zero.	0, 1, 2, 3, 4, 5, …	**número cabal** *(1)* Conjunto de los números naturales y cero.
X		
***x*-intercept** *(13)* The x-coordinate(s) of the point(s) where a graph intersects the x-axis.		**intersección con el eje *x*** *(13)* Coordenada/s x de uno o más puntos donde una gráfica corta el eje x.
Y		
***y*-intercept** *(13)* The y-coordinate(s) of the point(s) where a graph intersects the y-axis.		**intersección con el eje *y*** *(13)* Coordenada/s y de uno o más puntos donde una gráfica corta el eje y.
Z		
***z* score** *(80)* The value of a data point on the standard normal distribution.	z score of 1.4: value is 1.4 standard deviations above the mean z score of -2.3: value is 2.3 standard deviations below the mean	**puntaje *z*** *(80)* El valor de un punto de datos en la distribución normal estándar.

English	Example	Spanish
Z		
zero matrix *(5)* A matrix in which every element is 0.	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	**matriz cero** *(5)* Una matriz en la cual cada elemento es 0.
***z*-axis** *(Inv 3)* The third axis in a three-dimensional coordinate system.		**eje *z*** *(inv 3)* Tercer eje en un sistema de coordenadas tridimensional.
zero exponent property *(59)* For any nonzero real number x, $x^0 = 1$.	$5^0 = 1$	**propiedad del exponente cero** *(59)* Dado un número real distinto de cero x, $x^0 = 1$.
zero of a function *(27)* For the function f, any number x such that $f(x) = 0$.	 The zeros of $f(x) = x^2 + 2x - 3$ are -3 and 1.	**cero de una función** *(27)* Dada la función f, todo número x tal que $f(x) = 0$.

Index

A

absolute value
complex numbers, 489–490
equations/inequalities, 116–120

abstract equations, 617–619

accuracy, of measurements, 125, 819

addition
counting principle of, 240–241
of cubes, 437
of matrices, 29–30
of operations with functions, 136–138
of polynomials, 73
property of equality, 42
property of inequalities, 61, 62
of rational expressions, 266–269
of real numbers, 3, 4
of vectors, 691

additive inverses
of matrices, 30–31

algebra tiles, 42

algebraic expressions
definition, 8

amplitude, definition, 580

angle measures, 448

applications, 4, 5, 6, 7, 10, 11, 12, 16, 17, 18, 23, 24 25, 26, 32, 33, 34, 35, 38, 39, 40, 41, 45, 46, 47, 48, 49, 50, 51, 52, 53, 57, 58, 59, 65, 66, 67, 68, 69, 74, 75, 76, 79, 80, 81, 82, 89, 90, 91, 96, 97, 98, 102, 103, 104, 106, 109, 110, 111, 112, 113, 120, 126, 127, 128, 131, 132, 133, 134, 135, 139, 140, 142, 143, 144, 145, 149, 151, 152, 158, 159, 160, 161, 166, 169, 174, 175, 176, 183, 184, 185, 186, 187, 189, 190, 192, 193, 197, 198, 199, 200, 203, 204, 205, 209, 211, 212, 213, 214, 218, 219, 220, 221, 229, 230, 231, 232, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 249, 250, 251, 252, 255, 256, 257, 262, 265, 269, 270, 271, 272, 275, 276, 277, 278, 282, 283, 284, 285, 289, 290, 292, 295, 299, 300, 301, 308, 309, 310, 311, 314, 315, 316, 317, 320, 321, 326, 327, 328, 329, 333, 334, 335, 336, 339, 340, 341, 342, 344, 345, 347, 351, 352, 353, 357, 358, 360, 366, 367, 374, 375, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 392, 393, 394, 395, 396, 397, 398, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 416, 417, 418, 419, 423, 424, 425, 426, 428, 429, 430, 431, 432, 438, 439, 441, 445, 446, 451, 452, 453, 454, 458, 459, 460, 461, 465, 466, 467, 468, 472, 473, 474, 475, 479, 480, 481, 482, 485, 487, 488, 493, 494, 508, 509, 510, 511, 516, 517, 524, 525, 526, 530, 531, 532, 536, 537, 539, 542, 543, 544, 545, 548, 549, 550, 551, 560, 561, 562, 568, 569, 576, 577, 579, 582, 583, 588, 589, 595, 602, 608, 609, 614, 619, 620, 621, 626, 632, 643, 646, 647, 648, 652, 653, 654, 655, 656, 661, 667, 669, 676, 677, 681, 687, 688, 692, 693, 710, 716, 721, 728, 734, 735, 742, 743, 744, 748, 752, 753, 754, 760, 766, 767, 776, 783, 790, 795, 801, 809, 810, 815, 819, 820, 821, 822, 823, 827, 828, 829, 830

AquaDom (Berlin), 619

arc lengths, finding, 451

arches, 588

area under a curve, 572–573

arithmetic sequences, 646–648

arithmetic series, 732–735

associative property
of real numbers, 3, 4, 5, 6

astronomy, 643

asymptotes, 337, 745–748, 757

B

base, logarithms, 457, 513

bell curves, 565, 566

best fit models, 806–810

bias, sampling, 503, 505, 523–524, 819–821

binomial distribution, 361–363

binomial theorem. *See* **binomials**

binomials
dividing by, 365–366
as factors, 541
in fractions, 319–320
linear, 365–366, 436–437
multiplying, 130
overview, 348–351

boundary lines, 280, 313

box-and-whisker plots
organizing data with, 183

C

calculator. *See also* **graphing calculator**
for conditional probability, finding, 485
for determinants, finding, 95
for expression evaluation, 9
graphing, 27–28, 114–115
systems of equations, solving, 101

capture-recapture method, 524

Cartesian (rectangular) coordinate system, 672–674, 690

central tendency,
measures of, 180–184

change of base formula, 612–614

circles, equations of, 640–643

classification
of linear systems, 102
of lines, 88, 147
of polynomials, 73
of real numbers, 2

by solutions, 108
of systems of equations, 101

closure property
of real numbers, 3

cluster sample, 522

coefficient matrices, 107

coinciding lines, 147

combinations
overview, 306–308
permutations and, 303

common difference, 646–647

common ratio, 678

commutative property
and function composition, 379
of matrix addition, 29
of real numbers, 3, 4, 5
complex conjugate, 491
complex coordinate system, 690
complex expressions, 489–492
complex fractions, 343–345
complex numbers, 442–444, 770–773
complex zeros, 463–464
compositions, of functions, 378–381
compound events, 427
compound inequality, 63–64
compound interest, 339
compression, of parabolas, 217
computer Internet security, 293
conditional probability, 483–486
confidence intervals, 519–520
conic sections, 640, 793–795
conjugates, 289, 319, 491, 741
of denominators, 288
radical, 319
conjunction, definition, 70
conjunctions, absolute value inequalities with, 118
consecutive terms, 678
consistency
of solutions, 147, 170
of systems of equations, 101, 210
constant, of variation, 48
constant terms, definition, 10
constraints, in linear programming, 384
constructive interference, 608
continuous function
definition, 155
identifying, 156–157
contradiction, definition, 70
contrapositive, definition, 69
convenience sampling, 523
convergent series, 788–790
converse, definition, 69
conversion factors, units of measure, 124
coordinate planes, 280
coordinates, finding, 299, 671–673
coordinates of points, unit circle, 447–448
correlation
definition, 325
line fitting, 326
correlation coefficient (*r*), line fitting, 326–327
cosecant (csc), 332–333, 719
cosine (cos)
finding, 449–450
function graphing, 606
inverse relation of, 476–477
law of, 546–549
overview, 331–332, 373
cotangent (cot), 332–333, 720
coterminal angles, 400
counting principles, overview, 240–243
Cramer's Rule, overview, 107–110
critical points/values, 624
cryptography, 293–295
cubes
equations, 599
graphing, 706
rational functions with, 698–699
roots, 496–497, 536–537, 739–740
slant asymptotes and, 746
sums and differences, 437, 595

D

data
analysis, 327
collection, 502–505
displays, 820–821
fitting, 808–810
lists, 153–154
in matrices, 27–28
misleading, 819–821
organization, 29–33
statistical, 178–179
storage, 27–28
data storage
and analysis, 327
in matrices, 27–28
data storage and plotting, lists, 153–154
De Moivre's theorem, 770–773
decay, exponential, 406–410
decimals, percentages, changing to, 36
decreasing exponential equations, 653
degree of monomial/polynomial, 706
definition, 72
degrees, converting to radians, 448
denominators
of rational expressions, 266–267, 268
rationalizing, 288, 289, 318–320
slant asymptotes and, 745–746
of zero, 108–109
dependence
of events, 242, 393–394, 428–429
of systems of equations, 101
variables, 21
dependence/independence, of events, 242
dependent events, 393–394, 428–429
dependent variables, definition, 21
destructive interference, 608
determinants, overview, 93–97
diagonals, in matrices, 56
difference identities, 780–782
difference of two squares, 164, 165
dimensions, of matrices, 29, 54–55
direct variation
inverse variation vs., 77
overview, 48–51
discontinuous function
definition, 155
identifying, 156–157
discounts, as percentages, 37, 38, 41
discrete function
definition, 155
identifying, 156–157
discriminants, 528–531
disjoint events, 427–428
disjunction, definition, 69
disjunctions, absolute value inequalities with, 119
dispersion, measures of, 180–184
distance, 357
distance formula
circle equations and, 640–642

Pythagorean Theorem and, 296–300
distributive property
of real numbers, 3, 4
solving equations with, 44
divergent series, 788–789
dividends, definition, 273
division
of complex numbers, 491–492
of functions, 139
of polynomials, 273, 665
of rational expressions, 226–232
property of equality, 42
property of inequalities, 61, 62
synthetic, 364–367, 436
divisors, definition, 273
domain, of functions, 21–22, 24, 157
domains
excluded values and, 201
of function compositions, 378
of inverses, 355, 357
and maxima and minima, 216
dot product, 692–693
double-angle identities, 798–799
drawing tools, 114

E

echelon form, matrices in, 28
elements, of matrices, 29
elimination method, 170–174, 814
ellipse equations, 686
ellipses, 684, 795
end behavior, 706–710
equality, 42
properties of, 42
equations
absolute value, 116–120
abstract, 617–620
of circles, 640–643
cubic, 600, 739–740
determinant, 94
of ellipses, 686
exponential, 457–458, 652–655
exponential growth and decay, 407–408
graphing systems of, 100–104
of hyperbolas, 757–760
of a line, 187–190
linear, 42–45, 86–90, 107, 246–250
lines, horizontal/vertical, 262
logarithmic, 457–458, 612, 714–715
matrix, 31, 235
monomial factors, solving with, 542
nonlinear, 813–815
of parabolas, 218
parametric, 143–145
of piecewise functions, 558–560
polar, 673–675
polynomial, 469–473, 738–740
quadratic, 253–256, 413–414, 415
radical, 495–499, 618
rational, 618
of slant asymptotes, 745–748
trigonometric, 478–479, 825–828
trinomials, 164–165
even degree, 707
event, definition, 240–241, 391
events
probability of, 393–394
types of, 427–432
excluded values, 201
expansion by minors, 94–96
experiment, definition, 240
explicit formula, 647
exponential functions, 337–340, 406–410
exponential growth and decay, 406–410
exponents
equations, 652–655
expressions with, 8, 512, 574–575
functions, 337–340, 406, 727–728, 764
inequalities, 654
zlogarithms, 457–459
use of, 13–16
expressions, evaluating, 3, 4, 8–11, 443, 513
extraneous solutions, 495, 498

F

factor theorem, 436, 665–667
factorials, definition, 304
factoring
advanced, 436–438
definition, 163
dividing out-1, 202
polynomial equations, 469
polynomial roots, 540–543
simplifying expressions by, 202
factors, polynomials as, 275
Fahrenheit-Celsius formula, 188
feasible regions, 384–386
FOIL method, 129–132
fractional coefficients, 172
fractional constant and direct variations, 48–49
fractional exponents, 420–424
fractions
complex, 343–345
percentages, changing to, 36
solving equations with, 44
substituting with, 148
functions
compositions of, 378–380
cubic, 706
discrete/continuous, 155–159
exponential, 337–340, 406–410
graphing, 19–20
identifying/writing, 21–24
inverses of, 355–358
linear, 246–250, 356
operations with, 136–140
parent, 119
quadratic, 194–199, 255
trigonometric, 331–334, 402
fundamental counting principle, overview, 241–242
fundamental theorem of algebra, 738

G

geometric definition, ellipses, 686
geometric probability, 391–395
geometric sequences, 678–681
geometric series, 786–790
golden ratio in art, 166
graphing
circles, 640–641
complex numbers, 489–492
cosine function, 606–609
data plotting, 153–154, 504
ellipses, 684–687
equations, 100–104
exponential functions, 337–339
functions, 19
functions, discrete/continuous, 155–159

hyperbolas, 757–760
inequalities, 61–65
inverses, 355–358
line fitting, 325–328
linear equations, 86–90, 246–250
linear inequalities, 279–283
linear programming problems, 388
logarithmic functions, 764–766
maxima and minima, 84–85
nonlinear systems, 813–816
parametric equations, 143–144
piecewise functions, 558–560
polar equations, 671–675
polynomial functions, 706–710
quadratic functions, 196, 215–219
quadratic inequalities, 623–627
rational functions, 696–699, 745–748
reciprocal trigonometric functions, 719–722
regression, 323–324
sine (sin) functions, 580–583
slope-intercept form, 86–90
system solutions, 100–104
systems of linear inequalities, 312–315
tangent function, 630–633
trigonometric identities, 752–754
vectors, 690–693

graphing calculator, 7, 9, 11, 12, 18, 26, 33, 39, 45, 52, 58, 66, 76, 83, 95, 98, 102, 106, 110, 123, 127, 133, 140, 141, 143, 144, 145, 149, 152, 154, 162, 168, 176, 178, 179, 181, 185, 193, 196, 200, 206, 213, 220, 231, 234, 238, 245, 248, 251, 255, 258, 264, 271, 277, 282, 283, 284, 292, 295, 301, 303, 305, 310, 313, 316, 321, 323, 324, 328, 329, 333, 336, 340, 347, 351, 354, 355, 358, 368, 370, 371, 373, 377, 382, 390, 397, 405, 412, 419, 423, 425, 431, 440, 446, 453, 455, 456, 458, 459, 461, 468, 474, 480, 488, 493, 497, 498, 500, 511, 518, 527, 531, 539, 544, 550, 556, 562, 579, 583, 589, 592, 596, 600, 604, 610, 616, 622, 623, 625, 626, 629, 634, 635, 636, 641, 645, 650, 653, 655, 656, 661, 663, 670, 674, 675, 677, 683, 688, 695, 703, 704, 705, 711, 718, 722, 730, 734, 736, 740, 741, 744, 748, 749, 756, 761, 767, 771, 776, 779, 783, 787, 789, 790, 796, 803, 804, 805, 806, 807, 812, 815, 817, 825, 827, 828
labs, 19–20, 27–28, 84–85, 114–115, 153–154, 178–179, 303, 323–324, 370–371, 455–456, 519–520, 563–564, 638–639, 804–805, 817–818
local maximum/minimum, 709
logarithmic equations/inequalities, 714–716
polynomial equations, 469–473
quadratic equations, 462–465
rational inequalities, 658–662
side lengths, 371–374
trigonometric equations and, 825–828

graphs and system solutions, 100

gravitational forces, 77–81

gravity, 357

greatest common monomial factor, 163–166

greatest integer functions, 635–636

grouping, factoring by, 436–438

growth, exponential, 406–410

H

half-angle identities, 798–801

half-planes, 279–283

Heron's formula, 548

holes in rational functions, 747

horizontal asymptotes, 337, 745

horizontal line test, 356

horizontal lines, equations of, 259–263

horizontal orientation, hyperbolas, 757–760

horizontal shifts (transformations), 725–728, 774–776

horizontal stretch/compression (transformations), 774–776

horizontal/vertical lines, graphing, 249

hyperbolas, 757–760, 793–795

hyperbolic geometry, 760

hypotenuse
definition, 296
error checking, 298
in trigonometric functions, 331

I

identities, trigonometric equations with, 825–828

imaginary numbers, 442–443

inclusive events, 427–430

increasing exponential equations, 652

independence of systems of equations, 100–104

independent events, 391–394, 427–430

independent variables, definition, 21

index, definition, 286, 536

inequalities
absolute value, 116–120
exponential, 652–655
linear, 312–315
quadratic, 623–627
rational, 658–662
solving, 61–65
in two variables, 279–283

infinitely many solutions, 173

initial side (angles of rotation), definition, 399

integers
definition, 2
identifying, 2
negative, 2

intercepts
linear equations, 87
zeros and roots vs., 254–253

interest, 339

Internet security, 293–295

intersecting lines, 147

intersection points
graphing, 84–85
as solution, 100–104

inverse properties of logarithms and exponents, 714–716

inverse variation, overview, 77–80

inverses
cube root function, 535–536
of exponents, 512–516, 574
of functions, 21–24
of functions/relations, 355–358
of logarithms, 512, 575
matrix, 233–237
of real numbers, 4
square root function, 534–535

of trigonometric functions, 476–480
trigonometric relations, 476
irrational numbers
definition, 2
identifying, 2
irrational root theorem, 470

J

joint variation, 77–80

K

kinetic energy, 79

L

lasers, 632
law of cooling, 407, 614
law of cosines, 546–549
law of sines, 506–509
least common denominator (LCD)
of rational expressions, 267, 658
least integer functions, 635–637
like terms, definition, 10
line slope, graphing, 100–103
linear binomials, dividing by, 365–366
linear equations, 42–45
abstract, 617
Cramer's Rule and, 107
graphing, 246–252
overview, 42–45
in two variables, 86–90
linear functions, 355–358
linear inequalities
systems of, 312–315
in two variables, 279–282
linear models, 809
linear programming, 384–388
linear regression, 323–324
linear systems
classification of, 101
definition, 100
with matrix inverses, 233–237
lines
best fitting, 325–332
equations of, 187–190
of graphs, 114–115
horizontal/vertical, 249
parallel/perpendicular, 259–263
slope of, 86–90, 100
types of, 147–148
literal equations, 617–619
local maxima/minima, 709
locus of points, 686
logarithms
data fitting, 810
equations/inequalities, 714–716
expressions, 574, 612–614
function graphing, 764–767
natural, 574–577
overview, 457–459
properties of, 512–516
logic and truth tables, 69–71
logical implication, definition, 69
logically equivalent, definition, 71
long division of polynomials, 273–276

M

main diagonals in matrices, 56
markups as percentages, 36–39
math language, 4, 13, 37, 38, 50, 72, 107, 125, 156, 170, 202, 215, 234, 240, 241, 249, 296, 297, 307, 308, 312, 325, 326, 331, 393, 427, 448, 463, 489, 497, 506, 523, 552, 558, 565, 586, 617, 619, 635, 640, 647, 649, 671, 674, 679, 725, 732, 794, 807, 820
math reasoning, 14, 21, 22, 30, 32, 56, 61, 63, 65, 101, 102, 109, 119, 126, 136, 137, 139, 148, 171, 172, 174, 195, 201, 210, 217, 282, 286, 289, 299, 326, 332, 348, 380, 384, 391, 394, 437, 442, 443, 462, 464, 470, 472, 483, 485, 490, 492, 495, 496, 498, 507, 521, 522, 524, 529, 530, 546, 548, 560, 566, 588, 592, 593, 595, 610, 623, 624, 626, 640, 646, 648, 658, 659, 679, 680, 681, 735, 740, 741, 742, 790, 809, 813, 810, 814, 820
analyze, 7, 14, 18, 26, 32, 40, 52, 53, 67, 91, 127, 128, 132, 134, 148, 165, 167, 172, 186, 190, 201, 204, 206, 210, 217, 222, 229, 231, 232, 237, 239, 242, 245, 251, 252, 257, 268, 271, 278, 282, 285, 286, 291, 310, 315, 321, 330, 335, 338, 341, 352, 360, 362, 366, 374, 381, 382, 383, 386, 389, 391, 396, 397, 404, 411, 425, 433, 435, 438, 440, 443, 445, 446, 454, 461, 462, 466, 468, 470, 474, 481, 483, 485, 487, 492, 494, 495, 498, 502, 503, 507, 510, 511, 517, 518, 521, 522, 527, 529, 530, 532, 544, 545, 555, 556, 562, 566, 569, 570, 577, 582, 588, 590, 593, 595, 596, 597, 609, 620, 621, 624, 628, 634, 644, 648, 650, 651, 653, 657, 658, 659, 663, 667, 668, 677, 682, 683, 685, 687, 695, 701, 713, 717, 730, 737, 742, 743, 745, 750, 753, 755, 757, 761, 764, 766, 767, 768, 778, 779, 784, 790, 792, 796, 801, 803, 809, 811, 817, 820, 822, 828
connect, 14, 30, 80, 96, 131, 254, 402
estimate, 7, 35, 41, 45, 66, 74, 75, 81, 83, 91, 151, 174, 176, 220, 353, 368, 376, 415, 454, 485, 488, 500, 526, 533, 545, 567, 570, 590, 604, 621, 657, 669, 682, 737, 749, 791, 797
formulate, 21, 35, 47, 69, 75, 83, 90, 91, 98, 105, 127, 134, 142, 144, 151, 186, 256, 273, 336, 341, 346, 412, 418, 435, 453, 459, 468, 474, 493, 494, 511, 517, 524, 550, 562, 582, 597, 598, 627, 637, 644, 648, 649, 663, 677, 679, 723, 730, 744, 747, 751, 755, 761, 784, 812, 817, 830
generalize, 26, 44, 69, 70, 77, 102, 109, 119, 121, 133, 135, 139, 144, 193, 230, 259, 278, 293, 316, 321, 326, 330, 340, 356, 368, 369, 383, 390, 419, 431, 434, 435, 442, 475, 482, 487, 493, 501, 507, 533, 546, 549, 551, 557, 561, 570, 584, 589, 597, 608, 609, 635, 644, 656, 657, 663, 665, 666, 673, 688, 693, 698, 704, 713, 717, 743, 750, 755, 763, 764, 768, 784, 803, 816
justify, 5, 17, 26, 35, 46, 69, 76, 83, 91, 141, 152, 162, 168, 171, 176, 189, 193, 198, 199, 213, 216, 220, 221, 244, 257, 271, 277, 297, 299, 301, 302, 315, 316, 321, 335, 360, 386, 389, 403, 418, 419, 426, 432, 437, 461, 467, 533, 539, 548, 562, 623, 635, 640, 646, 658, 663, 669, 676, 681, 688, 701, 718, 723, 735, 740, 779, 802, 814, 816, 822, 823

model, 22, 35, 98, 110, 113, 175, 220, 285, 301, 311, 359, 376, 377, 411, 453, 481, 504, 541, 579, 585, 609, 615, 686, 689, 723, 730, 768, 791, 801, 813, 829
multiple representations, 134, 142, 636
predict, 67, 77, 126, 149, 291, 362, 363, 376, 398, 538, 556, 664, 670, 712, 778, 807
verify, 6, 12, 17, 18, 25, 35, 40, 41, 46, 47, 53, 58, 59, 67, 88, 98, 105, 122, 127, 130, 150, 163, 167, 188, 200, 206, 213, 231, 237, 252, 261, 272, 275, 283, 284, 285, 292, 309, 329, 332, 337, 341, 346, 347, 354, 356, 357, 361, 362, 363, 366, 368, 369, 387, 389, 394, 404, 424, 425, 428, 429, 432, 440, 441, 445, 446, 464, 472, 476, 482, 490, 496, 500, 501, 514, 515, 518, 533, 541, 545, 551, 561, 577, 584, 598, 606, 626, 630, 645, 650, 656, 677, 680, 690, 696, 735, 741, 746, 748, 752, 757, 759, 781, 790, 817
write, 5, 12, 17, 18, 25, 35, 41, 42, 53, 59, 68, 106, 112, 129, 140, 141, 144, 145, 162, 166, 167, 185, 186, 192, 193, 198, 204, 208, 213, 214, 232, 238, 244, 245, 252, 257, 266, 270, 271, 277, 291, 292, 301, 310, 314, 316, 322, 330, 336, 339, 341, 346, 347, 353, 360, 369, 375, 382, 389, 395, 405, 411, 416, 418, 426, 431, 439, 440, 446, 453, 460, 461, 466, 474, 475, 482, 488, 493, 500, 501, 510, 517, 526, 534, 535, 536, 538, 554, 556, 570, 577, 579, 585, 590, 603, 604, 611, 616, 619, 621, 627, 629, 634, 636, 645, 649, 651, 677, 681, 688, 689, 693, 694, 702, 710, 713, 716, 717, 724, 728, 731, 737, 744, 750, 755, 763, 778, 781, 785, 791, 796, 797, 811, 812, 822, 823, 829

mathematical modeling
exponential growth and decay, 406–407
with linear equations, 190

matrices
coefficient, 107
data organization, 29–35
data storage, 27–28
inverses, 233–239
multiplying, 54–60
rotation, 782
square, 93, 94–95

maxima
finding, 215–219
graphing, 84–85
local, 709
of objective functions, 385

mean
cautions, 182
confidence interval estimates, 519–520
definition, 180
as measure of center, 504
of normal distributions, 566–567
symbols of, 183

measure, units of, 124–126

measures of central tendency/ dispersion, 180–181, 504

median, 180, 504

median regression, 323–324

midpoint formula, 642

minima
finding, 215–219
graphing, 84–85
local, 709
of objective functions, 385

misleading data, recognizing, 819–821

mode, 180, 504

modeling, 77–78, 414

monomials
definition, 72
as divisors, 273
factoring/as factors, 163–164, 540–541, 542
in fractions, 318–319

multiple choice, 6, 12, 17, 25, 35, 40, 47, 52, 53, 59, 68, 76, 83, 92, 99, 104, 106, 113, 121, 127, 132, 142, 151, 161, 162, 168, 177, 184, 187, 191, 199, 205, 214, 221, 231, 232, 237, 245, 251, 252, 256, 258, 270, 277, 278, 283, 291, 301, 310, 316, 321, 329, 334, 335, 340, 342, 347, 353, 354, 359, 367, 369, 375, 376, 382, 383, 388, 390, 391, 396, 398, 404, 411, 417, 418, 425, 426, 430, 432, 440, 441, 445, 453, 461, 467, 473, 475, 481, 487, 488, 493, 500, 501, 510, 511, 517, 526, 527, 532, 533, 538, 539, 543, 544, 550, 555, 556, 561, 562, 569, 570, 573, 577, 578, 582, 588, 590, 595, 596, 604, 608, 614, 619, 620, 626, 627, 633, 645, 649, 650, 656, 662, 663, 669, 677, 683, 689, 695, 712, 718, 722, 723, 730, 735, 737, 743, 749, 755, 762, 767, 768, 778, 783, 785, 792, 796, 797, 802, 812, 816, 823, 829

multiplication
of complex numbers, 490
of inequalities, 62
of functions, 138
of polynomials, 129–132
properties of equality, 42
of rational expressions, 226–230
of real numbers, 3, 4

multiplicative identity matrices, 56–57

multiplicative inverses, 34

multi-step, 6, 12, 18, 26, 34, 40, 47, 52, 60, 67, 70, 75, 83, 90, 97, 104, 110, 122, 128, 134, 140, 152, 161, 162, 167, 175, 176, 186, 191, 192, 198, 199, 205, 206, 220, 221, 231, 232, 237, 238, 244, 245, 251, 270, 271, 272, 278, 284, 290, 292, 301, 310, 311, 317, 321, 322, 329, 335, 340, 342, 346, 352, 354, 359, 362, 363, 369, 376, 377, 382, 390, 396, 397, 404, 405, 410, 418, 419, 425, 431, 433, 441, 445, 454, 460, 461, 467, 474, 475, 481, 486, 488, 494, 500, 501, 511, 517, 526, 527, 533, 537, 538, 544, 550, 555, 556, 561, 569, 577, 583, 588, 589, 596, 603, 609, 610, 614, 620, 621, 627, 644, 649, 650, 657, 662, 668, 676, 682, 689, 695, 700, 702, 712, 718, 722, 723, 731, 737, 744, 750, 751, 756, 762, 767, 779, 783, 784, 791, 792, 796, 797, 803, 808, 815, 822, 823, 828, 830

multi-step logic, 70–71

multiplicity, 469–470

mutual exclusivity, 241, 427–430

N

_n_ terms of a series, 734
natural logarithm, 407, 574–577
navigation, 333, 479
negative discriminants, 530
negative exponents, 13–15
Newton, Isaac, 79, 80
Newton's Law of Cooling, 407, 614
nonlinear equations, 813–815
normal distribution, curves, 565–568
notation
  of functions, 23–24
  of operations with functions, 136–137
***n*th roots,** 420
***n*th terms**
  of arithmetic sequences, 647–649
  of geometric sequences, 679–681
numerators, slant asymptotes, 746–747

O

objective functions, 384–385
odd degree, 707
odds, calculating, 394
operations, order of, 8
ordered pairs, 86, 279, 355
ordered triples, 207–208
orientation, hyperbolas, 757–759
outcomes, definition, 240
outliers, effects of, 181–182

P

parabolas, 194–197, 215–219, 793
parallel lines, 148, 259–265, 313
parametric equations, 143–145
parent functions, 119, 337
parentheses, in expressions, 9
Pascal's Triangle, 308, 348
percentages, changes in, 36–39
perfect square trinomials, 165, 434
perfect squares, 413–415
periodic functions, 605–609, 630–632
permutations
  combinations and, 303, 307
  counting with, 304–309
  probability and, 392–393
perpendicular lines, 259–265
phase shift right, 781
piecewise functions, 558–562
planets, rotation of, 451, 643, 687
point of discontinuity, 156
point-slope form, 188–189, 246–247
polar coordinates, 671–675
polar equations, 673–675
polar forms, 771–772
polygon vertices, 314
polynomials
  cubic, 595
  data fitting, 806–809
  division of, 273–276, 364, 665–666
  equations, 469–473, 617–618, 738–741
  factoring, 163–165
  functions, 706–710, 774–777
  linear, 593
  multiplying, 129–131
  overview, 72–74
  quadratic, 594
  roots, 540–543, 598–603, 738–741
  synthetic substitution, evaluating with, 366
population sampling, 502–505, 519–520, 521–525
positive discriminants, 529
potential energy (PE), 81
power property of logarithms, 515, 575–576, 714
power rule, for exponents, 14–15
power of *i*, 491
precision (accuracy), 125
prime factors/numbers, 163
probability
  conditional, 483–486
  finding, 351
  overview, 390–395
  sampling, 521–523
product property of logarithms, 514, 576, 714–715
product rule, 14, 287
programming, linear, 384–387
projectile motion, 542
properties of *n*th roots, 420
Pythagorean Theorem, 296–300, 781, 800

Q

quadratic equations
  with imaginary numbers, 443
  with real zeros, 463
  from roots, 586–589
  roots of, 528–529, 739
  solving, 253–254, 444, 552–555
  as squares, 413–415
quadratic form, 825–826
quadratic formula, 433–435, 462–465, 826
quadratic functions, 194–197, 706
quadratic inequalities, 623–629
quadratic model, 806–807
quadratic polynomials, 594
quadratic terms, 698–699, 746
quadratic trinomial equations, 164–165
quality control, 485
quartic functions, 706
quartic model, 809
questioning, data, 820
quintic functions, 706
quotient of powers property, 201–202
quotient property of logarithms, 514–515, 575–576, 714–715
quotients, definition, 273

R

radian measures, 447–452
radians, converting to degrees, 448
radical conjugates, 319
radical equations, 495–499, 619
radical expressions, 286–290, 356–357, 420–421
radical functions, graphing, 534–537
radicands, 286, 287
random digit tables, 523
random events, 391
random sampling, 522–523

ranges
definition, 181
excluded values and, 201
of functions, 21–22, 157, 378
of inverses, 355
and maxima and minima, 216

rate conversion, 125

rational equations, 592–595, 618

rational exponents, 420, 421–423

rational expressions
addition/subtraction, 266–269
multiplication/division, 226–229
rational functions and, 592
simplifying, 201–204

rational functions, 592, 696–699, 745–748

rational inequalities, 658–664

rational numbers
definition, 2
identifying, 2

rational root theorem, 470, 600–602, 698, 739

ratios, direct variation as, 49

reading math, 23, 29, 64, 78, 109, 116, 118, 119, 124, 173, 183, 254, 280, 378, 379, 392, 420, 447, 457, 476, 477, 504, 709, 799

real numbers, properties of, 2–5

real zeros, 463

reasoning. *See* **math reasoning**

reciprocals
of real numbers, 228
of trigonometric functions, 331–333, 719–722

reduced-row-echelon form, 28

reference angles
rotation, angles of, 401
trigonometric functions and, 449–450

reflections, transformations, 725–729, 774–776

regression, 323–324, 325, 804–805

relations, inverses of, 355–358

remainder, definition, 273

Remainder Theorem, 366, 665–667

right triangles, 262, 296–298, 372–377

roots
definition, 286
of polynomial equations, 470–471, 738–741
of polynomials, 540–543, 598–603
of quadratic equations, 528–529
quadratic equations derived from, 586–589
of quadratic functions, 195
testing for, 666–667
types of, 287
zeros and *x*-intercepts vs., 254–255

rotation
angles of, 399–403

rotation matrix, 782

S

sample space, definition, 240–241

sampling
confidence intervals, 519–520
data collection, 502–505
errors, 819–821
overview, 521–525

scalar multiplication, 31

scalar quantity, 691

scale factor, 372

scientific notation, 15

secant (sec), 332–333

secant function, 719–720

sequence, definition, 646, 732

shift, in graphing, 215

side lengths, finding, 373–374

side-angle-side (SAS), 546

side-side-side (SSS), 546

sigma notation, 732–733

sign tables, 660

significant digits, in measurements, 125–126

simple random sample (SRS), 522–523

simultaneous equations, 207–212

sine (sin), 331–332, 373, 449–450, 476–477, 506–509

sine (sin) function, 580–583

slant asymptotes, 745–748

slopes
intercepts, 88, 187, 188, 281
of lines, 100
undefined, 249

solution tables, 86

solutions; infinite/none, 173

sound waves, 608

special product patterns, 131

special right triangles, 372–374

speed equation, 269

square matrices, 56, 93

square roots
of equations, 495–499
in functions, 534–535, 725–727
graphing calculators and, 181
of negative numbers, 442
property of, 413

squares, completing, 413–417

standard deviation
cautions, 182
data collection and, 504
definition, 181
deviations, 566
and range, 181–183
unknown, 520

standard form, 72–73, 187–188, 194–195, 247

standard normal distribution, 567

standard position (angles of rotation), 399–400

statistical accuracy, 819–821

statistics, definition, 180

step functions, 558, 635–637

stratified samples, 503, 505, 522

stretching, of parabolas, 217

substitution ciphers, 293

substitution method
solving by, 146–149, 813
synthetic, 366

subtraction
of cubes, 437
of matrices, 30–31
of functions, 136–138
of polynomials, 73, 273–275
properties of equality, 42
properties of inequalities, 61, 62
of rational expressions, 266–269
vectors, 691

summation notation, 732–733, 735

sum and difference identities, 780–783

symmetry, of quadratic functions, 216
synthetic division, 364–366, 436
synthetic substitution, 366
system consistency, 170
systematic samples, 503, 505, 522
systems, solving, 100–103
systems of equations
definition, 100
nonlinear, 812–815
with quadratic inequalities, 626
solving by elimination method, 171
solving by substitution, 146–149
in three variables, 207–211
systems of linear inequalities, 312–314

T

tables
building, 19–20
system solutions and, 100
of values, 280
tangent (tan), 331–333, 373, 448–451, 476–479
tangent function, 630–632
tautology, definition, 69
technology. *See* **graphing calculator**
term, definition, 646
terminal side (angles of rotation), 399
test point values, 659
theoretical probability, 391
three-dimensional figures, 438
transformations
definition, 215
of exponential functions, 338–339
finding, 725–729
of functions, 119–120
of $f(x)$, 248
of polynomial functions, 774–777
transitive property, 61, 62
tree diagrams, 240
trial, definition, 240
triangle inequality theorem, 65
triangles. *See also* **trigonometric functions**
angles of, 508, 547–548
area of, 506–507, 548
Pascal's, 308
right, 262, 296–298
sides of, 507, 546–547
triangulation, 508
trigonometric functions
inverses of, 476–480
overview, 402
reciprocals, 331–334, 719–722
unit circle, 449
trigonometry
De Moivre's theorem, 772–773
equations, 825–828
graphing calculator and, 370–371
identities, 752–754
ratios, 447–448
special right triangles, 372–373
trinomials
binomials, multiplying by, 130
rational expressions, simplifying, 203
solving equations, 164–165
truth tables, 69–71
turning points, 709

U

unit circles, 447–452
units of measure, 124–126
Universal Law of Gravitation, 80

V

values, of rational expressions, 268
variables
in algebraic expressions, 8
for data, statistical, 178–179
definition, 8
dependent/independent, 21–22
in matrices, 31
on one/both sides, 43
in polynomials, 72
variance, 504
variation
direct, 48–51
inverse, 77–80
vectors, 690–693
vertex form, of quadratic functions, 194–195, 216
vertical asymptotes, 745
vertical lines, 22–23, 262, 356
vertical orientation, hyperbolas, 757–759
vertical shifts (transformations), 725–727, 729, 776
vertical stretch/compression (transformations), 725–728, 774–776
vertical/horizontal lines, 249
vertices
of feasible regions, 384
of hyperbolas, 757
voluntary response sampling, 523

W

whole numbers, 2, 4
windows, of graphs, 114–115

X

***x*-intercepts,** 254–255

Z

zero discriminants, 529
zero exponent property, 421
zero matrices, 30
zero product property, 164, 253, 470, 586
zero remainder, 665
zeros
of factored polynomials, 540
finding, 253
real/complex, 463–464
roots and *x*-intercepts vs., 254–255
z-scores, 567

INDEX